On energy accumulation in a class of piecewise-defined Hamiltonian systems

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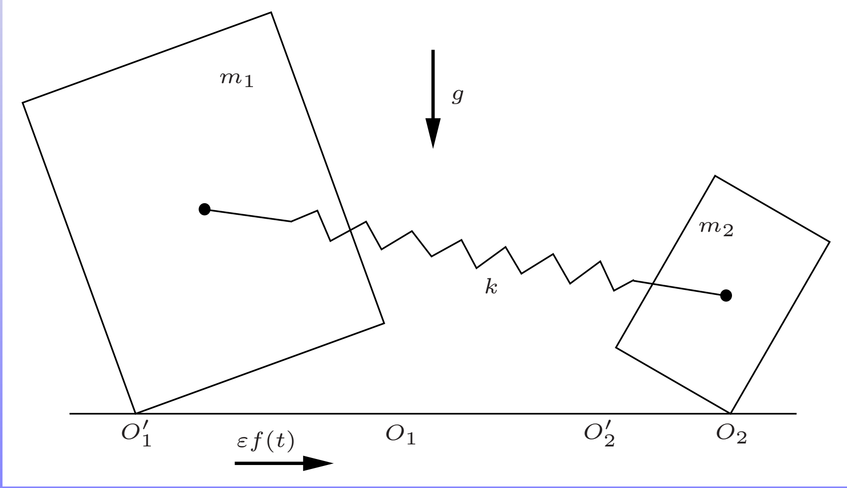
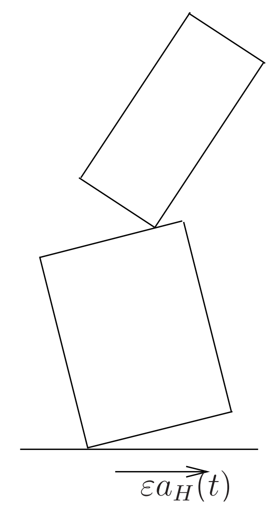
Motivation

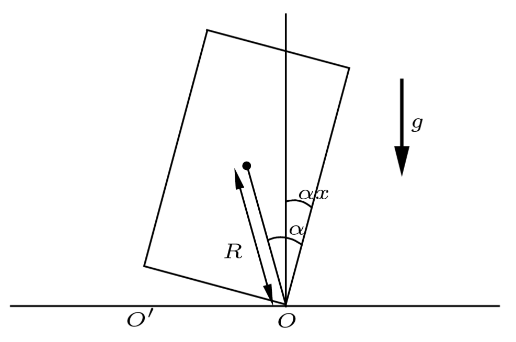


Motivation

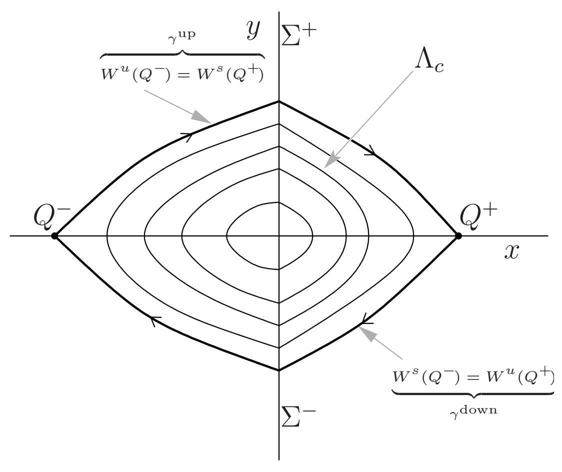


The rocking block

$$\alpha\ddot{x} + \sin(\alpha(1-x)) = 0 \quad \text{if } x > 0$$
$$\alpha\ddot{x} - \sin(\alpha(1+x)) = 0 \quad \text{if } x < 0$$



Phase portrait



Piecewise-defined system

Let us split the plane in two sets

$$S^+ = \left\{(x,y) \in \mathbb{R}^2 \,|\, x > 0\right\}$$
$$S^- = \left\{(x,y) \in \mathbb{R}^2 \,|\, x < 0\right\}$$

separated by the switching manifold

$$\Sigma = \Sigma^+ \cup \Sigma^- \cup (0,0)$$

where

$$\Sigma^+ = \left\{(x,y) \in \mathbb{R}^2 \,|\, x = 0,\ y > 0\right\}$$
$$\Sigma^- = \left\{(x,y) \in \mathbb{R}^2 \,|\, x = 0\, y < 0\right\}.$$

Piecewise-defined system

We consider the piecewise smooth system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \mathcal{X}_0^+(x,y) & \text{if } (x,y) \in S^+ \cup \Sigma^+ \\ \mathcal{X}_0^-(x,y) & \text{if } (x,y) \in S^- \cup \Sigma^- \end{cases}$$

which we assume "Hamiltonian"

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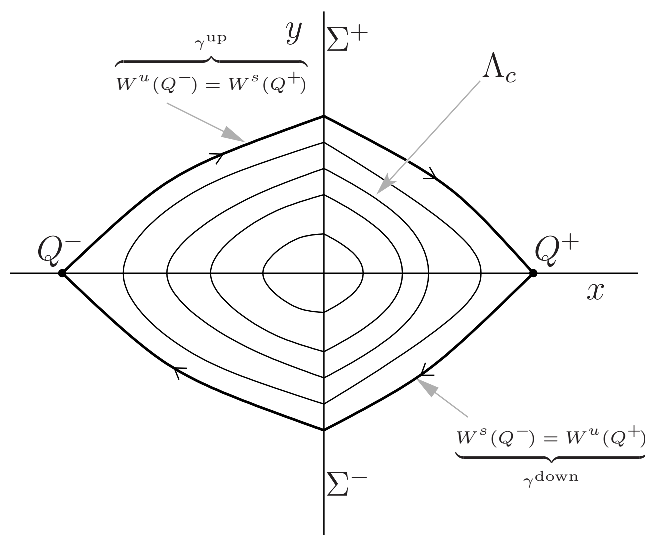
which we assume "Hamiltonian", with Hamiltonian

$$X(x,y) := \frac{y^2}{2} + Y(x) := \begin{cases} \dfrac{y^2}{2} + Y^+(x) & \text{if } (x,y) \in S^+ \cup \Sigma^+ \\ \dfrac{y^2}{2} + Y^-(x) & \text{if } (x,y) \in S^- \cup \Sigma^- \end{cases}$$

with $Y^\pm \in C^\infty(\mathbb{R})$ satisfying $Y^+(0) = Y^-(0)$ but $(Y^+)'(0) \neq (Y^-)'(0)$.


$\gamma^{\rm up}$
y
Σ^+
Λ_c
$W^u(Q^-) = W^s(Q^+)$
Q^-
Q^+
x
$W^s(Q^-) = W^u(Q^+)$
$\gamma^{\rm down}$
Σ^-

The coupled system

We consider

$$H_\varepsilon(x,y,u,v,t) := X(x,y) + U(u,v) + \varepsilon h(x,y,u,v,t),\ \varepsilon > 0,$$

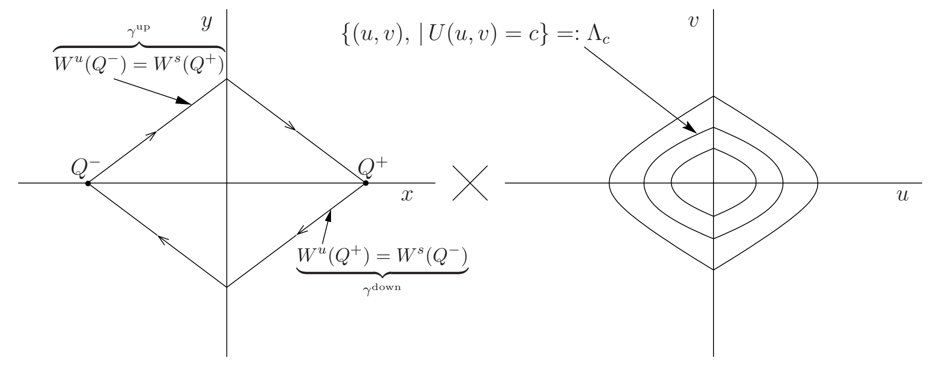
with h T-periodic in t:

$$h(x,y,u,v,t) = h(x,y,u,v,t+T)$$

and X and U are piecewise-defined Hamiltonian modelling two rocking blocks.

Invariant objects for the unperturbed case

If $\varepsilon = 0$, the coupled system is a cross product of two rocking blocks$\times\mathbb{T}$



Invariant objects for the unperturbed case

- $\tilde{\mathcal{T}}_c^{\pm} = Q^{+} \times \Lambda_c \times \mathbb{T} =$
  $\{(x, y, u, v, s) \,|\, (x, y) = Q^{\pm}, U(u, v) = c,\; s \in \mathbb{T}\}$

Invariant objects for the unperturbed case

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- $W^s(\tilde{\mathcal{T}}_c^{+}) = W^u(\tilde{\mathcal{T}}_c^{-}) = W^s(Q^{+}) \times \Lambda_c \times \mathbb{T} = W^u(Q^{-}) \times \Lambda_c \times \mathbb{T}$

Invariant objects for the unperturbed case

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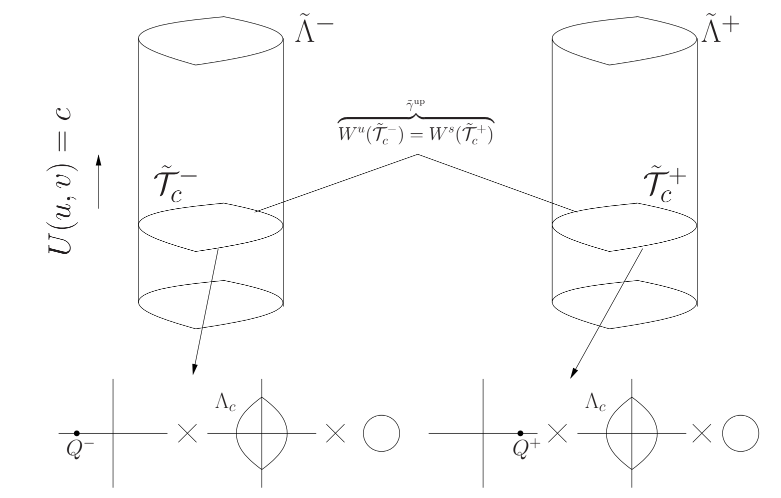
  $\{(x, y, u, v, s) \,|\, (x, y) = Q^{\pm}, U(u, v) = c,\ s \in \mathbb{T}\}$
- $W^s(\tilde{\mathcal{T}}_c^{+}) = W^u(\tilde{\mathcal{T}}_c^{-})$
  $= W^s(Q^{+}) \times \Lambda_c \times \mathbb{T} = W^u(Q^{-}) \times \Lambda_c \times \mathbb{T}$
- $\tilde{\Lambda}^{-} = \bigcup_{c \in [c_1, c_2]} \tilde{\mathcal{T}}_c^{-} = \bigcup_{c \in [c_1, c_2]} Q^{-} \times \Lambda_c \times \mathbb{T}$
- $\tilde{\Lambda}^{+} = \bigcup_{c \in [c_1, c_2]} \tilde{\mathcal{T}}_c^{+} = \bigcup_{c \in [c_1, c_2]} Q^{+} \times \Lambda_c \times \mathbb{T}$

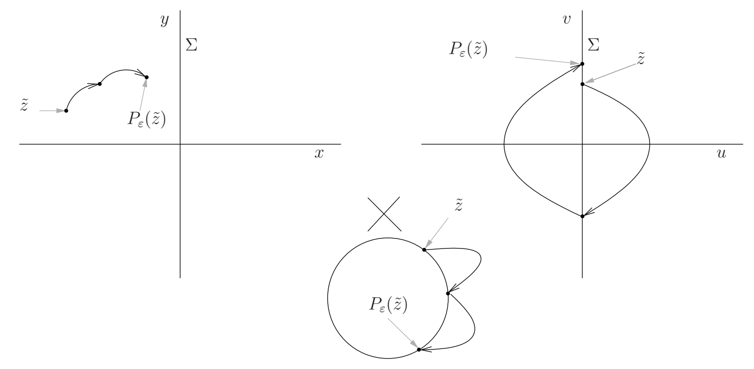
C^0 Invariant manifolds



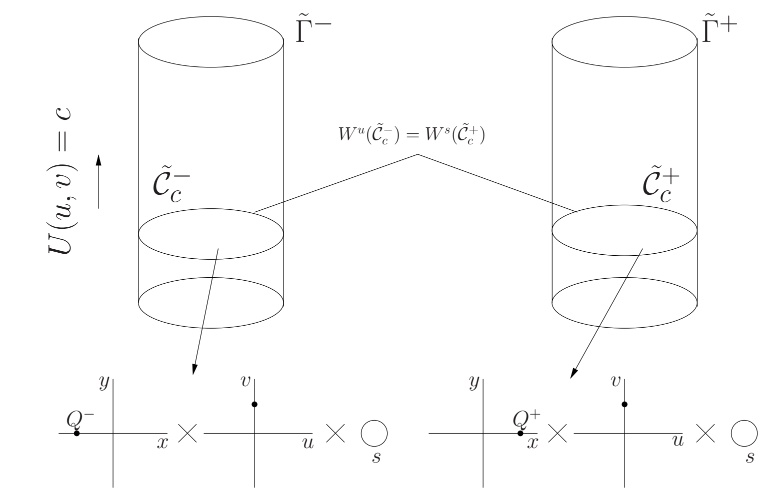
Persistence when $\varepsilon > 0$

We don't have theory that provides the persistence of these invariant manifolds because of the non-smoothness. Instead we use the impact map:

$$P_\varepsilon : \mathbb{R}^2 \times \Sigma \times \mathbb{T} \longrightarrow \mathbb{R}^2 \times \Sigma \times \mathbb{T}$$



NHIM's for the impact map



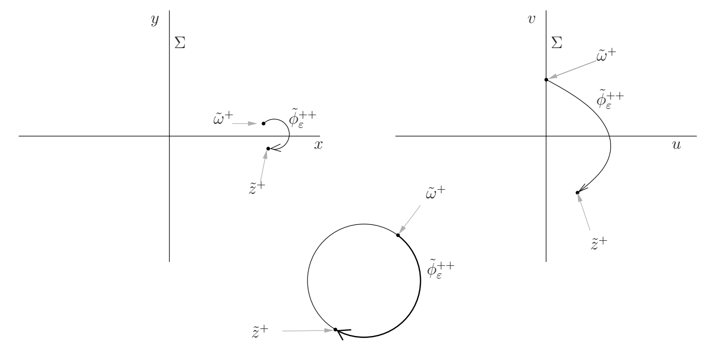
Perturbed invariant manifolds for the map

Using that P_ε is smooth, we can apply classical theory of normally hyperbolic manifolds to have the existence of

$$\tilde{\Gamma}_\varepsilon^{\pm},\ W^{u,s}(\tilde{\Gamma}_\varepsilon^{\pm})$$

Perturbed invariant manifolds for the flow

$$\tilde{\omega}^{+}=\left(\omega^{+},s^{+}\right)\in\tilde{\Gamma}_{\varepsilon}^{+}\longrightarrow\tilde{z}\in\tilde{\Lambda}_{\varepsilon}^{+}$$
$$\tilde{z}^{+}=\left(z^{+},s^{+}+\tau\right)=\tilde{\phi}(\tau;\tilde{\omega}^{+};\varepsilon),\ 0\leq\tau<\Pi_{s}\left(P_{\varepsilon}(\tilde{\omega}^{+})-s^{+}\right)$$



Perturbed invariant manifolds for the flow

$$\tilde{\omega}^+ = (\omega^+, s^+) \in \tilde{\Gamma}^+_\varepsilon$$

$$\tilde{\omega}^s = (\omega^s, s^s) \in W^s(\tilde{\Gamma}^+_\varepsilon)$$

$$\left| P^n_\varepsilon(\tilde{\omega}^+) - P^n_\varepsilon(\tilde{\omega}^s) \right| \longrightarrow 0$$

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$$\tilde{z}^+ = \tilde{\phi}(\tau; \tilde{\omega}^+; \varepsilon) \in \tilde{\Lambda}_\varepsilon^+$$
$$\tilde{z}^s = \tilde{\phi}(\tau \qquad ; \tilde{\omega}^s; \varepsilon) \notin W^s(\tilde{z}^+)$$

Perturbed invariant manifolds for the flow

$$\tilde{\omega}^{+} = (\omega^{+}, s^{+}) \in \tilde{\Gamma}_{\varepsilon}^{+}$$
$$\tilde{\omega}^{s} = (\omega^{s}, s^{s}) \in W^{s}(\tilde{\Gamma}_{\varepsilon}^{+})$$

$$\left| P_{\varepsilon}^{n}(\tilde{\omega}^{+}) - P_{\varepsilon}^{n}(\tilde{\omega}^{s}) \right| \longrightarrow 0$$

$$\tilde{z}^{+} = \tilde{\phi}(\tau; \tilde{\omega}^{+}; \varepsilon) \in \tilde{\Lambda}_{\varepsilon}^{+}$$
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$$\tau' = s^+ - s^s$$

Perturbed invariant manifolds for the flow

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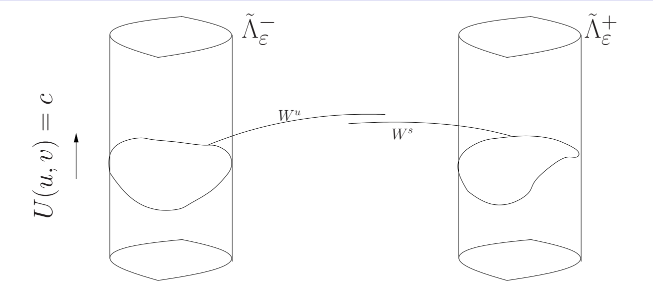
$$\tau' = s^+ - s^s$$

$$\left| \tilde{\phi}(t; \tilde{z}^+; \varepsilon) - \tilde{\phi}(t; \tilde{z}^s; \varepsilon) \right| \longrightarrow 0$$

Intersection of the heteroclinic manifolds

When $\varepsilon > 0$ there exist 3-dimensional invariant manifolds

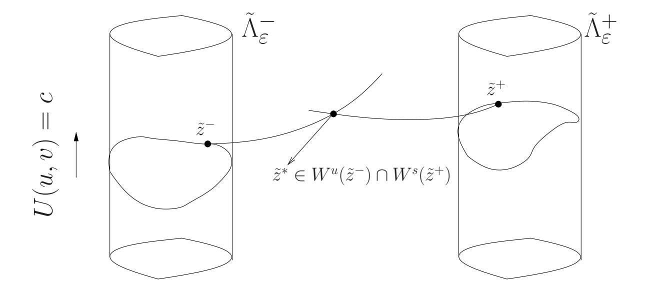


with 4-dimensional $W^u(\tilde{\Lambda}_\varepsilon^-)$ and $W^s(\tilde{\Lambda}_\varepsilon^+)$

Do they intersect??

Intersection of the heteroclinic manifolds

Yes, under certain conditions



$W^u(\tilde{\Lambda}_\varepsilon^-) \cap W^s(\tilde{\Lambda}_\varepsilon^+)$ is 3-dimensional

Intersection of the heteroclinic manifolds

A sufficient condition to have (transversal) intersection of the heteroclinic manifolds is that the Melnikov function

$$\zeta \longmapsto M(\zeta, \tau, v, s),$$

has simple zeros.

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In the smooth case,

$$M(\zeta,\tau,v,s) := \int_{-\infty}^{\infty} \{X,h\} \Big(\phi \Big(t; \underbrace{\left(\sigma\left(\zeta\right), \phi_{\mathcal{U}}\left(\tau;0,v\right), s\right)}_{\tilde{z}_0}; 0 \Big) \Big) dt$$

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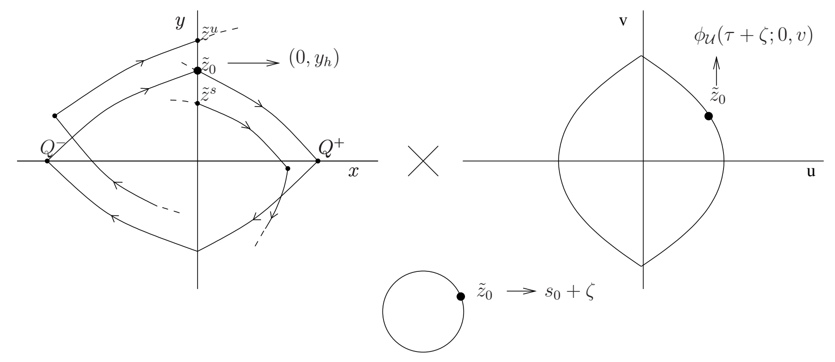
In our case,

$$M(\zeta,\tau,v,s) := \int_{-\infty}^{\infty} \{X,h\}\left(\phi\Big(t;\underbrace{\left(0,y_h,\phi_{\mathcal{U}}\left(\tau+\zeta;0,v\right),s+\zeta\right)}_{\tilde{z}_0};0\Big)\right)dt$$

Intersection of the heteroclinic manifolds

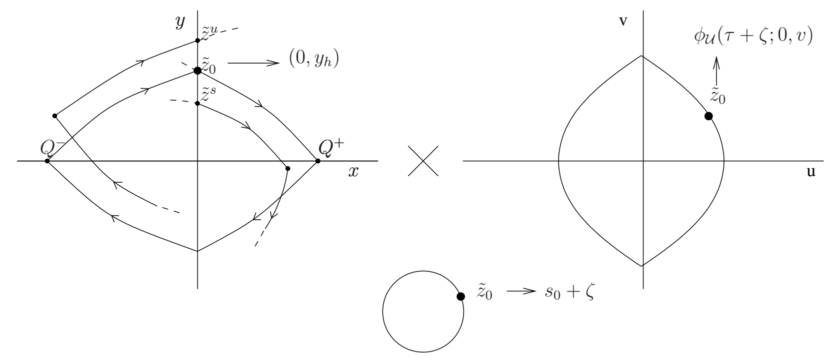
$$\tilde{z}_0 = (0, y_h, \phi_{\mathcal{U}}(\tau + \zeta; 0, v), s + \zeta)$$

$$\tilde{N} = \{\tilde{z}_0 + l(0,1,0,0,0),\ l \in \mathbb{R}\}$$



Intersection of the heteroclinic manifolds

$$\Delta(\zeta,\tau,v,s;\varepsilon) = X(\tilde{z}^u) - X(\tilde{z}^s) = \varepsilon M(\zeta,\tau,v,s) + O(\varepsilon^2) = 0$$



Intersection of the heteroclinic manifolds

Let $\bar{\zeta}$ be a simple zero of

$$\zeta \longmapsto M(\zeta, \tau, v, s)$$

$$\Longrightarrow \exists\ \tilde{z}_0^* \in W^u(\tilde{\Lambda}_\varepsilon^-) \cap W^s(\tilde{\Lambda}_\varepsilon^+) \cap \{x = 0\},\ \tilde{z}^\pm \in \tilde{\Lambda}_\varepsilon^\pm$$

$$\tilde{z}_0^* = (0, y_h, \phi_{\mathcal{U}}(\tau + \bar{\zeta}; 0, v), s + \bar{\zeta}) + O(\varepsilon)$$
$$\tilde{z}^\pm = (Q^\pm, \phi_{\mathcal{U}}(\tau + \bar{\zeta}; 0, v), s + \bar{\zeta}) + O(\varepsilon)$$

$$\left|\tilde{\phi}(t; \tilde{z}_0^*; \varepsilon) - \tilde{\phi}(t; \tilde{z}^\pm; \varepsilon)\right| \to 0,\ t \to \pm\infty$$

Properties of the scattering map

The scattering map becomes

$$\begin{array}{cccc} S_\varepsilon : & \tilde{\Lambda}_\varepsilon^- & \longrightarrow & \tilde{\Lambda}_\varepsilon^+ \\ & \tilde{z}^-(\tau, v, s) & \longmapsto & \tilde{z}^+(\tau, v, s) \end{array}$$

We want to measure the difference of the energy of the points $\tilde{z}^+$ and $\tilde{z}^-$. We use the unperturbed Hamiltonians

$$\begin{aligned} X(\tilde{z}^+) + U(\tilde{z}^+) - \left(X\left(\tilde{z}^-\right) + U(\tilde{z}^-)\right) \\ = \underbrace{X(\tilde{z}^+) - X(\tilde{z}^-)}_{O(\varepsilon^2)} + \underbrace{U(\tilde{z}^+) - U(\tilde{z}^-)}_{\Delta U \varepsilon + O(\varepsilon^{1+\rho})} \end{aligned}$$

Properties of the scattering map

$$U(\tilde{z}^+) - U(\tilde{z}^-) = \varepsilon \Delta U + O(\varepsilon^{1+\rho})$$

$$\begin{aligned}\Delta U(\tau, v, s) = &\int_{-\infty}^{0} \Big(\{U,h\}\,(\text{unperturbed heteroclinic}) \\ &- \{U,h\}\Big(\text{ assympthotic solution in } \tilde{\Lambda}^-\Big)\Big)dt \\ &+ \int_{0}^{+\infty} \Big(\{U,h\}\,(\text{unperturbed heteroclinic}) \\ &- \{U,h\}\Big(\text{ assympthotic solution in } \tilde{\Lambda}^+\Big)\Big)dt\end{aligned}$$

Properties of the scattering map

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(U\left(\tilde{\phi}\left(t;\tilde{z}^+;\varepsilon\right)\right)-U\left(\tilde{\phi}\left(-t;\tilde{z}^-;\varepsilon\right)\right)\right)dt$$

Properties of the scattering map

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$$=\varepsilon\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\int_{-t}^t\{U,h\}\left(\sigma(-\bar{\zeta}+r),\phi_{\mathcal{U}}(\tau+r;0,v),s+r\right)dr\right)dt\right)$$

$$+O(\varepsilon^{1+\rho})$$

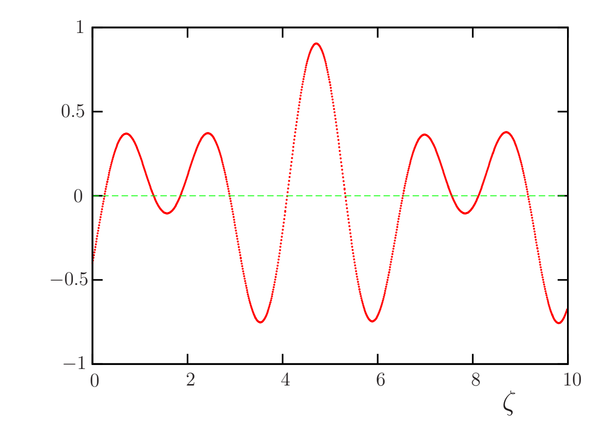
System equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - \mathsf{sign}(x) \\ \qquad + \varepsilon\big(k(u-x) - \delta\cos(s)\big) \\ \dot{u} = v \\ \dot{v} = u - \mathsf{sign}(u) \\ \qquad + \varepsilon\big(k(x-u) - \delta\cos(s)\big) \\ \dot{s} = 1, \end{cases}$$

$$H(x,y,u,v,s) = \frac{y^2}{2} - \frac{x}{2} + |x| + \frac{v^2}{2} - \frac{u}{2} + |u| \\ + \varepsilon\left(k\left(\frac{u^2}{2} + \frac{x^2}{2} - ux\right) + \delta(x+u)\cos(s)\right),$$

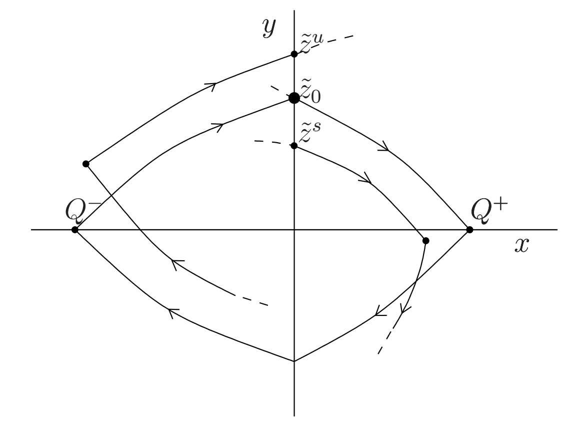
Melnikov function

Let us fix $v = 0.5$ and $\tau = s = 0$.



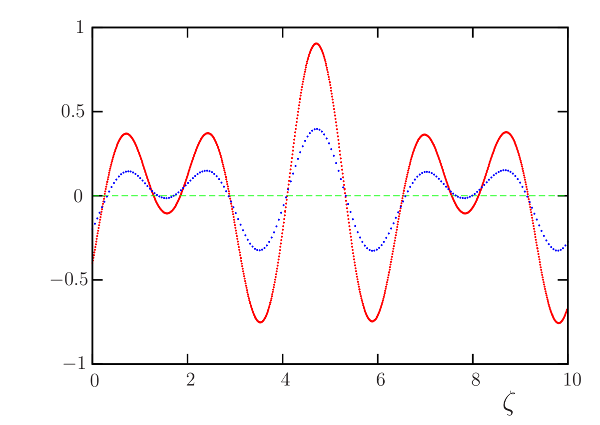
Real distance

Recalling that $\Delta(\zeta,\tau,v,s;\varepsilon)=X(\tilde{z}^u)-X(\tilde{z}^s)$.



Real distance

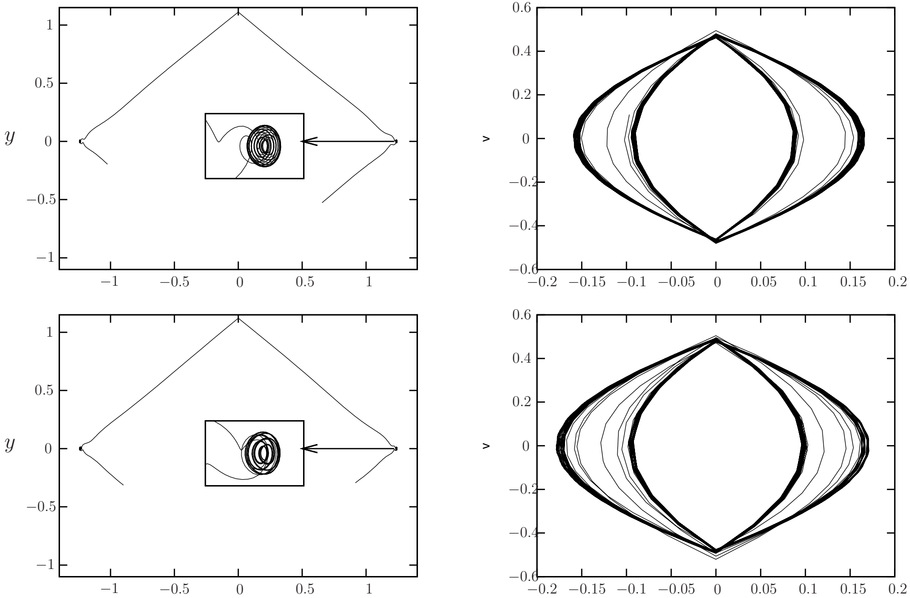
we get, for $\varepsilon = 0.01$ ($v = 0.5$ and $\tau = s = 0$)

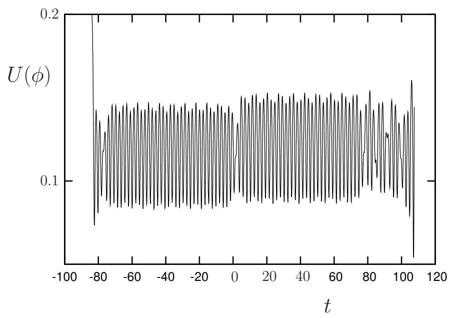


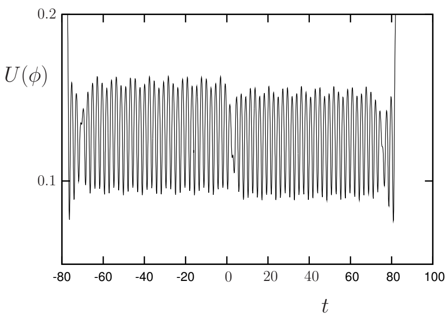
$$\Delta <>= \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\int_{-t}^{t} \{U,h\} \left(\sigma(-\bar{\zeta}+r), \phi_{\mathcal{U}}(\tau+r;0,v), s+r\right) dr \right) dt$$

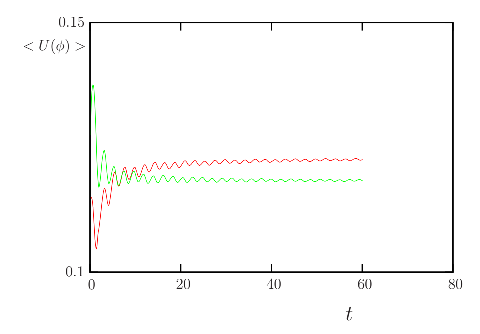
Third zero $\longrightarrow \Delta <>\simeq 0.4$

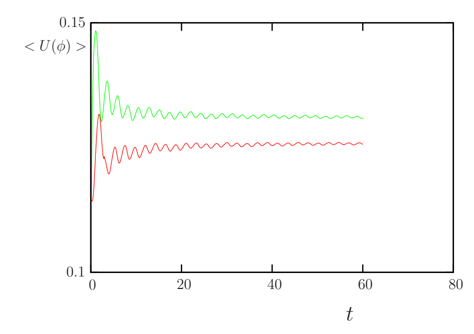
Fourth zero $\longrightarrow \Delta <>\simeq -0.3$

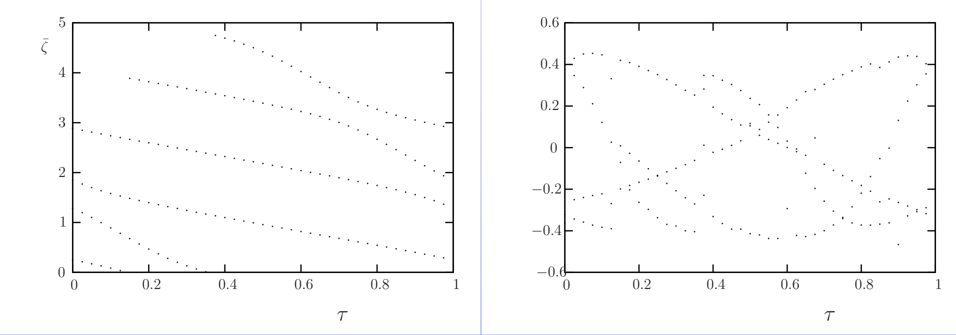

y
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
v
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.2
−0.15
−0.1
−0.05
0
0.05
0.1
0.15
0.2

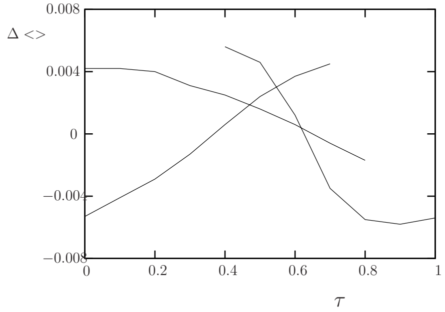

0.2
0.1
$U(\phi)$
-100 -80 -60 -40 -20 0 20 40 60 80 100 120
t

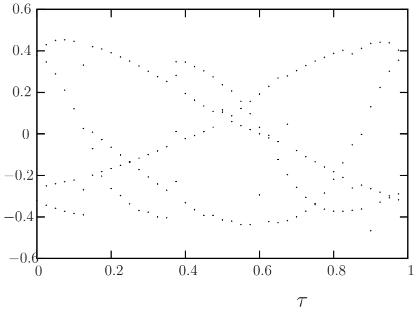

0.2
0.1
$U(\phi)$
-80 -60 -40 -20 0 20 40 60 80 100
t


0.15
< U(φ) >
0.1
0
20
40
60
80
t


0.15
< U(φ) >
0.1
0
20
40
60
80
t


5
4
3
2
1
0
$\bar{\zeta}$
0
0.2
0.4
0.6
0.8
1
τ
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
0
0.2
0.4
0.6
0.8
1
τ


0.008
Δ <>
0.004
0
−0.004
−0.008
0
0.2
0.4
0.6
0.8
1
τ


0.6
0.4
0.2
0
−0.2
−0.4
−0.6
0
0.2
0.4
0.6
0.8
1
τ