A global study of 2D dissipative diffeomorphisms with a Poincaré homoclinic figure-eight.

DDAYS 2012, Benicàssim,

October 24–26, 2012

A. Vieiro

(Univ. Barcelona)

Joint work with S. Gonchenko (Univ. Nizhny Nóvgorod) and C. Simó (Univ. Barcelona)

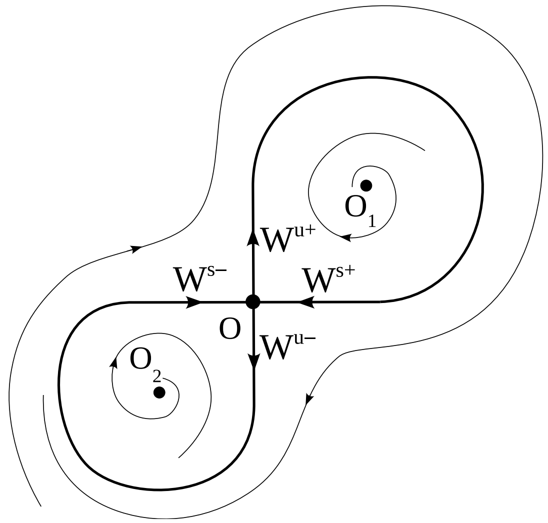
Introduction

We consider a family $T_{\mu,\epsilon}$, $\mu \in \mathbb{R}^2$, $\epsilon \in \mathbb{R}$, of 2D analytic diffeomorphisms.

$T_{\mu,\epsilon}$ can be seen as the **Poincaré map** of a **non-autonomous** (2π-periodic in time) $\mathcal{O}(\epsilon)$-perturbation of an **autonomous** family of vector fields f_μ.

- The non-autonomous perturbation is assumed to be fixed and sufficiently small (equivalently, ϵ is a small given value).
- The family of autonomous systems f_μ is a 2-parameter unfolding of the system f_0, which we assume to posses a ***homoclinic figure-eight*** to a **dissipative** saddle point.

Let $\Gamma^+ = W^{u+} = W^{s+}$ and $\Gamma^- = W^{u-} = W^{s-}$ be the homoclinic loops of the flow f_0. Then $\Gamma_0 = \Gamma^+ \cup \Gamma^-$ is the (unperturbed) homoclinic figure-eight of the saddle O.

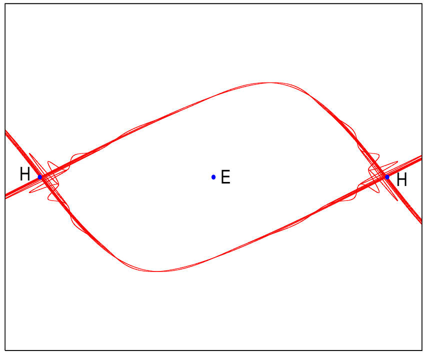


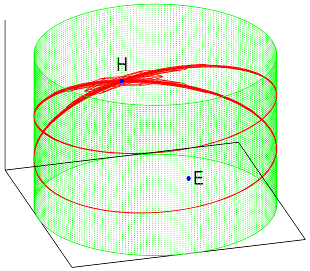
Motivation

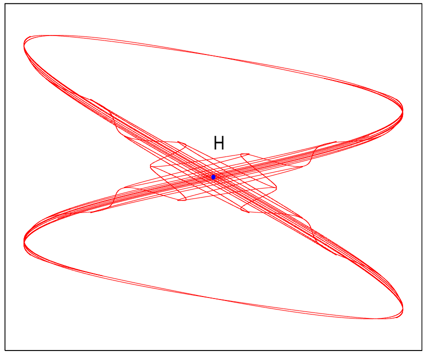
$T_{\mu,\epsilon}$ are **pendulum-like systems** under **forcing and dissipation**.

- **Forcing** $\Rightarrow$ elliptic point becomes a repellor.
- **Dissipation** $\Rightarrow$ the dynamics is towards the separatrix.

The **cylinder-sphere-stereo projection** identifies with a **dissipative figure-eight**.







Device? pendulum + dissipation proportional to velocity (assymetric if different bulk left/right shapes) + magnetic field kicks at the minimum to make the fixed points unstable.

Idea of this talk

We want to study the **parameter space** of $T_{\mu,\epsilon}$ for ϵ small fixed. Concretely, we consider:

1. A **qualitative** approach to the full bifurcation diagram.
   - → Different dynamics and regions.
   - → Homoclinic dynamics. Lobe dynamics.
   - → MS & SA boundaries.
2. A **quantitative** approach to the full bifurcation diagram.

   We use a separatrix map model.
   - → Size of the main regions having different dynamics.
   - → Scaling properties of the bifurcation diagram.
   - → Stability regions related to cubic tangencies.

This talk is based on (some of) the results that can be found in

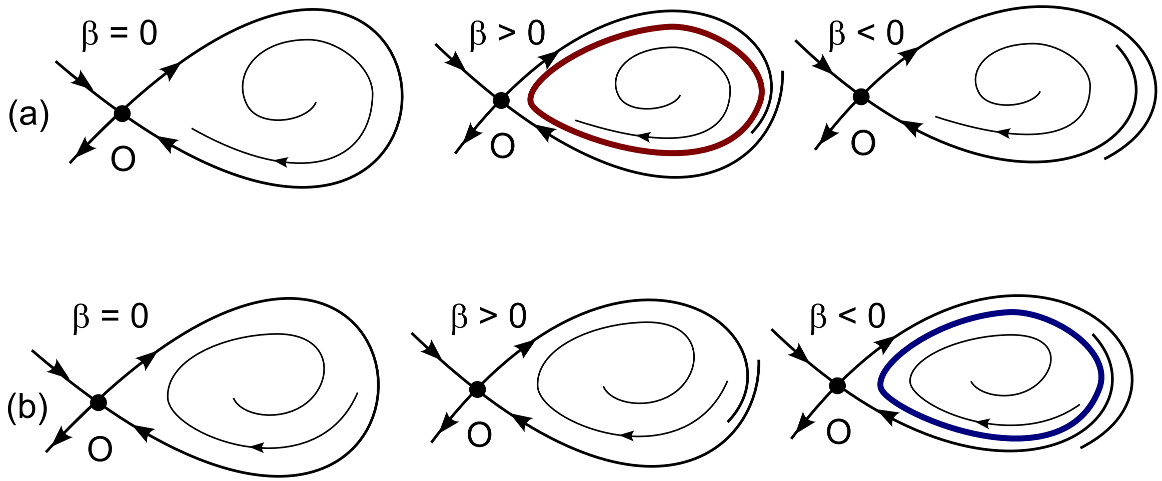
Richness of dynamics and global bifurcations in systems with a homoclinic figure-eight.

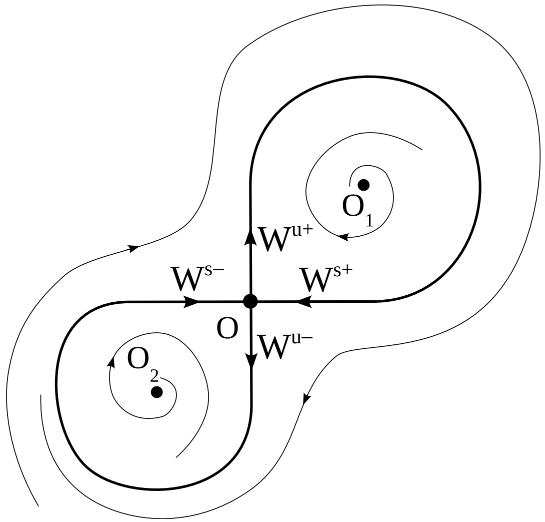
S.V. Gonchenko , C. Simó and AV, submitted to Nonlinearity.

The flow (autonomous) case

Bifurcations of limit cycles from a homoclinic loop to a saddle:

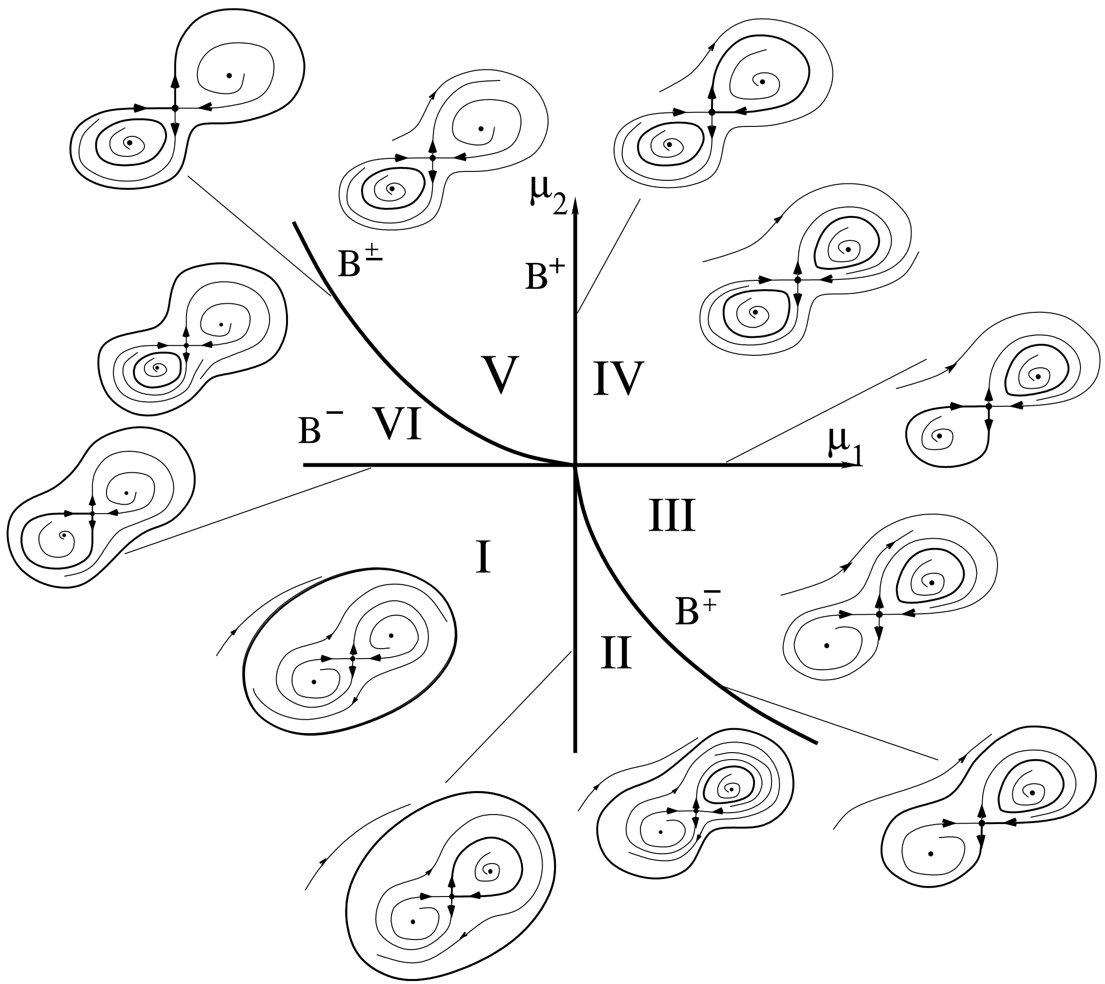
- Let $\lambda > 0$ and $-\gamma < 0$ the characteristic roots of the saddle.
- If $\sigma = \lambda - \gamma \neq 0$ exactly one limit cycle is born (Andronov-Leontovich).





Left: unfolding a dissipative loop. Right: figure-eight before unfolding.

The flow case: bifurcation diagram



- **Six** regions.
- **Boundaries:**

  $W^{u+} = W^{s+}$ (I→II);

  $W^{u-} = W^{s+}$ (II→III);

  $W^{u-} = W^{s-}$ (III→IV);

  $W^{u+} = W^{s+}$ (IV→V);

  $W^{u+} = W^{s-}$ (V→VI);

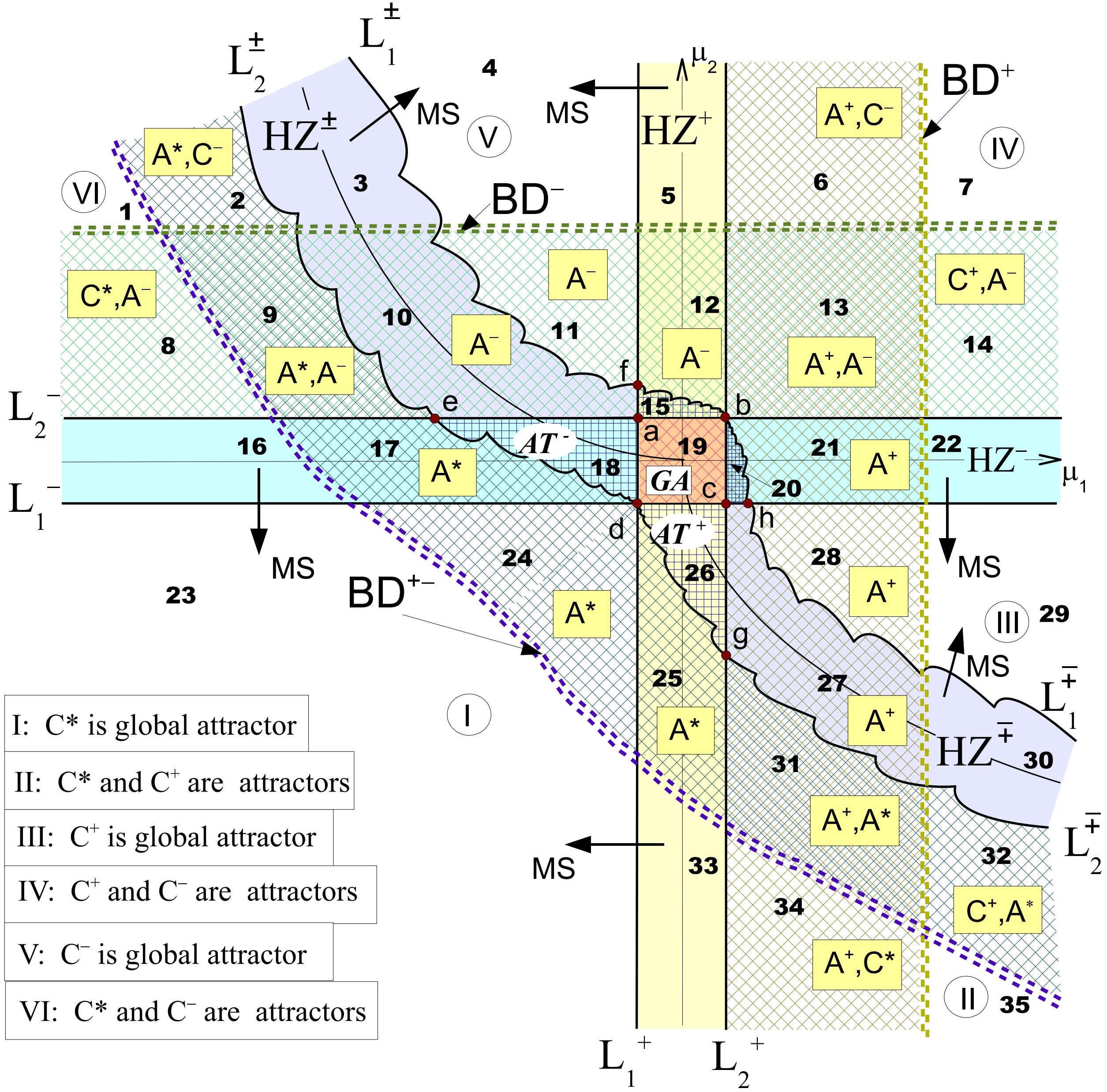
  $W^{u-} = W^{s-}$ (VI→I);

D. Turaev. On a case of bifurcation of a contour composed by two homoclinic curves of a saddle.

Methods of the qualitative theory of differential equations, Ed. Gorki, 1984, 162–175.

The diffeomorphism case: bifurcation diagram

We consider the effect of the non-autonomous perturbation and we look at the Poincaré map.



Properties of the bifurcation diagram of $T_{\mu,\epsilon}$

1. There appear **35 regions** with different dynamics!
2. These regions are separated by first/last tangency curves

$$\mathrm{L}_1^+,\ \mathrm{L}_2^+,\ \mathrm{L}_1^-,\ \mathrm{L}_2^-,\mathrm{L}_1^\pm,\ \mathrm{L}_2^\pm,\ \mathrm{L}_1^\mp,\ \mathrm{L}_2^\mp,$$

   and/or by "curves" that indicate transitions from "simple" dynamics to strange attractor (e.g. folding of an invariant curve can cause collision between tangent/normal bundles and create a SA)

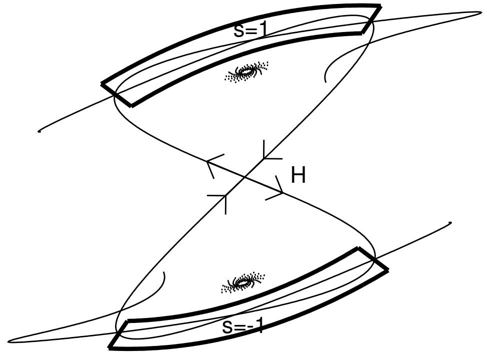
$$\mathrm{BD}^+,\mathrm{BD}^-,\mathrm{BD}^{+-}.$$

3. Only the $\mathrm{L}_{1,2}^{+,-}$ are smooth. The curves $\mathrm{L}_{1,2}^{\pm,\mp}$ have a complicated structure (later) with infinitely many intervals of smoothness.
4. Multiple attractors can coexist.

$\longrightarrow$ For a detailed analysis we introduce the following return map model...

A quantitative model: dissipative separatrix map

$$M_{a,b,\psi,A,\omega} : \begin{pmatrix} z \\ \eta \\ s \end{pmatrix} \mapsto \begin{pmatrix} z + \omega_j + A\log(|y|) \\ \text{sign}(y)|y|^{\psi} \\ \text{sign}(y)s \end{pmatrix}$$



→ FD = **two annuli**: the index j equals 1 if $s = 1$ and $j = 2$ if $s = -1$.

→ $\psi = \lambda/\gamma$ accounts for the dissipation in the passage near the saddle.

→ **Returning time** = constant ω_j +“flying” time $A\log(y)$ near the saddle.

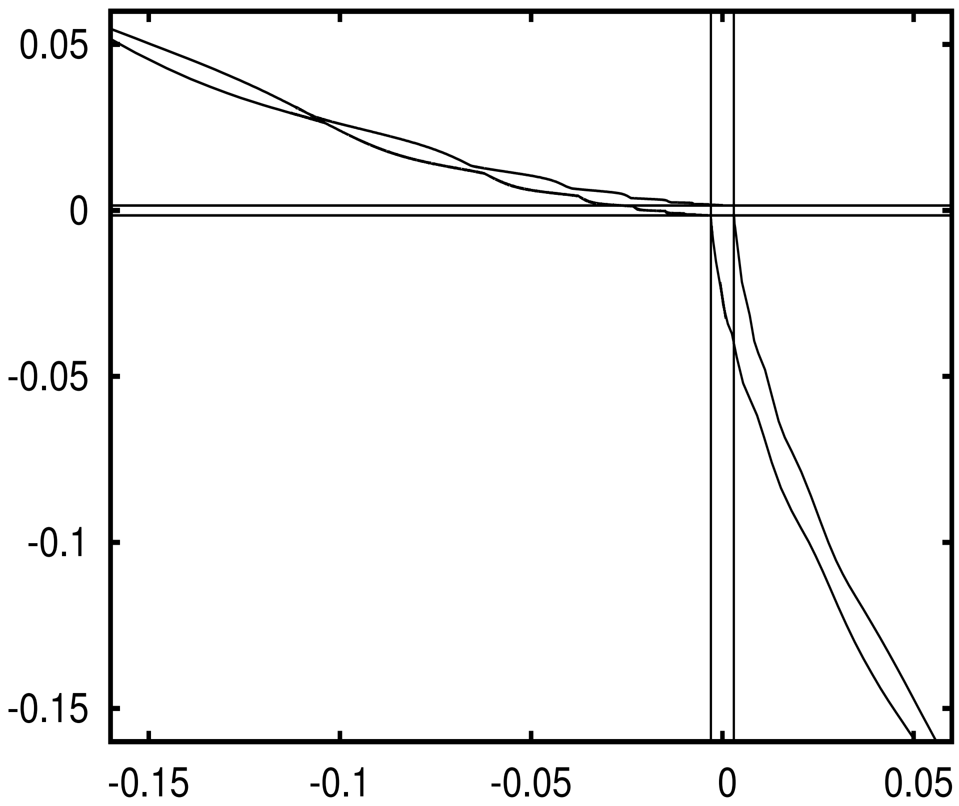
→ $y = a_j + \eta + b_j \sin(2\pi z)$, and for both η (distance w.r.t. W^u) and y (distance w.r.t. W^s) the positive orientation points towards the saddle.

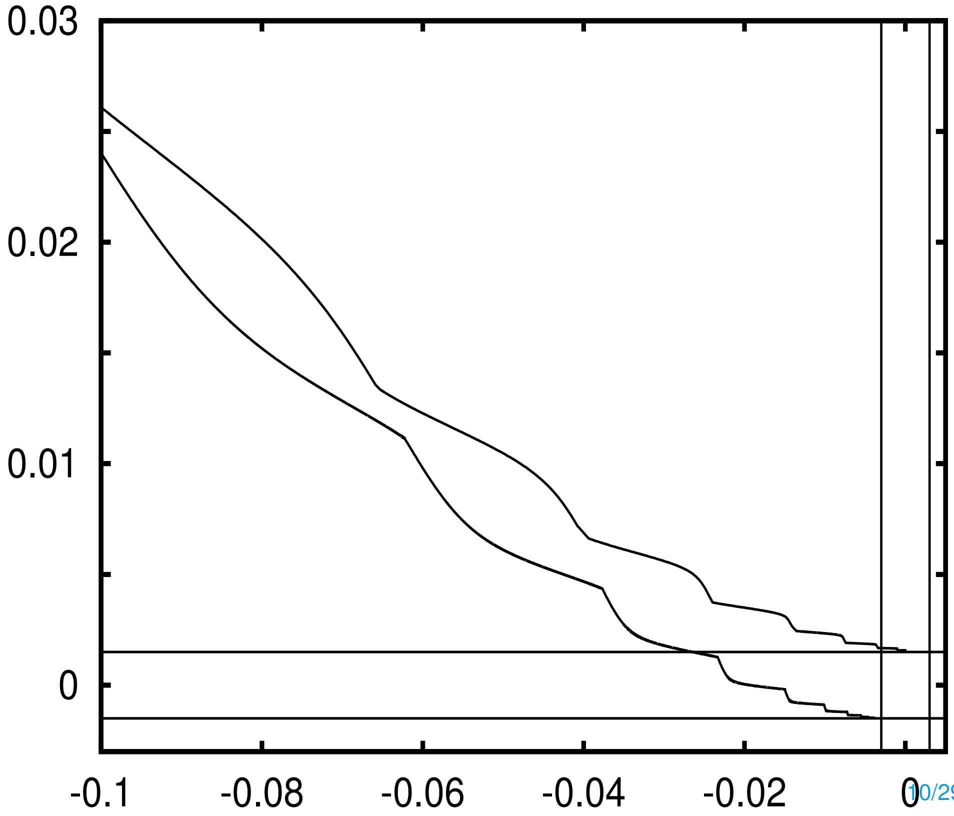
→ If $a_j = b_j = 0$ both branches $W^{u/s}$ coincide.
For $b_j = 0$ it mimics the vector field provided $|a_j| < (\psi - 1)/\psi^{\psi/(\psi-1)}$.
Then b_j play the role of ϵ (they undulate the inv. manifolds).

→ In the **simulations**: $\omega_j = 0$, $A = 2$, $\psi = 1.6$, $b_1 = 0.003$, $b_2 = 0.0015$.
Then a_1, a_2 are taken as **leading parameters** ranging in $[-0.15, 0.15]$.

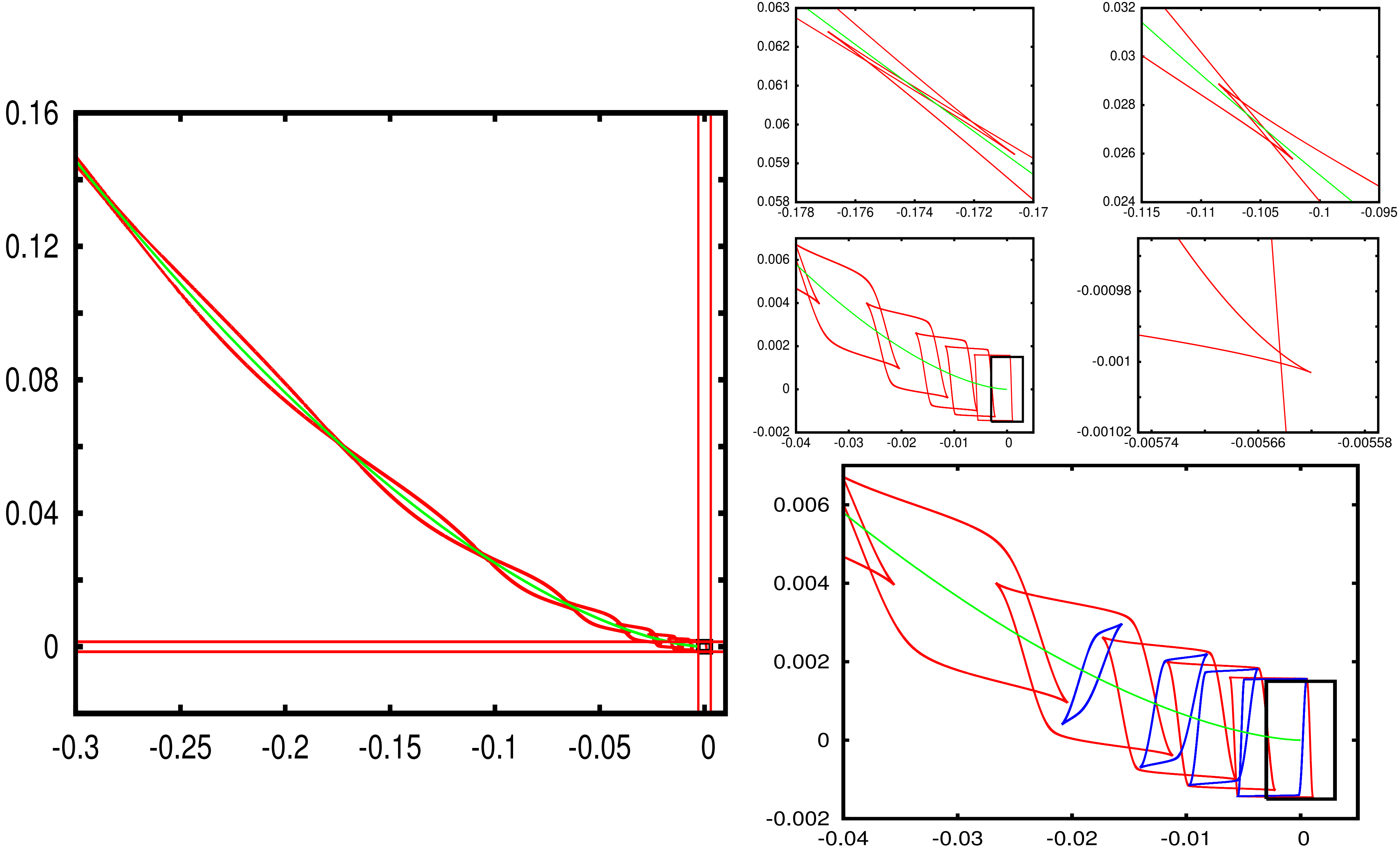
A preliminary numerical exploration of the model

In the (a_1, a_2)-parameter space we compute first/last primary homoclinic quadratic tangency curves between $W^{u\pm} = \{\eta = 0, s = \pm 1\}$ and $W^{s\pm} = \{y = 0, s = \pm 1\}$. The curves $L^{\pm}_{1,2}$ and $L^{\mp}_{1,2}$ are the envelope of different bifurcating curves (related to different primary quadratic tangencies) that bound a “diagonal” strip with “stair-type” structure. Essentially **8 curves**.

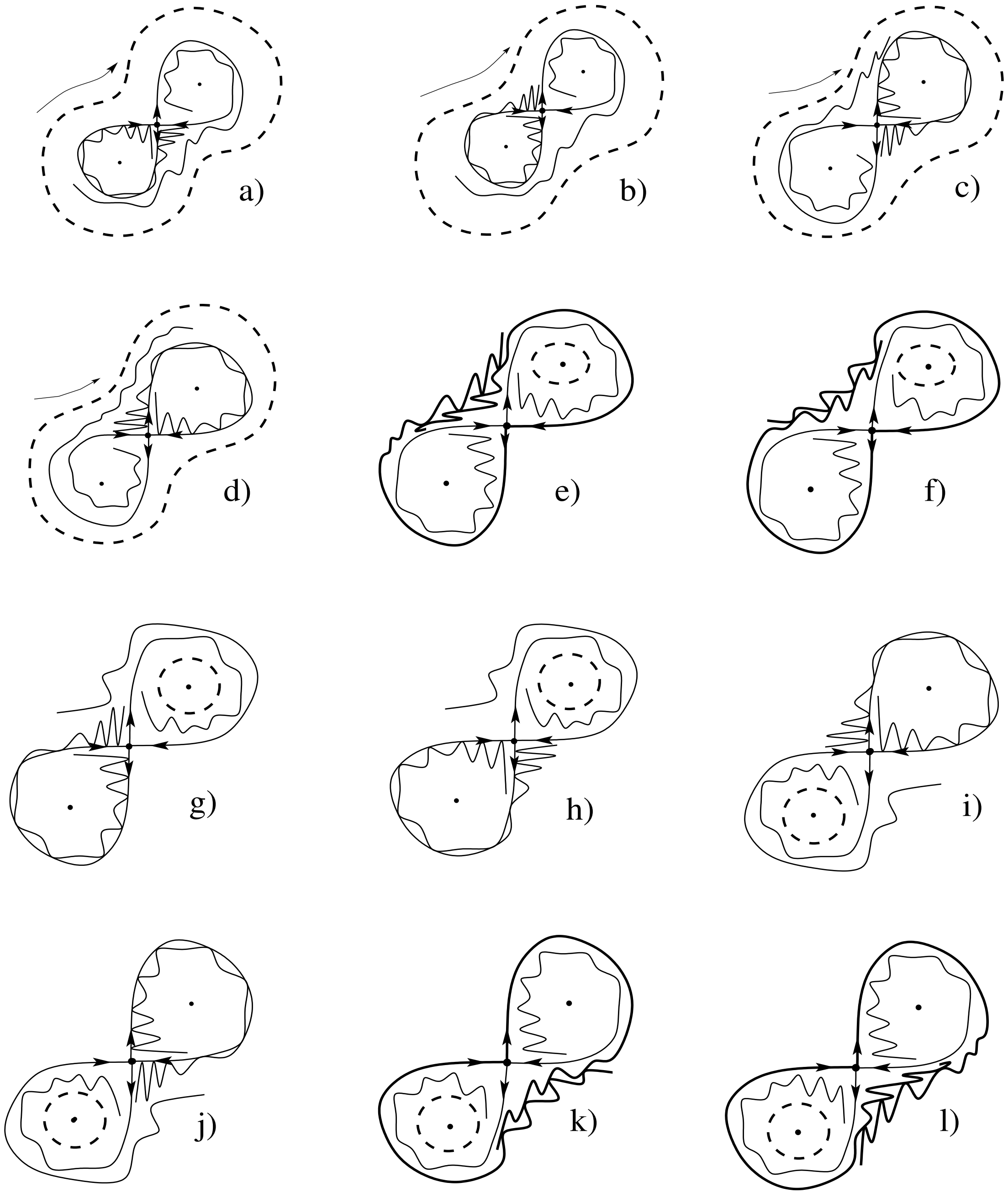




Bifurcating curves within $HZ^{\pm}$



Homoclinic tangencies – phase space



a) $\mu \in L_2^-$, $\mu_1 < 0$;

b) $\mu \in L_1^-$, $\mu_1 < 0$;

c) $\mu \in L_1^+$, $\mu_2 < 0$;

d) $\mu \in L_2^+$, $\mu_2 < 0$;

e) $\mu \in L_2^{\mp}$;

f) $\mu \in L_1^{\mp}$;

g) $\mu \in L_1^-$, $\mu_1 > 0$;

h) $\mu \in L_2^-$, $\mu_1 > 0$;

i) $\mu \in L_2^+$, $\mu_2 > 0$;

j) $\mu \in L_1^+$, $\mu_2 > 0$;

k) $\mu \in L_1^{\pm}$;

l) $\mu \in L_2^{\pm}$.

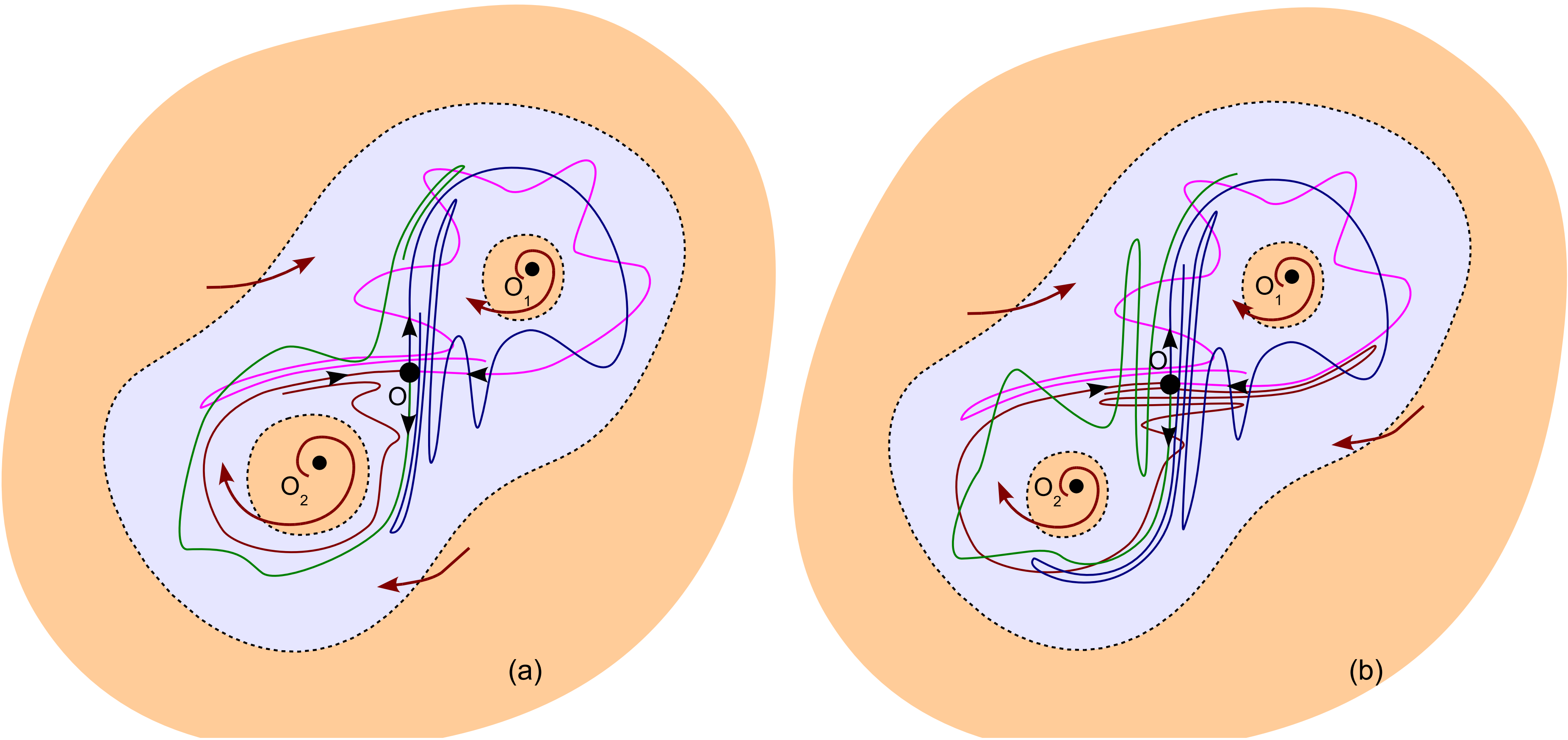
Comments on the attractors

1. Only the regions I,II,...,VI are related to non-chaotic dynamics (like the flow). The global attractors are invariant curves C^+, C^- and/or C^*.

2. In the chaotic regions, the closure of the invariant manifolds can contain a **quasi-attractor**: a nontrivial attracting invariant set which contains stable p.o. (sinks) and/or SA (maybe made by several pieces). Arbitrarily small perturbations of the parameters when a SA is found can give rise to sinks.

3. There appear strange attractors of different nature:
   - → A^+, A^- and A^* are born under the break-down of the closed invariant curves C^+, C^- and C^*: Due to the folding of the curve it becomes tangent to stable foliation of the saddle fixed point.
   - → The global attractors AT^+, AT^- and GA are "homoclinic attractors" related to the intersection of (some or all) the invariant manifolds.
   - → SA can also appear at the end of a period doubling cascade of sinks. These attractors have local character.

Tail attractors

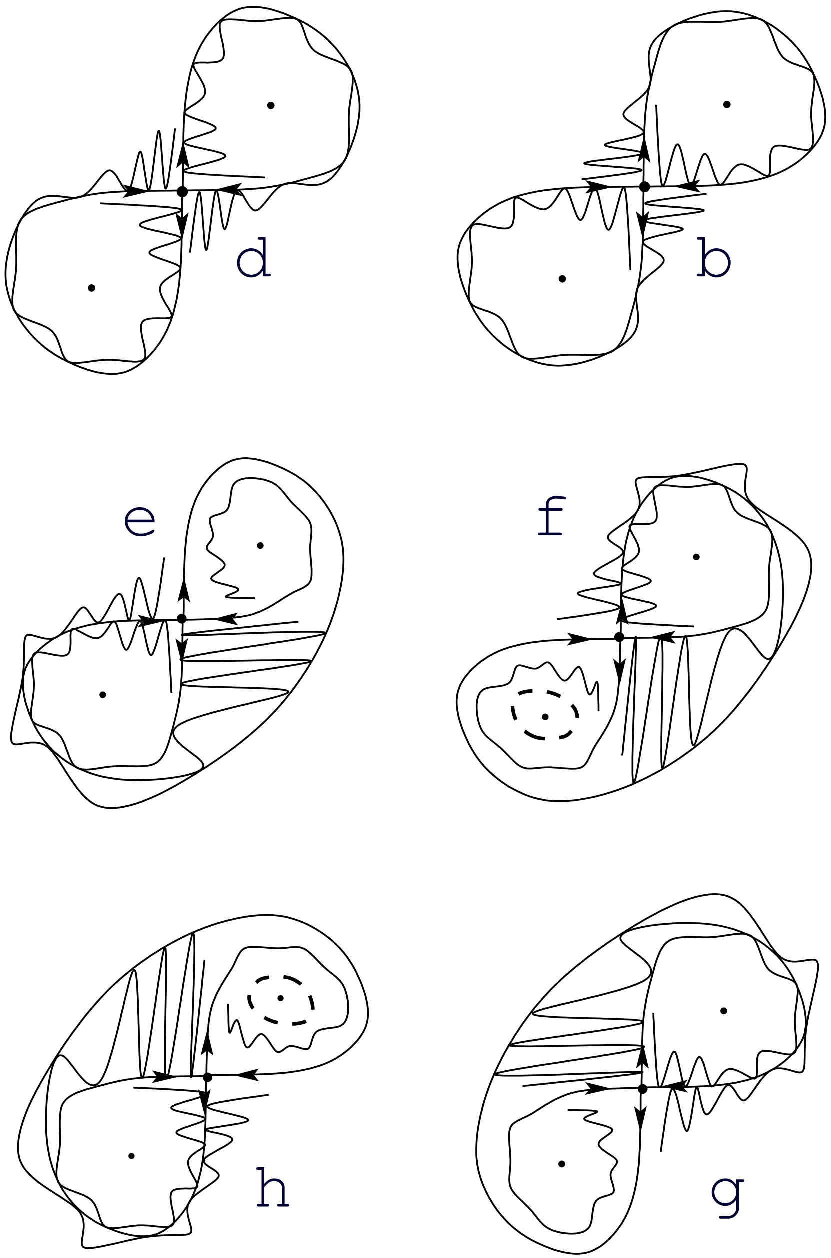


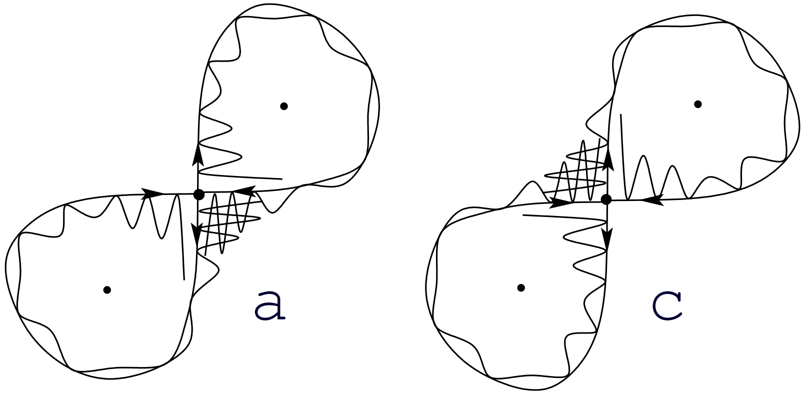
Homoclinic intersections:

(a) "Tail" strange attractor AT^{+} $\quad(\mu \in \mathbf{26})$

(b) Global strange attractor GA $\quad(\mu \in \mathbf{19})$

Double homoclinic tangencies



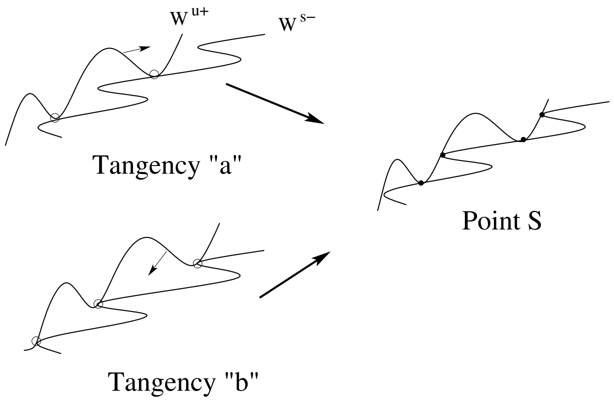


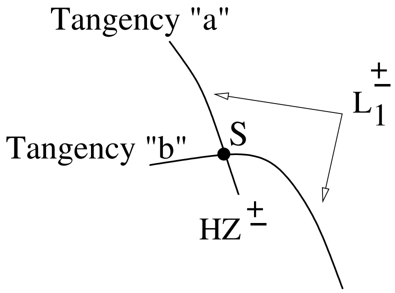
The boundaries of $\mathrm{HZ}^{+,-,\pm,\mp}$ intersect at

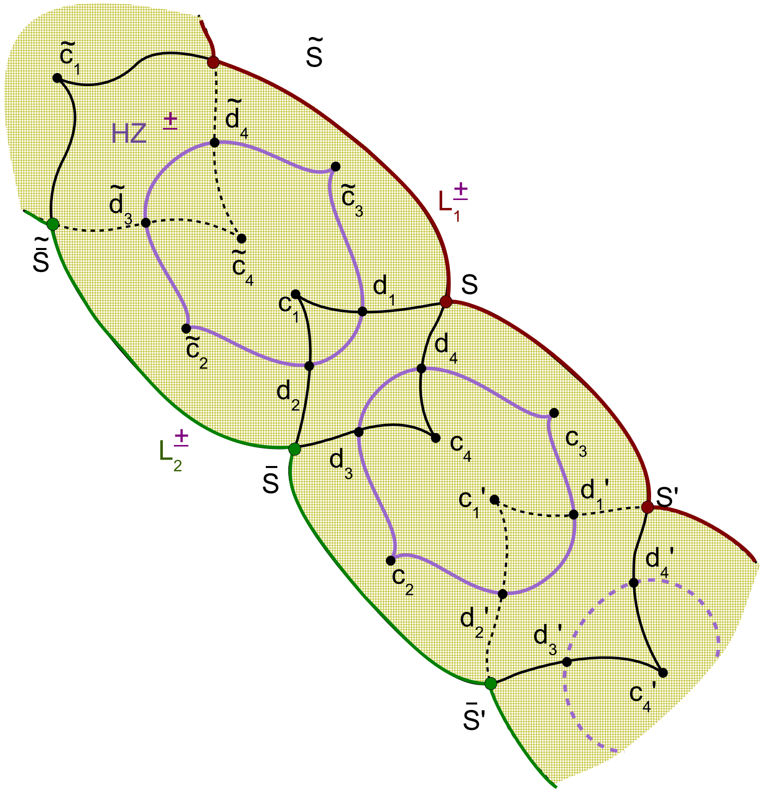
$\rightarrow$ double primary tangencies b, d, e, f, g, h

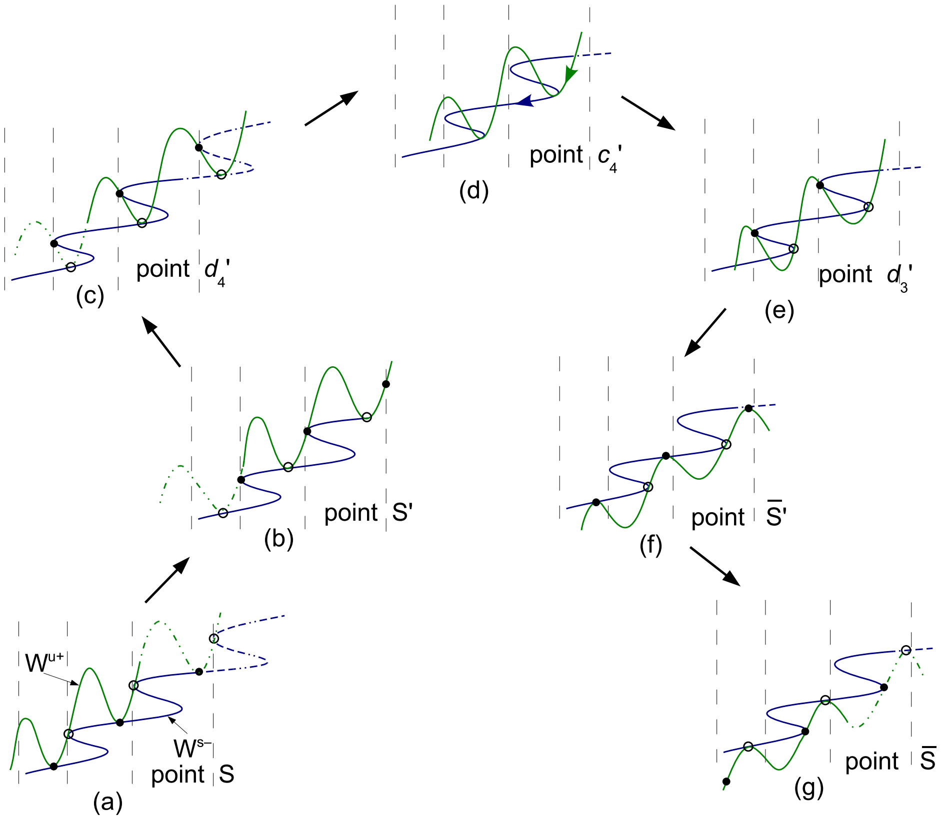
$\rightarrow$ double non-primary tangencies a, c.

The stepness of $HZ^{\pm,\mp}$



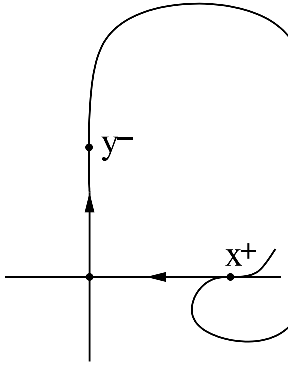






cubic tangencies!

Cubic single-round homoclinic tangencies

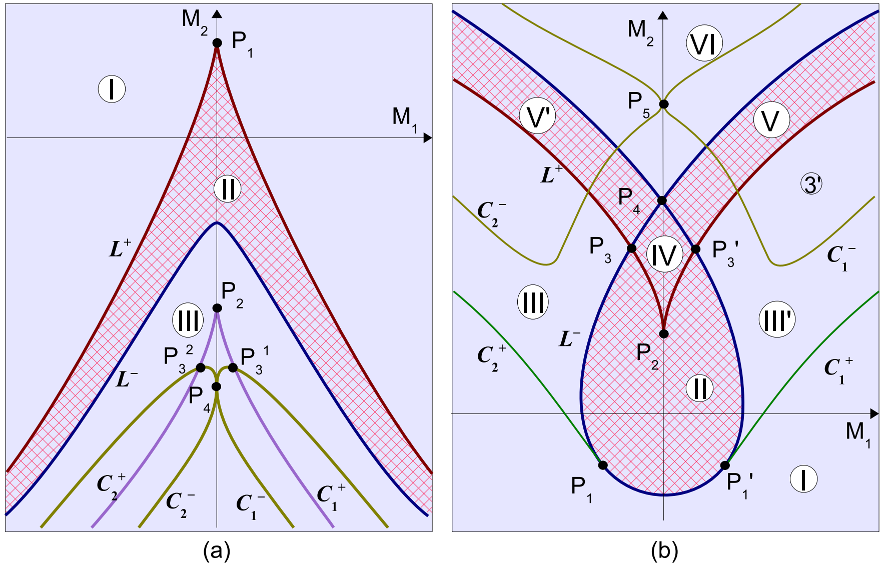


Outer map:

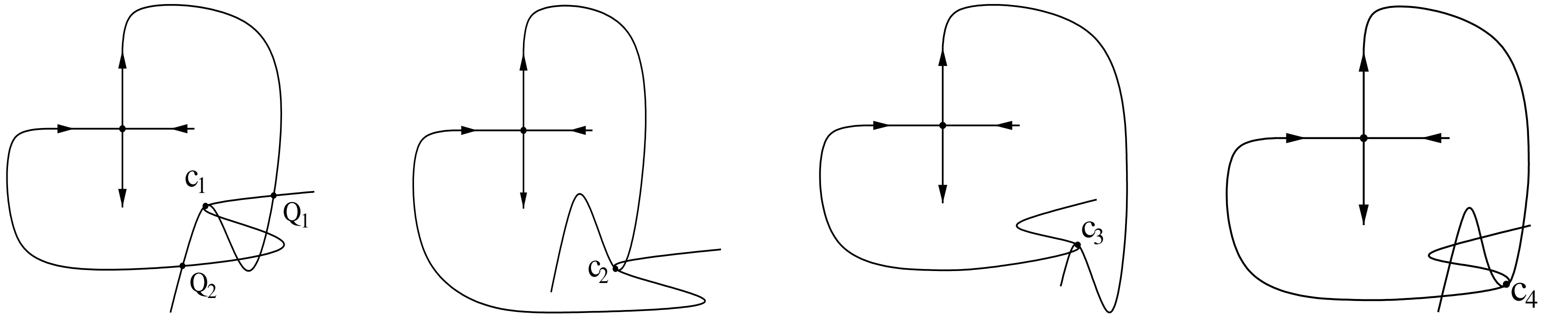
$$\begin{aligned} \bar{x} - x^+ &= ax + b(y - y^-), \\ \bar{y} &= cx + d(y - y^-)^3. \end{aligned}$$

Single round k-p.o, k large,
limit return map:

$$\begin{aligned} \bar{X} &= Y, \\ \bar{Y} &= M_1 + M_2 Y + \text{sign}(d) Y^3. \end{aligned}$$

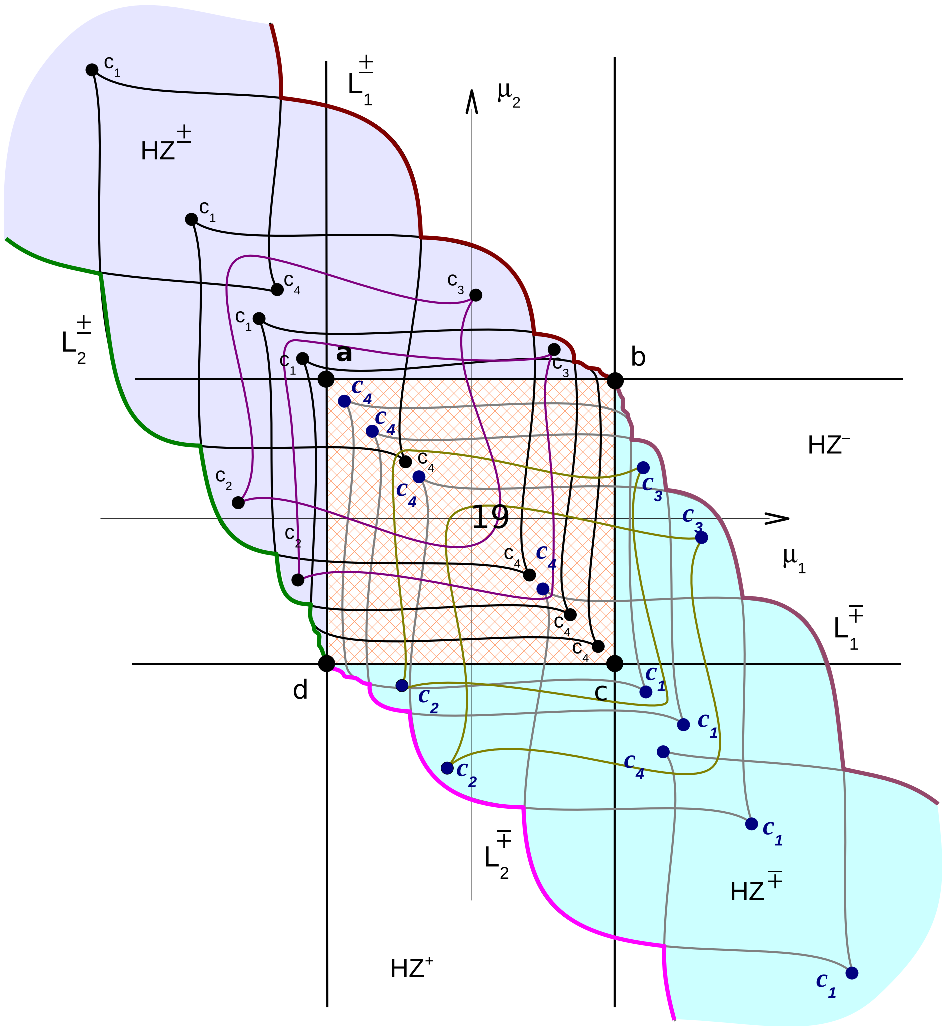


In our system, $c_1, \ldots, c_4$ cubic tangencies inside $HZ^{\pm}$ and $HZ^{\mp}$.



Lemma. All the cubic tangencies $c_1, \ldots, c_4$ are of **spring-area type** $(d < 0)$.

Accumulation of links inside $HZ^{\pm}$



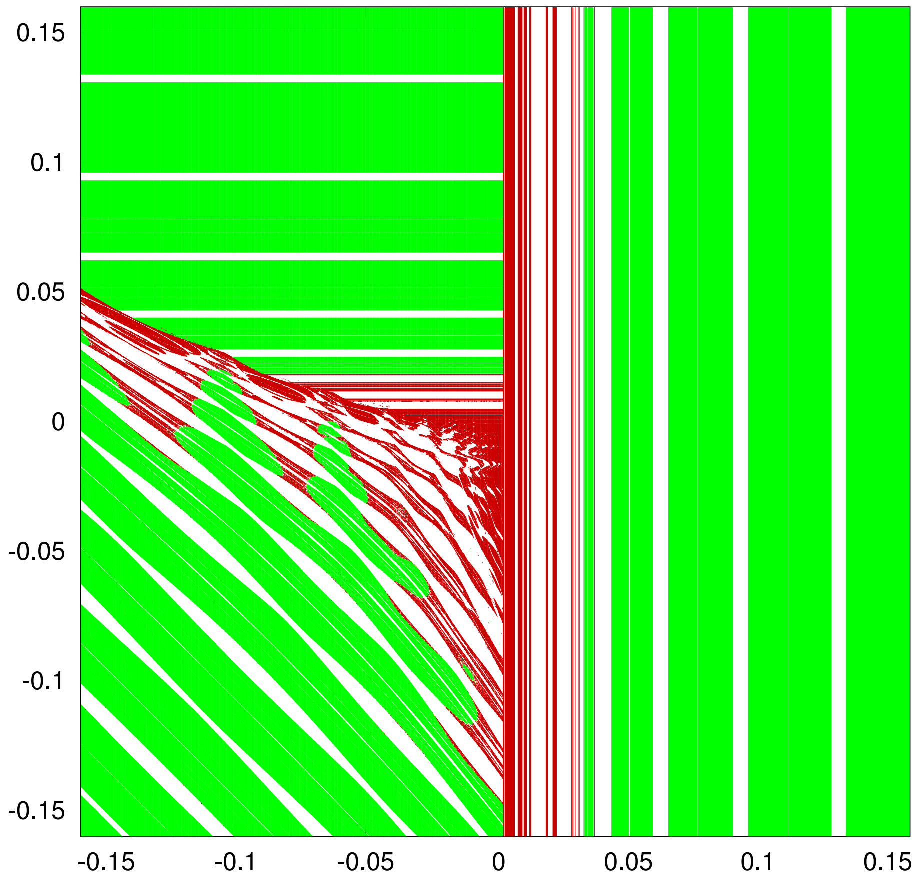
Lemma.

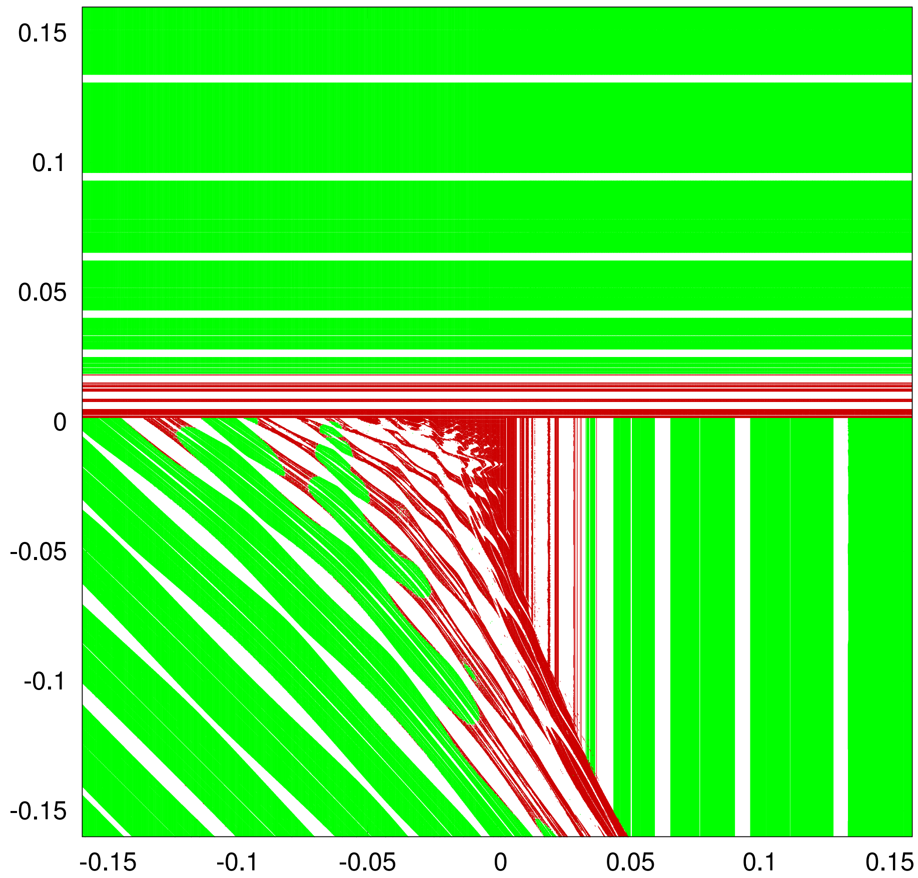
1. The primary cubic tangencies c_1 can exist only if $W^{u+} \cap W^{s+} = \emptyset$ and $W^{u-} \cap W^{s-} = \emptyset$ (i.e. in the regions **3** and **10** of the bif. diagram).
2. The primary cubic tangencies c_2 can exist if $W^{s+} \cap W^{u+} = \emptyset$ (i.e. in the regions **3**, **10** and **18**).
3. The primary cubic tangencies c_3 can exist if $W^{s-} \cap W^{u-} = \emptyset$ (i.e. in the regions **3**, **10** and **15**).
4. In the region **19** of the bif. diagram only primary cubic tangencies c_4 can exist.

Corollary. The cusp points c_1, c_2, c_3 and c_4 accumulate to the points $\mathtt{a}$, $\mathtt{d}$, $\mathtt{b}$ and $\mathtt{c}$ resp.

Further analysis of the model: MLE

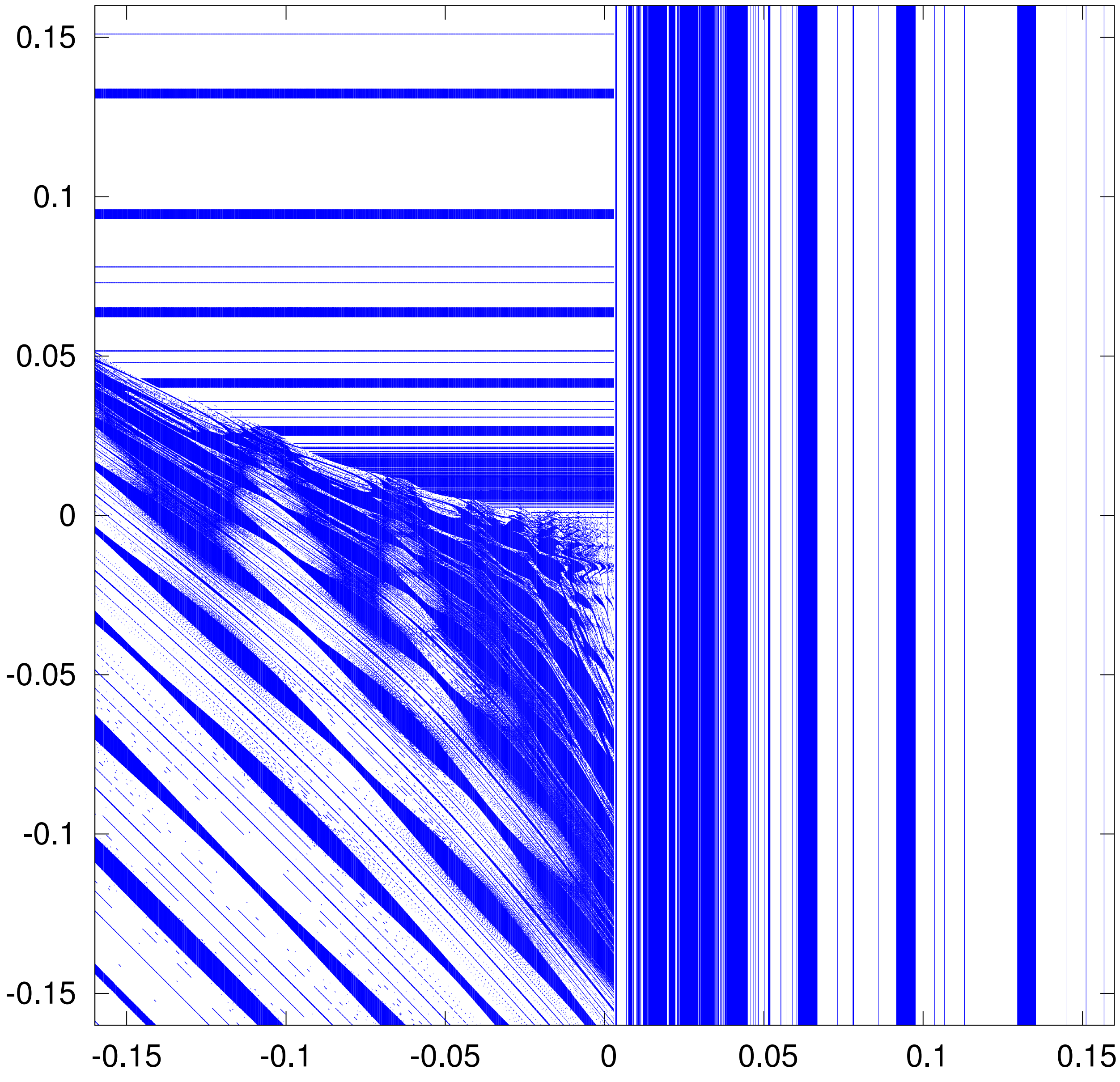
For each (a_1, a_2)-parameters we take $z_0 = 0.5$, $\eta_0 = 0$ and $s_0 = 1$ (left) or $s_0 = 1$ (right) as i.c. (i.e. on W^u) and compute the Max. Lyap. exp. Λ.



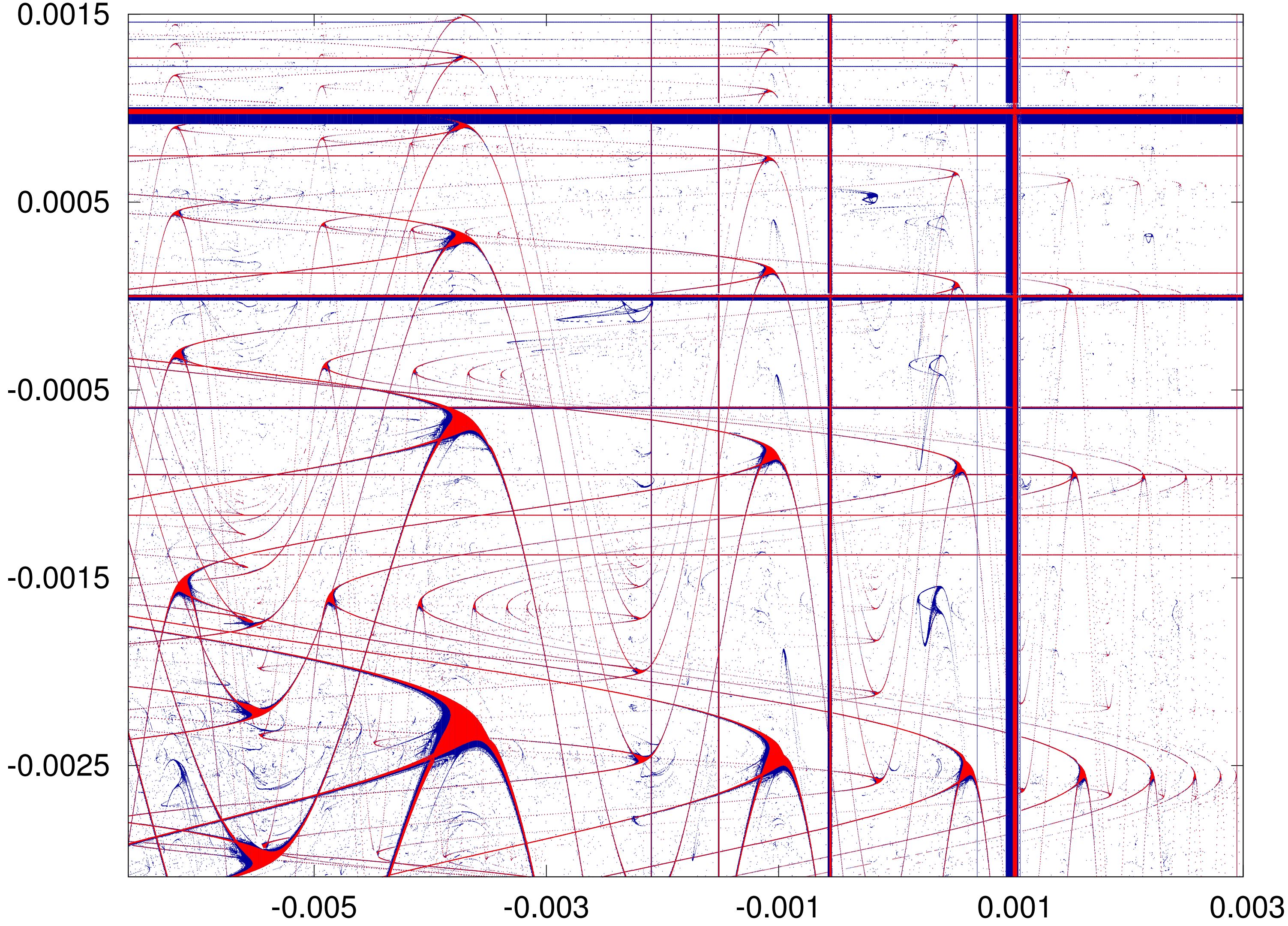


Red points correspond to $\Lambda > 0$ (**chaotic attractor**), **green** points to $\Lambda = 0$ (**invariant curve**) and **white** points to $\Lambda < 0$ (**periodic sink**).

Stability regions ($\Lambda < 0$) related to periodic sinks



Stability region: magnification

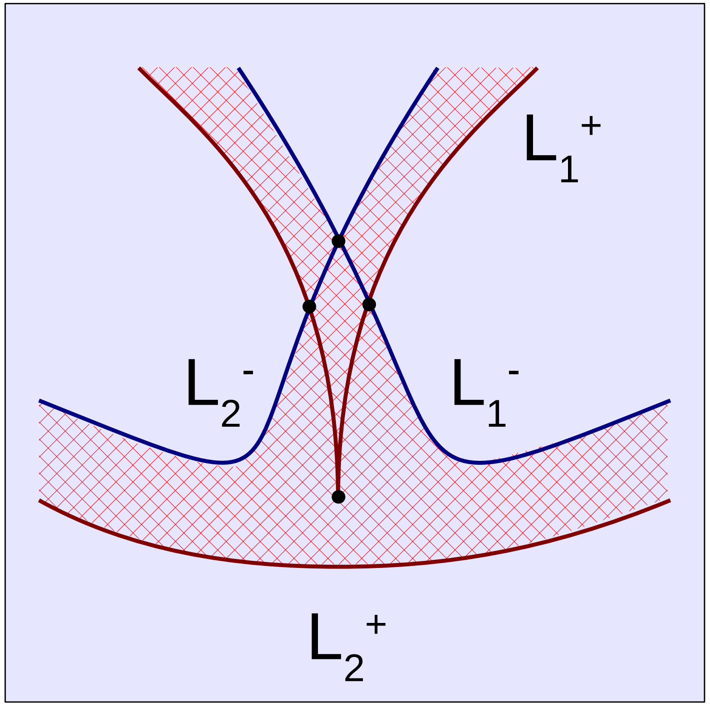


Blue: set of (a_1, a_2)-parameters with $\Lambda < 0$ for the i.c. $(0.5, 0, 1)$. The attractor is a **periodic sink**.

Red: parameters for which there is a **2-periodic sink** as attractor.

The cross-road scenario

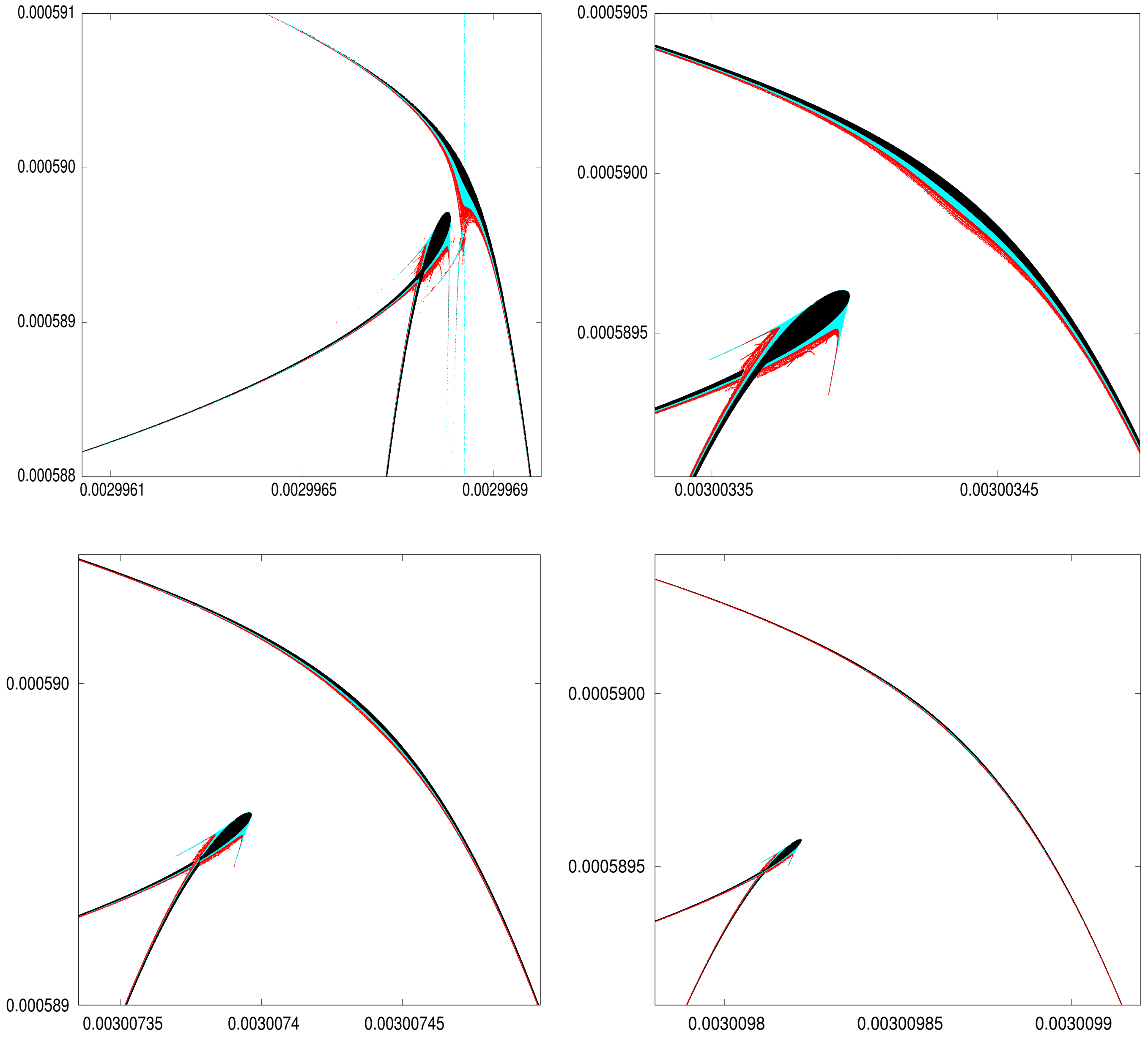
If k is not large enough (depending on the parameters) other configurations might appear (non-local effects and role of high order terms in the return map). One of this, which is commonly observed in numerical explorations and related to the spring-area configuration, is the cross-road scenario.

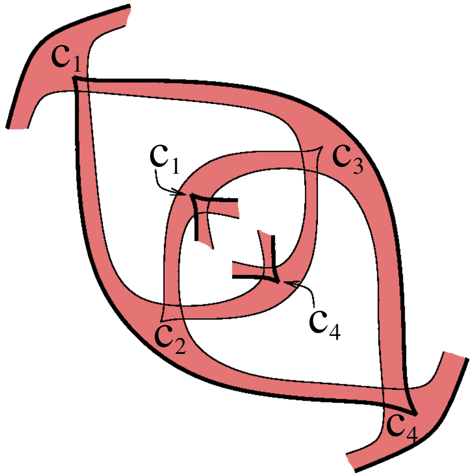


H. Broer, C. Simó and J.C. Tatjer. *Towards global models near homoclinic tangencies of dissipative diffeomorphisms.* Nonlinearity, 1998, 11, 667–770.

J.P. Carcassès, C. Mira, M. Bosch, C. Simó and J.C. Tatjer. *"Crossroad area-spring area" transition (I)-(II). Parameter plane representation.* Int. J. Bifur. and Chaos, 1991, 1.

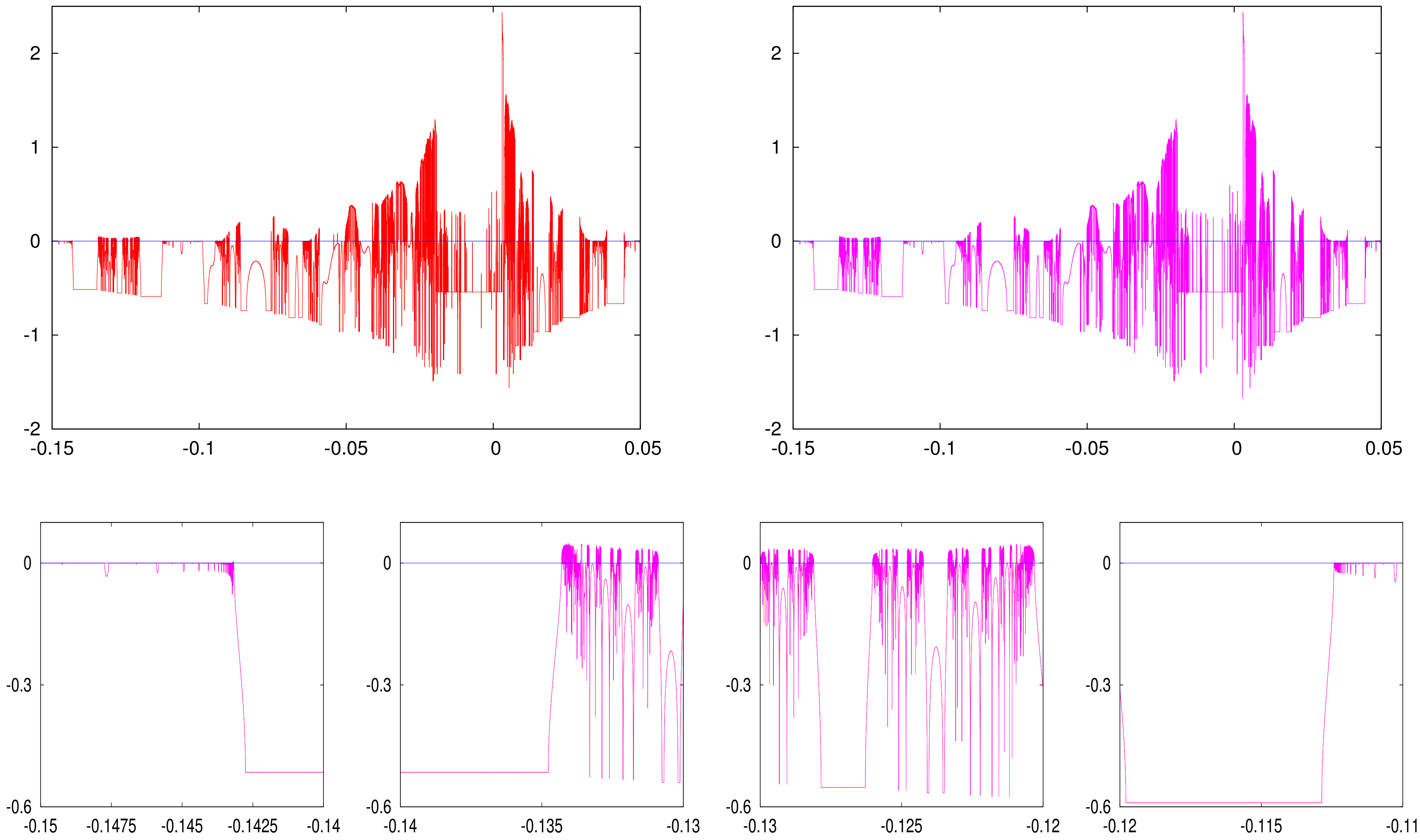
Transition to spring-area: larger (return) periods



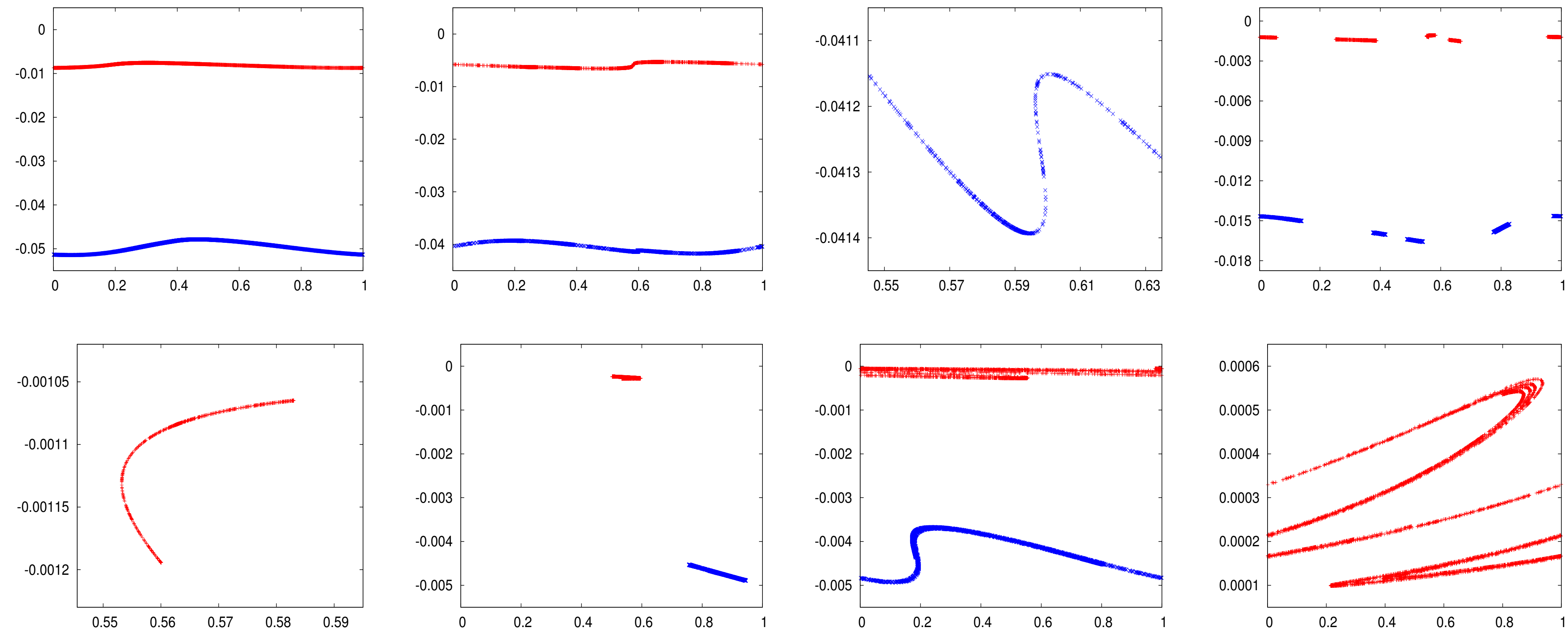


Note the progressive destruction of the previous cross-road domain.

Lyapunov exponents

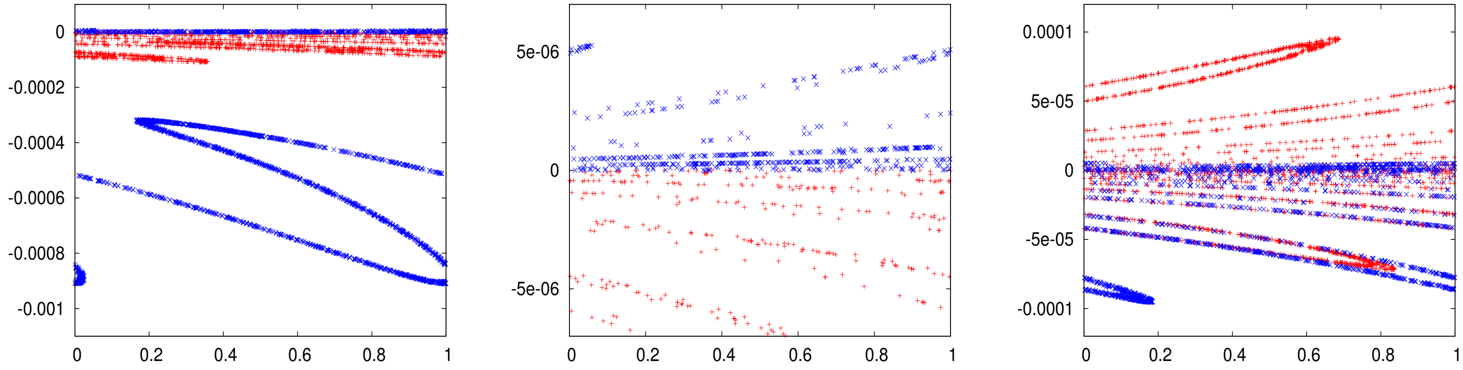


A sample of attractors I $(a_2 = 0)$

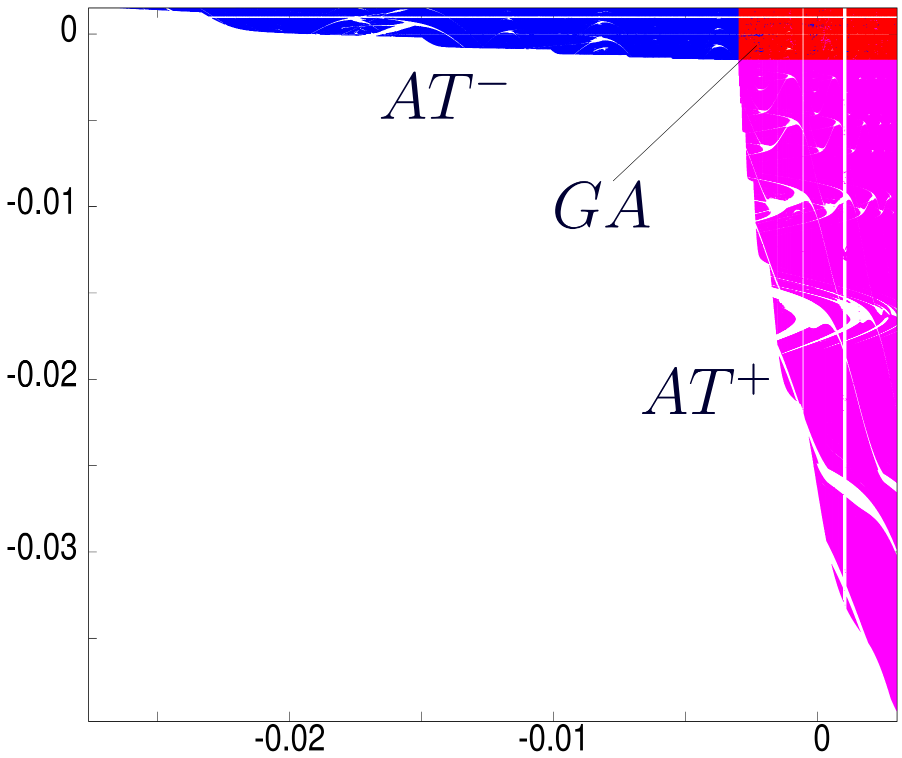


1st row: invariant curve ($a_1 = -0.145$), SA of type A^* with a global nature ($a_1 = -0.129$), detail of the fold in the previous SA ($a_1 = -0.129$) and a SA of type A^* with a local periodic nature ($a_1 = -0.073$). **2nd row:** Detail of the Hénon-like structure of the previous SA ($a_1 = -0.073$), SA of type A^* with a local nature ($a_1 = -0.034$), globalization of the previous SA ($a_1 = -0.033$) and a SA of type A^- ($a_1 = 0.006$).

A sample of attractors II $(a_2 = -0.001)$



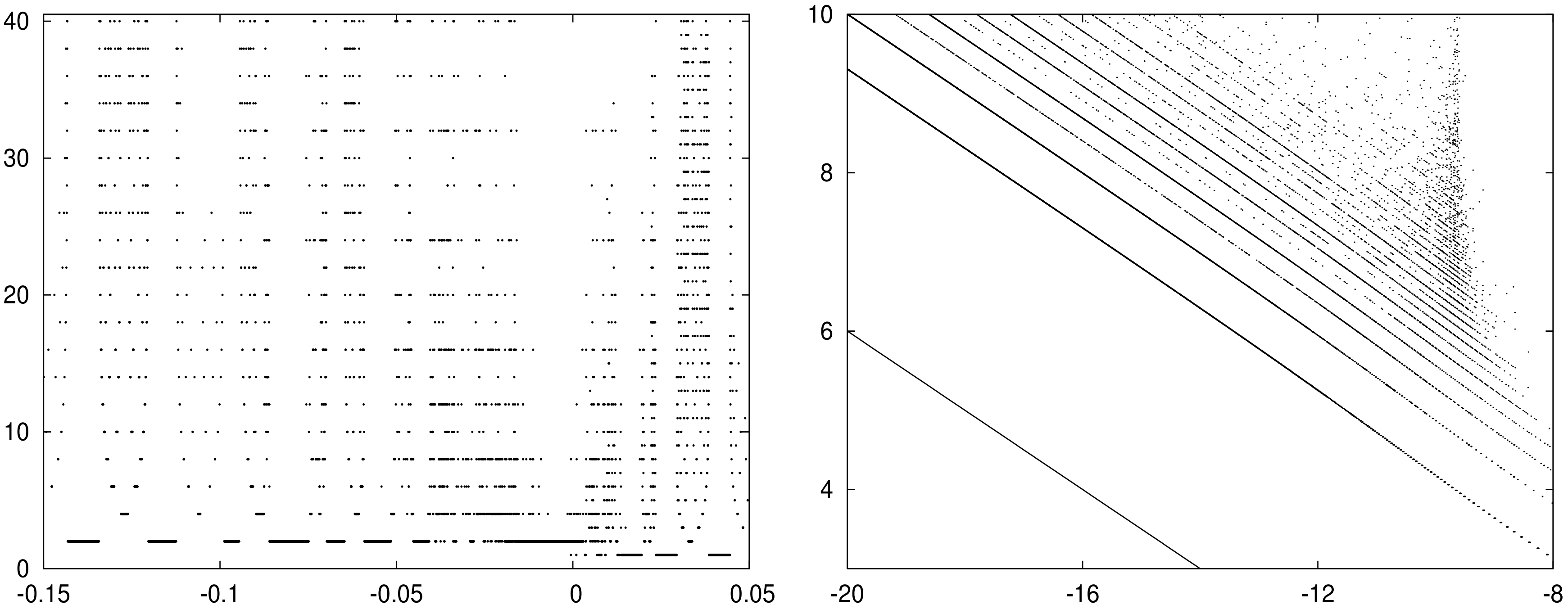
Left: Tail attractor of type AT^- ($a_1 = -0.0095$). Center: Magnification of the previous figure. Right: Global SA of type GA ($a_1 = 0$).



We can identify the points $\mathbb{e}$ and $\mathbb{g}$ of the bif. diagram. The white domains contained in these colored regions correspond to sinks.

The period of the sinks

Lemma. If a s-n appears for a critical value $a_1 = a_{1,c}$, then the period of nearby sinks behaves as ctant$\times|a_1 - a_{1,c}|^{-1/2}$.



$a_2 = 0$. We plot Per vs. a_1 (left) and $\log(\text{Per})$ vs $\log(a_1 - a_{1,c})$ (right).

$a_{1,c} \approx -0.143170413565918$ is the value for the first appearance of period 2 orbits with $a_1 > -0.15$.

All periods (under M) from 24 to 11026 have been detected!

Open problems and extensions

Several questions remain **open**, like

- The **creation/destruction** of SA by **folding** of IC. In particular the **boundary marked as BD** in the bifurcation diagram.
- The **abundance** of sinks, taking into account the existence of **cross-road and spring** areas.
- **Links with s-n boundaries** connecting different cross-road and spring areas.
- Relative **size** of the **basins of attraction** when there is **multiplicity of attractors**.

... and possible **extensions** to **3D and higher dimension** diffeomorphisms.

E.g.: Shilnikov-like, Hopf-Shilnikov-like maps, etc.

Thanks for your attention!!