Formal averaging of periodic and quasi-periodic vector fields

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Example (Fermi-Pasta-Ulam type problem)

Hamiltonian system with Hamiltonian function

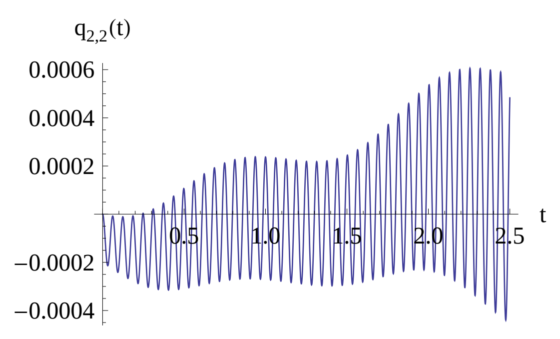
$$\begin{aligned} H(p,\bar{p},q,\bar{q}) &= \frac{1}{2}(p^T p + \bar{p}^T \bar{p}) + \frac{1}{2\epsilon^2} q^T q + U(q,\bar{q}), \\ U(q,\bar{q}) &= \frac{1}{4}\left((\bar{q}_1 - q_1)^4 + (\bar{q}_m + q_m)^4\right) \\ &\quad + \frac{1}{4}\sum_{j=1}^{m-1} (\bar{q}_{j+1} - q_{j+1} - \bar{q}_j - q_j)^4. \end{aligned}$$

We consider $m = 3$, $\epsilon = 1/100$, and initial values

$$\bar{p}(0) = p(0) = \bar{q}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ q(0) = \begin{pmatrix} \epsilon \\ 0 \\ 0 \end{pmatrix}.$$

Example (Fermi-Pasta-Ulam time problem (cont.))

Solution for the component $q_2(t)$,



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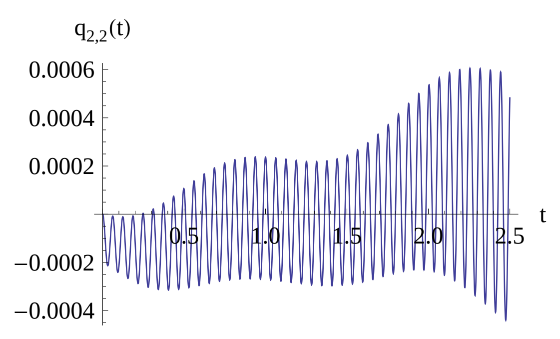

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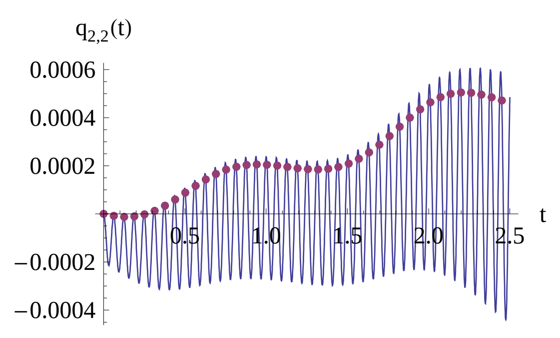

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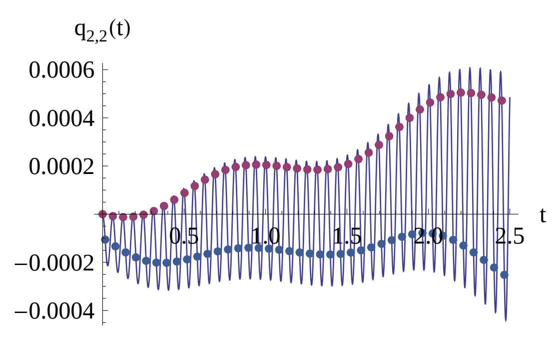

and for $n = 0, 1, 2, 3, \ldots,$

$$q_2(2\pi\epsilon n), \quad \text{and} \quad q_2(\frac{\pi\epsilon}{2} + 2\pi\epsilon n).$$

Consider a Hamiltonian system

$$\frac{d}{dt}y = J^{-1}\nabla H(y;\epsilon), \quad H(y;\epsilon) := \epsilon^{-1}H_F(y) + H_S(y),$$

Let $\varphi_\tau^\epsilon : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be such that $\varphi_{t/\epsilon}^\epsilon$ is the t-flow of that system. Assume that φ_τ^0 is (2π)-periodic.

Stroboscopic averaging

There exists $\mathcal{H}(Y;\epsilon) = \mathcal{H}_0(Y) + \epsilon\mathcal{H}_1(Y) + \epsilon^2\mathcal{H}_2(y) + \cdots$,

$$\frac{d}{dt}Y = J^{-1}\nabla\mathcal{H}(Y;\epsilon),$$

such that, $Y(2\pi\epsilon n) = y(2\pi\epsilon n)$ if $Y(0) = y(0) = y_0$.

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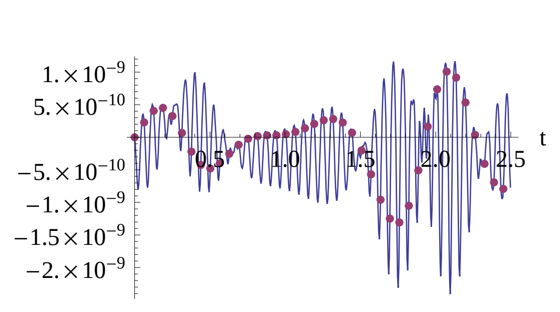
Example (Fermi-Pasta-Ulam type problem)

We consider

$$\begin{aligned} \tilde{\mathcal{H}}(Y,\epsilon) &:= \mathcal{H}_0(Y) + \epsilon^2 \mathcal{H}_2(Y) + \epsilon^4 \mathcal{H}_4(Y) \\ &= \mathcal{H}(Y;\epsilon) + \mathcal{O}(\epsilon^6), \end{aligned}$$

and plot the variation $\tilde{\mathcal{H}}(y(t);\epsilon) - \tilde{\mathcal{H}}(y(0);\epsilon)$



Smooth invariant

Under the previous assumtions for

$$\frac{d}{dt}y = J^{-1}\nabla H(y;\epsilon), \quad H(y;\epsilon) := \epsilon^{-1}H_F(y) + H_S(y),$$

consider $\mathcal{H}(Y;\epsilon) = \mathcal{H}_0(Y) + \epsilon\mathcal{H}_1(Y) + \epsilon^2\mathcal{H}_2(y) + \cdots$ as before, then $\mathcal{H}(y;\epsilon)$ is a first integral of the original system.

Indeed, for $t_n = 2\pi\epsilon n$, $n = 1, 2, \ldots$

$$H(Y(t_n);\epsilon) = H(y(t_n);\epsilon) = \mathrm{Const}$$

and by a interpolating argument, $H(Y(t);\epsilon) = \mathrm{Const}$, and thus

$$\{H, \mathcal{H}\} \equiv 0.$$

Can this be generalized to the multi-frequency case?

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Numerical integration of HOS with ϵ-independent time-steps

Integrate the smooth system

$$\frac{d}{dt}Y = J^{-1}\nabla\mathcal{H}(Y;\epsilon), \quad Y(0) = y_0$$

instead of the highly oscillatory one. Different options

- Symbolic-numeric algorithms using explicit knowledge of $\mathcal{H}$
- Purely numerical schemes that try to approximate $Y(t)$ by using H as input (HMSM, SAM, ...).

Motivated by that, we aim at

- Obtaining formulae for $\mathcal{H}(Y;\epsilon)$ and its solutions $Y(t)$
- Such formulae should be as explicit as possible and of universal character
- Knowledge about possible (formal) invariants of the original system.

Standard high order averaging [Bogoliubov and Mitropolski 1958, Perko 1969, Sanders, Verhulst, Murdock 2007]

Under suitable assumptions on the HOS

$$\frac{d}{dt}y \quad = \quad f(y, t/\epsilon).$$

there exists a formal $(2\pi\epsilon)$-periodic change of variables $y = K(Y, t/\epsilon)$ that transforms the original HOS into the (*averaged*) autonomous equations

$$\frac{d}{dt}Y \quad = \quad F(Y;\epsilon) := F_0(Y) + \epsilon F_1(Y) + \epsilon^2 F_2(Y) + \cdots$$

The change of variables $y = K(Y, \tau)$ is not unique:

- Stroboscopic averaging: $K(Y, 0) = Y$, which implies $Y(2\pi\epsilon n) = y(2\pi\epsilon n)$ for all $n \in \mathbb{Z}$.
- $\int_0^{2\pi} K(Y, \tau)\, d\tau = Y$,

Autonomous form for stroboscopic averaging

Consider an autonomous system (in terms of slow time $\tau = t/\epsilon$),

$$\frac{d}{d\tau} y = f_F(y) + \epsilon f_S(y),$$

and denote φ_τ^ϵ its τ-flow. Assume that $\varphi_\tau^0(y)$ is (2π)-periodic in τ.

The periodic change of variables $y = \varphi_\tau^0(x)$ leads to a system in standard form for periodic averaging

$$\frac{d}{d\tau} x = \epsilon f(x, \tau), \qquad f(x, \tau) = \frac{\partial}{\partial x} \varphi_{-\tau}^0(x)\, f_S(\varphi_\tau^0(x))$$

where $f(y, \tau)$ is (2π)-periodic in τ. In turn, a system in standard form can be written in autonomous form, with $y = (x, \theta) \in \mathbb{R} \times \mathbb{T}$, and

$$\frac{d}{d\tau} \begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \epsilon \begin{pmatrix} f(x, \theta) \\ 0 \end{pmatrix}$$

Recall that we consider an autonomous system

$$\frac{d}{d\tau} y = f_F(y) + \epsilon f_S(y), \tag{1}$$

and denote φ_τ^ϵ its τ-flow. ($\varphi_\tau^0(y)$ is (2π)-periodic in τ.)

General idea for formal averaging

- Obtain a formal representation of the τ-flow φ_τ^ϵ in the form $\varphi_\tau^\epsilon(y) = \Phi_{\tau,\tau}(y)$, where

$$\Phi_{\tau,\theta}(y) = \varphi_\theta^0(y + \epsilon G_1(y,\tau,\theta) + \epsilon^2 G_2(y,\tau,\theta) + \cdots),$$

  and each $G_j(y,\tau,\theta)$ is polynomial in τ and (2π)-periodic in θ.
- Under general assumptions, it holds $\forall(\tau,\theta),(\tau',\theta') \in \mathbb{R} \times \mathbb{T}$,

$$\Phi_{\tau,\theta} \circ \Phi_{\tau',\theta'} = \Phi_{\tau+\tau',\theta+\theta'}(y).$$

- In particular, both $\Phi_{\tau,0}$ and $\Phi_{0,\tau}$ are flows of autonomous vector fields, the former a smooth near-to-identity map, and the later a periodic map.
- If both $f_F(y)$ and $f_S(y)$ are Hamiltonian, then $\Phi_{\tau,0}$ and $\Phi_{0,\tau}$ are Hamiltonian flows (with Hamiltonian functions $\epsilon\mathcal{H}(y)$ and $H_F(y) + \epsilon(H_S(y) - \mathcal{H}(y)))$, and since they commute with the flow $\varphi^{\epsilon}_{\tau} = \Phi_{\tau,\tau}$, we have that $\{\mathcal{H}, H_F + \epsilon H_S\} = 0$ (i.e., $\mathcal{H}$ is a formal invariant of the system).
- The (stroboscopically) averaged ODE is

$$\frac{d}{d\tau}Y = F(Y) := \frac{d}{d\tau}\Phi_{\tau,0}(Y)\Big|_{\tau=0}, \quad Y(\tau) = \Phi_{\tau,0}(Y(0)),$$

  and the (2π)-periodic change of variables is $y = K(Y,\tau) := \Phi_{0,\tau}(Y)$, so that, if $Y(0) = y(0) = y_0$,

$$y(\tau) = \varphi^{\epsilon}_{\tau}(y_0) = \Phi_{\tau,\tau}(y_0) = \Phi_{0,\tau}(\Phi_{\tau,0}(y_0)) = K(Y(\tau),\tau).$$

- The idea generalizes nicely to the quasiperiodic case!

Multifrequency case

Given $\omega \in \mathbb{R}^d$ non-resonant ($k \cdot \omega \neq 0$ for all $k \in \mathbb{Z}^d \backslash \{0\}$),

$$\frac{d}{dt} y = f(y, \omega\, t), \quad y(0) = y_0,$$

where $f(y, \theta)$ is

- smooth in y
- 2π-periodic in each component of $\theta \in \mathbb{R}^d$, with Fourier expansion

$$f(y, \theta) = \sum_{k \in \mathbb{Z}^d} e^{i(k \cdot \theta)} f_k(y).$$

- The map f itself may depend on the frequencies ω but we do not reflect that in the notation.

High order averaging of quasi-periodic vector fields (Perko 1969)

Given the quasi-periodic vector field

$$\frac{d}{dt}y \;=\; \epsilon f(y, \omega\, t) = \epsilon \sum_{k \in \mathbb{Z}^d} e^{i(k\cdot\omega)t} f_k(y).$$

there exists a formal quasi-periodic change of variables $y = K(Y, \omega\, t)$ that transforms the QP system into

$$\frac{d}{dt}Y \;=\; \epsilon F_1(Y) + \epsilon^2 F_2(Y) + \cdots$$

- The first term $F_1(y)$ is uniquely determined as
$$F_1(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(y, \theta)\, d\theta = f_0(y).$$
- $K(Y, \theta)$ is not unique. Classical choice:
$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} K(y, \theta)\, d\theta = y$$

B-series expansion of solution of the QP system

For the solutions of $\dot{y} = \sum_k e^{i(k\cdot\omega)t} f_k(y)$,

$$y(t) = y(0) + \sum_{u\in\mathbb{T}} \frac{\alpha_u(t)}{\sigma_u} \mathcal{F}_u(y(0)),$$

$\mathbb{T}$ is the set of rooted trees labelled by $k \in \mathbb{Z}^d$, and for each $u \in \mathbb{T}$,

- the coefficients $\alpha_u(t)$ are linear combinations of $t^j e^{i(k\cdot\omega)t}$,
- the elementary differentials $\mathcal{F}_u : \mathbb{R}^d \to \mathbb{R}^d$ are smooth maps,

  ($\sigma_u \in \mathbb{Z}$ is a normalization factor.)

Elementary coefficients

$$\alpha_u(t) = \int_0^t e^{i(k\cdot\omega)t'} \alpha_{u_1}(t') \cdots \alpha_{u_m}(t')\, dt', \qquad u = [u_1 \cdots u_m]_k.$$

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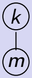
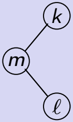
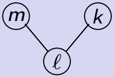
Examples for rooted trees with less than 4 vertices

u	$\mathcal{F}_u(y)$	$\alpha_u(t)$
	$f_k(y)$	$\int_0^t e^{i(k\cdot\omega)t_1}\, dt_1$
	$f'_m(y) f_k(y)$	$\int_0^t \int_0^{t_2} e^{i(kt_1+mt_2)\cdot\omega}\, dt_1\, dt_2$
	$f'_\ell(y) f'_m(y) f_k(y)$	$\int_0^t \int_0^{t_3} \int_0^{t_2} e^{i(kt_1+mt_2+\ell t_3)\cdot\omega}\, dt_1\, dt_2\, dt_3$
	$f''_\ell(y)(f_m(y), f_k(y))$	$\int_0^t \int_0^{t_2} e^{i(kt_1+mt_1+\ell t_2)\cdot\omega}\, dt_1\, dt_2$

For each $u \in \mathbb{T}$,

$$\alpha_u(t) = \sum_{k\in\mathbb{Z}^d} \alpha_u^k(t) e^{i(k\cdot\omega)t},$$

where each $\alpha_u^k(t)$ is a polynomial in t.

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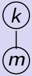
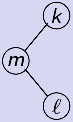
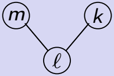
	$f'_m(y)f_k(y)$	$\int_0^t \int_0^{t_2} e^{i(kt_1+mt_2)\cdot\omega}\,dt_1\,dt_2$
	$f'_\ell(y)f'_m(y)f_k(y)$	$\int_0^t \int_0^{t_3} \int_0^{t_2} e^{i(kt_1+mt_2+\ell t_3)\cdot\omega}\,dt_1\,dt_2\,dt_3$
	$f''_\ell(y)(f_m(y),f_k(y))$	$\int_0^t \int_0^{t_2} e^{i(kt_1+mt_1+\ell t_2)\cdot\omega}\,dt_1\,dt_2$

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$$\alpha_u(t) = \sum_{k\in\mathbb{Z}^d} \alpha_u^k(t)e^{i(k\cdot\omega)t},$$

where each $\alpha_u^k(t)$ is a polynomial in t.

Averaging with B-series

There exist $\bar{\beta}_u, \bar{\alpha}_u(t), \kappa_u(\theta)$, $u \in \mathbb{T}$, ($\bar{\alpha}_u(t)$ polynomial, $\kappa_u(\theta)$ (2π)-periodic) such that for any solution $y(t)$ of the QP system

$$y(t) = K(Y(t), \omega t), \quad \frac{d}{dt} Y(t) = F(Y(t)),$$

where

$$\begin{aligned} F(Y) &= \sum_{u \in \mathbb{T}} \frac{\bar{\beta}_u}{\sigma_u} \mathcal{F}_u(Y), \\ Y(t) &= Y(0) + \sum_{u \in \mathbb{T}} \frac{\bar{\alpha}_u(t)}{\sigma_u} \mathcal{F}_u(Y(0)), \\ K(Y, \theta) &= Y + \sum_{u \in \mathbb{T}} \frac{\kappa_u(\theta)}{\sigma_u} \mathcal{F}_u(Y), \end{aligned}$$

That is, $\alpha(t) = \bar{\alpha}(t) * \kappa(t\omega)$ with $\frac{d}{dt}\bar{\alpha}(t) = \bar{\alpha}(t) * \bar{\beta}$.

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Observe that $\alpha(t)$ is such that

$$\frac{d}{dt}\alpha(t) = \alpha(t) * \beta(\omega t), \quad \alpha(0) = \mathbb{1},$$

where $\beta_k(\theta) = e^{ik\cdot\theta}$ and $\beta_u(\theta) = 0$ if u has more than one vertices. Recall that, for each $u \in \mathbb{T}$,

$$\alpha_u(t) = \sum_{k\in\mathbb{Z}^d} \alpha_u^k(t) e^{i(k\cdot\omega)t},$$

where each $\alpha_u^k(t)$ is a polynomial in t. Consider now

$$\gamma_u(t,\theta) = \sum_{k\in\mathbb{Z}^d} \alpha_u^k(t) e^{i(k\cdot\theta)},$$

so that $\alpha(t) = \gamma(t, t\omega)$. It is not difficult to see that

$$(\tfrac{\partial}{\partial t} + \omega\cdot\nabla_\theta)\gamma(t,\theta) = \gamma(t,\theta) * \beta(\theta), \quad \gamma(0,0) = \mathbb{1}.$$

The multifrequency case

Consider non-resonant $\omega \in \mathbb{R}^d$ ($k \cdot \omega \neq 0$ if $0 \neq k \in \mathbb{Z}^d$) and

$$\frac{d}{d\tau} y = f_F(y) + \epsilon f_S(y)$$

where its τ-flow φ_τ^ϵ is such that $\varphi_\tau^0 = \Psi_{\tau\omega}$, where $\forall \theta, \theta' \in \mathbb{T}^d$

$$\Psi_\theta \circ \Psi_{\theta'} = \Psi_{\theta+\theta'}.$$

The change of variables $y = \varphi_\tau^0(x) = \Psi_{\tau\omega}(x)$ transforms the autonomous system into

$$\frac{d}{d\tau} x = \epsilon f(x,\tau), \qquad f(x,\tau) = \frac{\partial}{\partial x} \Psi_{-\tau\omega}(x)\, f_S(\Psi_{\tau\omega}(x)).$$

We want to expand the solutions $x(\tau)$ of the later in the form

$$x(\tau) = x(0) + \epsilon G_1(x(0), \tau, \tau\omega) + \epsilon^2 G_2(x(0), \tau, \tau\omega) + \cdots$$

Where each $G_j(x, \tau, \theta)$ is (2π)-periodic in each component of θ, it is polynomial in τ, and $G_j(x, 0, 0) = 0$.

This will give an expansion of the solutions of the original autonomous system of the form

$$y(\tau) = \Phi_{\tau,\tau\omega}(y(0))$$

,

$$\Phi_{\tau,\theta}(y) = \Psi_{\theta}(y + \epsilon G_1(y,\tau,\theta) + \epsilon^2 G_2(y,\tau,\theta) + \cdots)$$

There are many options of write such expansion. Among then, we choose first Fourier expanding

$$f(x,\theta) = \frac{\partial}{\partial x}\Psi_{-\theta}(x)\, f_S(\Psi_{\theta}(x)) = \sum_{k\in\mathbb{Z}^d} e^{ik\cdot\theta} f_k(x),$$

and then, either using series indexed by rooted trees (B-series), or series indexed by words.

Given $\omega \in \mathbb{R}^d$ non-resonant ($k \cdot \omega \neq 0$ if $0 \neq k \in \mathbb{Z}^d$)

$$\frac{d}{d\tau} x := \epsilon f(x, \tau\omega) = \epsilon \sum_{k \in \mathbb{Z}^d} e^{i\tau(k\cdot\omega)} f_k(x),$$

Expansion of solutions of quasi-periodic system

$$x(t) = x(0) + \sum_{w \in \mathcal{W}} \epsilon^{|w|} \gamma_w(\tau, \omega\tau)\ f_w(x(0)),$$

where $\mathcal{W}$ is the set of 'words' $w = k_1 \cdots k_r$ on the alphabet $\mathbb{Z}^d$, and for each word w,

- $|w|$ is the number of letters in the word w, $\gamma_w(\tau, \theta)$ depends polynomially in τ and it is a Laurent polynomial on each $e^{i\theta_j}$,
- and $f_{mk} = f_k' f_m$, $f_{\ell mk} = f_k''(f_m, f_\ell) + f_k' f_m' f_\ell$, and in general,

$$f_{k_1 \cdots k_r}(x) = \frac{\partial}{\partial x} f_{k_2 \cdots k_r}(x) f_{k_1}(x).$$

Given $\omega \in \mathbb{R}^d$ non-resonant ($k \cdot \omega \neq 0$ if $0 \neq k \in \mathbb{Z}^d$)

$$\frac{d}{d\tau} x := \epsilon f(x, \tau\omega) = \epsilon \sum_{k \in \mathbb{Z}^d} e^{i\tau(k\cdot\omega)} f_k(x),$$

Expansion of solutions of quasi-periodic system

$$x(t) = x(0) + \sum_{w \in \mathcal{W}} \epsilon^{|w|} \gamma_w(\tau, \omega\tau)\; f_w(x(0)),$$

where $\mathcal{W}$ is the set of 'words' $w = k_1 \cdots k_r$ on the alphabet $\mathbb{Z}^d$, and for each word w,

- $|w|$ is the number of letters in the word w, $\gamma_w(\tau, \theta)$ depends polynomially in τ and it is a Laurent polynomial on each $e^{i\theta_j}$,
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Each $\gamma_w(\tau,\theta)$ is a linear combination of terms of the form $\tau^j e^{i(k\cdot\theta)}$. The coefficients $\gamma_w(\tau,\theta)$ are a solution of

$$\frac{\partial}{\partial\tau}\gamma_{wk}(\tau,\theta)+\omega\cdot\nabla_\theta\,\gamma_{wk}(\tau,\theta)=e^{ik\cdot\theta}\gamma_w(\tau,\theta),\quad \gamma_w(0,0)=0,$$

which is unique if we require that for each word w, $\gamma_w(\tau,\theta)$ depends polynomially on τ.

Recursive formulae for $\gamma_w(\tau,\theta)$

If $r\in\mathbb{Z}^+$, $k\in\mathbb{Z}^d-\{0\}$, $l\in\mathbb{Z}^d$, and $w\in\mathcal{W}\cup\{\emptyset\}$,

$$\begin{aligned}
\gamma_k(\tau,\theta) &= \frac{i}{k\cdot\omega}(1-e^{i(k\cdot\theta)}),\\
\gamma_{0^r}(\tau,\theta) &= \frac{\tau^r}{r!},\\
\gamma_{0^rk}(\tau,\theta) &= \frac{i}{k\cdot\omega}(\gamma_{0^{r-1}k}(\tau,\theta)-\gamma_{0^r}(\tau,\theta)e^{i(k\cdot\theta)}),\\
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\end{aligned}$$

Definition

For each $(\tau, \theta) \in \mathbb{R} \times \mathbb{T}^d$ and each $y \in \mathbb{R}^n$,

$$\Phi_{\tau,\theta}(y) = \Psi_\theta(y + \sum_{w \in \mathcal{W}} \epsilon^{|w|} \gamma_w(\tau, \omega\tau)\ f_w(y)).$$

Main result

$\forall (\tau, \theta), (\tau', \theta') \in \mathbb{R} \times \mathbb{T}^d$

$$\Phi_{\tau,\theta} \circ \Phi_{\tau',\theta'} = \Phi_{\tau+\tau',\theta+\theta'}.$$

In particular, high-order averaging decomposition:

$$\varphi_\tau^\epsilon = \Phi_{\tau,\omega\tau} = \Phi_{\tau,0} \circ \Phi_{0,\omega\tau},$$

where $\Phi_{0,\omega\tau}$ is a quasiperiodic flow, and $\Phi_{\tau,0}$ is a smooth near-to-identity flow.
Furthermore,

$$\Phi_{0,\omega\tau} = \Phi_\tau^{[1]} \circ \cdots \Phi_\tau^{[d]},$$

where each $\Phi_\tau^{[j]}$ is a $(2\pi/\omega_j)$-periodic flow of a periodic vector field.
In the Hamiltonian case, this gives d oscillatory formal invariants $I^j(y)$ of the original system (in addition to a smooth formal invariant $I^S(y)$ such that $H_F(y) + \epsilon H_S(y) = I^S(y) + \sum_j I^{[j]}(y)$).

Explict expression of the formal invariants (Hamiltonian case)

Let us denote $\omega^{[j]} = (0, \ldots, 0, \omega_j, 0, \ldots, 0)$. Then

$$I^{[j]}(y) = \frac{d}{d\tau} \Psi_{\tau\omega^{[j]}}(Y)\Big|_{\tau=0} + \sum_{w \in \mathcal{W}} \epsilon^{|w|} \beta_w^{[j]} H_w(y),$$

where

$$\begin{aligned} H_{k_1 \cdots k_m} &= \frac{1}{m} \{H_{k_1}, \{H_{k_2}, \ldots \{H_{k_{m-1}}, H_{k_m}\} \cdots \}, \\ \beta_w^{[j]} &= \frac{d}{d\tau} \gamma_w(0, \omega^{[j]} \tau)\Big|_{\tau=0}. \end{aligned}$$

Simple recursions from the recursions of $\gamma_w(\tau, \theta)$.

Recursion for coefficients of averaged equation

$$
\begin{aligned}
\beta_k^{[j]} &= -\frac{k \cdot \omega^{[j]}}{k \cdot \omega}, \\
\beta_{0^r}^{[j]} &= 0, \\
\beta_{0^r k}^{[j]} &= \frac{i}{k \cdot \omega}\left(\beta_{0^{r-1} k}^{[j]} - \beta_{0^r}^{[j]}\right), \\
\beta_{klw}^{[j]} &= \frac{i}{k \cdot \omega}\left(\beta_{lw}^{[j]} - \beta_{(k+l)w}^{[j]}\right), \\
\beta_{0^r klw}^{[j]} &= \frac{i}{k \cdot \omega}\left(\beta_{0^{r-1} klw}^{[j]} - \beta_{0^r (k+l) w}^{[j]}\right),
\end{aligned}
$$