



Piecewise Linear Systems: twenty years on



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DDAYS 2012, Benicasim, October 24-26th.

Introduction

- The interest on the analysis of piecewise linear differential systems (or simply piecewise linear systems) has increased in the last decades, as modern engineering applications require the piecewise linear modeling of a wide range of problems in mechanics, power electronics, control theory, biology...
- On the one hand piecewise linear systems are the natural extension of the linear ones in order to cope with nonlinear phenomena, for they can reproduce much of the complex behavior observed in smooth nonlinear systems: multi-stability, self-sustained oscillations, hysteretic behavior, homoclinic and heteroclinic connections and of course, chaotic behavior. On the other hand, piecewise linear systems turn out to be the most accurate models for some realistic applications.
- Piecewise linear systems can be classified in two big classes depending on the continuity of the associated vector field. Discontinuous cases constitute nowadays the subject of intense research, and there is not yet a total agreement about basic concepts and definitions. Even for continuous piecewise linear systems (CPWL, for short), there are still unsolved issues as the seemingly simple problem of stability of the only equilibrium point.

Introduction

- Apart from equilibria, it is very important to characterize the periodic orbits of such systems, since they constitute the next step in complexity for observed behavior in practice.
- We have paid special attention to the study of existence of periodic orbits for piecewise linear systems, following a point of view which is typical in bifurcation theory, that is, we will study degenerated situations and after parameter variations we will look for the appearance of limit cycles.
- Unfortunately the non-smoothness of continuous piecewise linear systems requires that limit cycle bifurcations must be analyzed in a case-by-case approach for the different families of systems which are relevant in applications.
- Only planar and certain three dimensional cases will be here reviewed.

Summary

- A historical review
- A general setting for planar PWL systems with 2 zones
- Some unexpected results

A historical review

- The Russian school. Andronov et al. (1934)
- The Berkeley ERL reports by Chua & Lum (1990)
- A paper by Llibre & Sotomayor (1992...1996) stimulated by some mexican authors
- Ph. D. dissertations of Rodrigo, Teruel (1998...2000), and then Carmona, Fernández-G, Fernández-S, Ros, García-Medina, Vela... (initiating a series of publications still unfinished, much of them under supervision of Emilio Freire and F. Torres)
- A good source: the DiBernardo-Budd-Champneys-Kowalczyk book (2008)
- Seminal papers by Kuznetsov-Gragnani-Grinaldi (2003) and Guardia-M.Seara-Teixeira (2011)

A general setting for planar PWL systems with 2 zones

- We consider the non-smoothness boundary at the vertical axis

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$$

- The boundary induces the partition of the phase plane into

$$S^- = \{(x, y) \in \mathbb{R}^2 : x < 0\},$$

$$S^+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}.$$

The systems to be studied become

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{F}^+(\mathbf{x}) = (F_1^+(\mathbf{x}), F_2^+(\mathbf{x}))^T = A^+\mathbf{x} + \mathbf{b}^+, & \text{if } \mathbf{x} \in S^+, \\ \mathbf{F}^-(\mathbf{x}) = (F_1^-(\mathbf{x}), F_2^-(\mathbf{x}))^T = A^-\mathbf{x} + \mathbf{b}^-, & \text{if } \mathbf{x} \in S^-. \end{cases}$$

Tangencies and sliding set

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^- x + a_{12}^- y + b_1^- \\ a_{21}^- x + a_{22}^- y + b_2^- \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^+ x + a_{12}^+ y + b_1^+ \\ a_{21}^+ x + a_{22}^+ y + b_2^+ \end{pmatrix}$$

We will assume $a_{12}^-, a_{12}^+ \neq 0$ to avoid ‘wall’ cases.

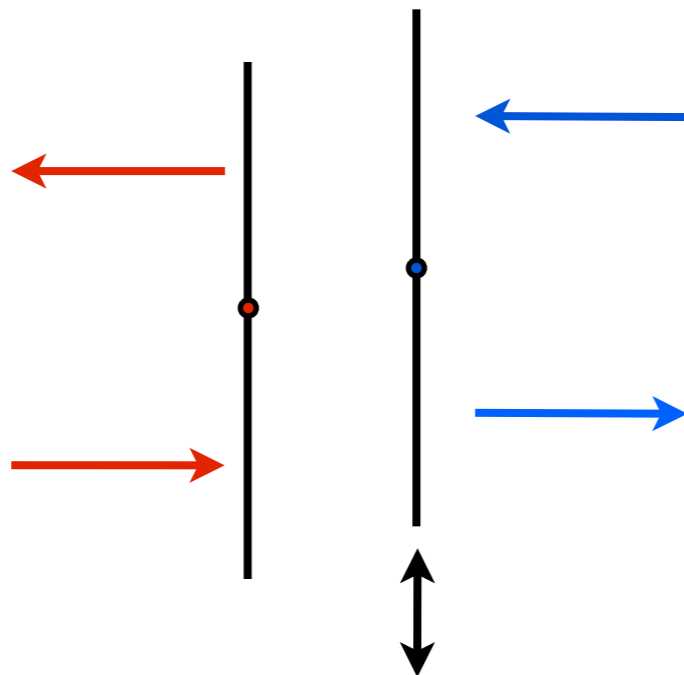
We have a tangency point in Σ when $\dot{x}|_{x=0} = a_{12}y + b_1$ vanishes.

At tangency points, we speak of visible (invisible) tangency depending on the sign of \ddot{x} . Since $\ddot{x}|_{x=0} = a_{11}(a_{12}y + b_1) + a_{12}(a_{21}y + b_2)$, we obtain

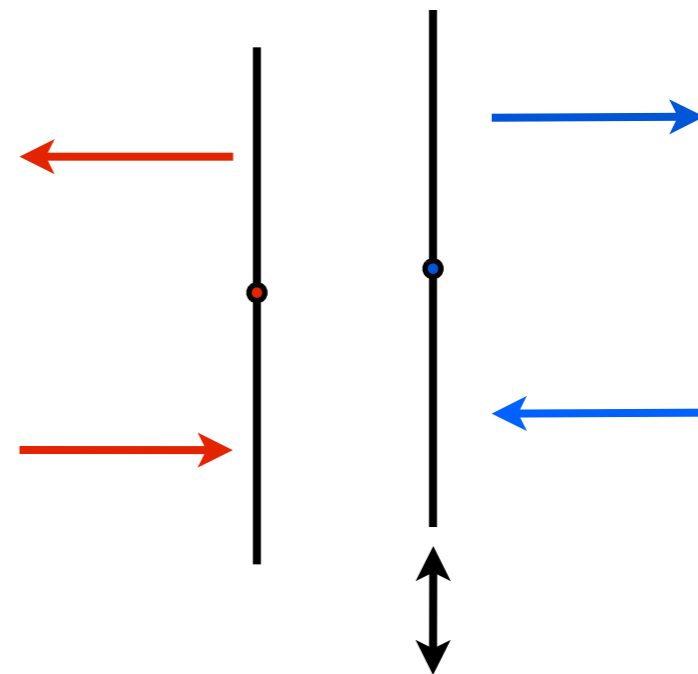
$$\ddot{x}|_{\dot{x}=0} = a_{12}b_2 - a_{21}b_1$$

Tangencies and sliding set (cont'd)

Assuming $a_{12}^- < 0$, there are two possibilities for a_{12}^+ :



($a_{12}^+ < 0$: bounded sliding)



($a_{12}^+ > 0$: bounded crossing)

For a non-smooth system... a non-smooth change!

We do a continuous piecewise linear change of variables $\mathbf{u} = f(\mathbf{x})$, where

$$\mathbf{u} = -a_{12}^+ \begin{pmatrix} x \\ a_{22}^- x - a_{12}^- y \end{pmatrix} + a_{12}^+ \begin{pmatrix} 0 \\ b_1^- \end{pmatrix}, \quad x < 0,$$

and

$$\mathbf{u} = -a_{12}^- \begin{pmatrix} x \\ a_{22}^+ x - a_{12}^+ y \end{pmatrix} + a_{12}^+ \begin{pmatrix} 0 \\ b_1^- \end{pmatrix}, \quad x > 0,$$

and afterwards rename the variable \mathbf{u} to \mathbf{x} .

This change is a global homeomorphism that conjugates the vector field in each halfplane, separately. Such a conjugacy cannot be extended to the sliding vector field.

The (possibly) discontinuous canonical form

Liénard canonical form for DPWL systems. Assume that $a_{12}^+ a_{12}^- > 0$ (bounded sliding set). Then the system can be written in the form,

$$\dot{\mathbf{x}} = \begin{pmatrix} T^- & -1 \\ D^- & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ a^- \end{pmatrix} \text{ if } \mathbf{x} \in S^-,$$

$$\dot{\mathbf{x}} = \begin{pmatrix} T^+ & -1 \\ D^+ & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} -b \\ a^+ \end{pmatrix} \text{ if } \mathbf{x} \in S^+,$$

where T, D stand for trace and determinant, and

$$a^- = a_{12}^+ (a_{12}^- b_2^- - a_{22}^- b_1^-), \quad a^+ = a_{12}^- (a_{22}^+ b_1^+ - a_{12}^+ b_2^+), \quad b = a_{12}^+ b_1^- - a_{12}^- b_1^+.$$

This system has as its tangency points $(0,0)$ and $(0,b)$.

Apart from the linear invariants, the other three parameters are associated to the x -coordinates of the equilibrium points (a^+ and a^-) and the size and stability of the sliding set (b).

Remarks

- For continuous vector fields one has $\mathbf{A}^+ \cdot (0,1)^\top = \mathbf{A}^- \cdot (0,1)^\top$ and $\mathbf{b}^+ = \mathbf{b}^-$, so that $a^+ = a^-$ and $b = 0$, automatically.
- When $a^+ = a^-$ and $b = 0$, we get a continuous piecewise linear system even if the original system was discontinuous.
- In particular, homogeneous systems ($\mathbf{b}^+ = \mathbf{b}^- = 0$) with bounded sliding set can always be transformed in a continuous system. Thus the class of bimodal systems considered in

Y. ZOU, T. KUPPER AND W. J. BEYN, *Generalized Hopf Bifurcations for Planar Filippov Systems Continuous at the origin*, J. Nonlinear Science **16** (2006), 159–177.

Y. IWATANI AND S. HARA, *Stability Analysis and Stabilization for Bimodal Piecewise Linear Systems Based on Eigenvalue Loci* Mathematical Engineering Technical Reports Web page <http://www.i.u.-tokyo.ac.jp/mi-e.htm>.

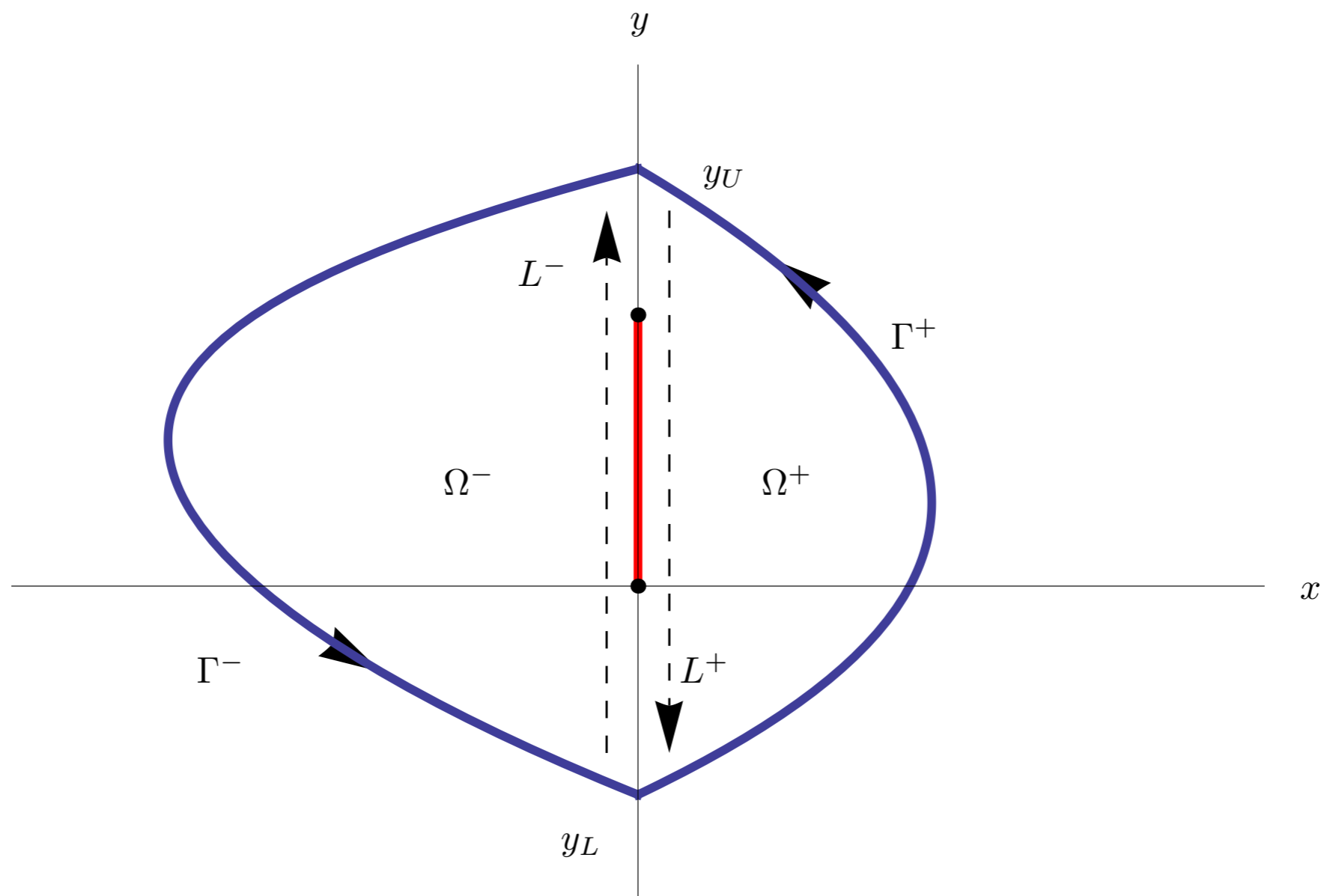
could be analyzed just by using the results in

E. FREIRE, E. P., F. RODRIGO AND F. TORRES, *Bifurcation Sets of Continuous Piecewise Linear Systems with Two Zones*, Int. J. Bifurcation and Chaos **8** (1998), 2073–2097.

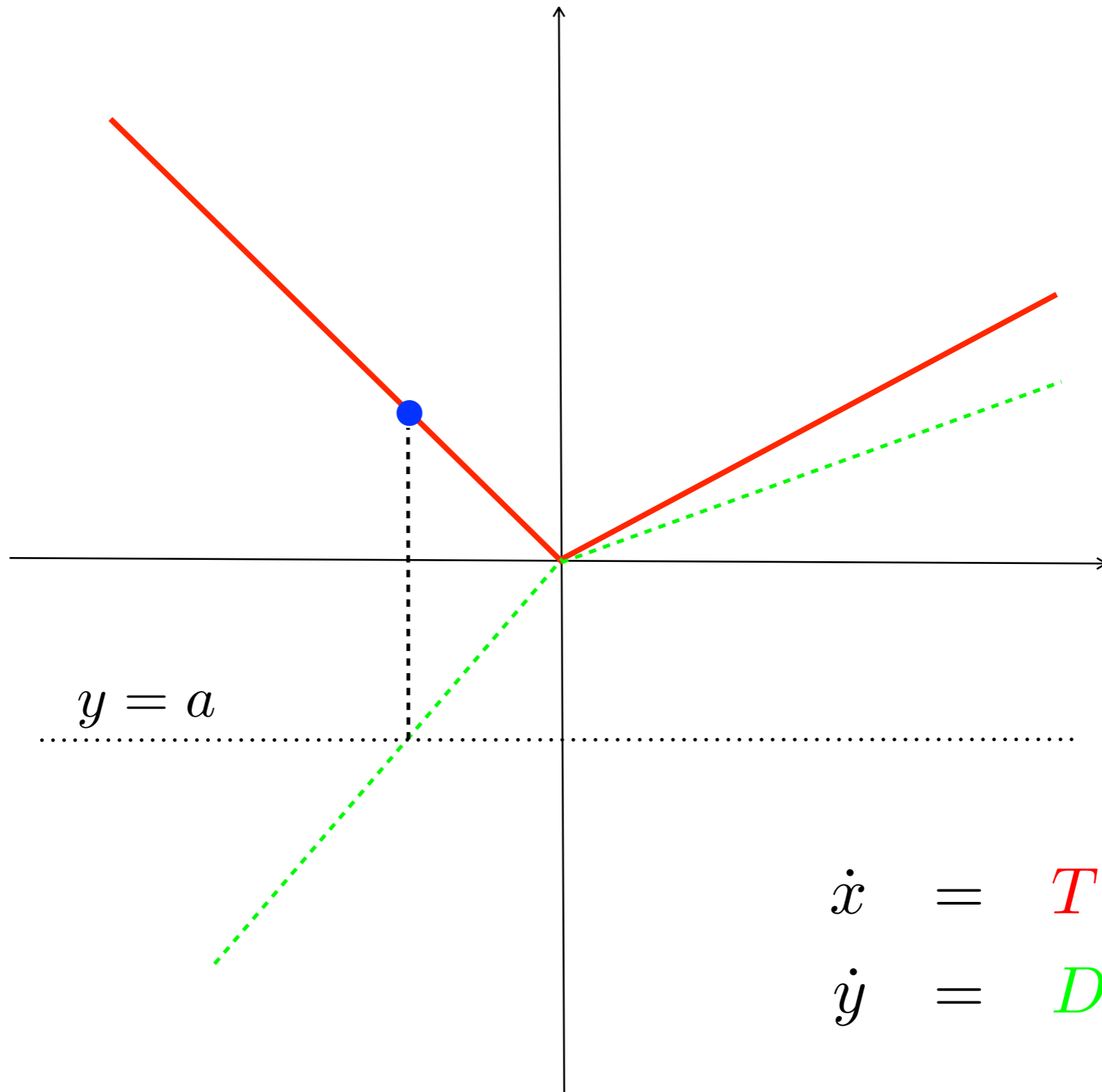
A necessary condition for crossing periodic orbits

Proposition Defining the values $\sigma^- = \text{area}(\Omega^-)$, $\sigma^+ = \text{area}(\Omega^+)$ and $h = y_U - y_L$, then we have

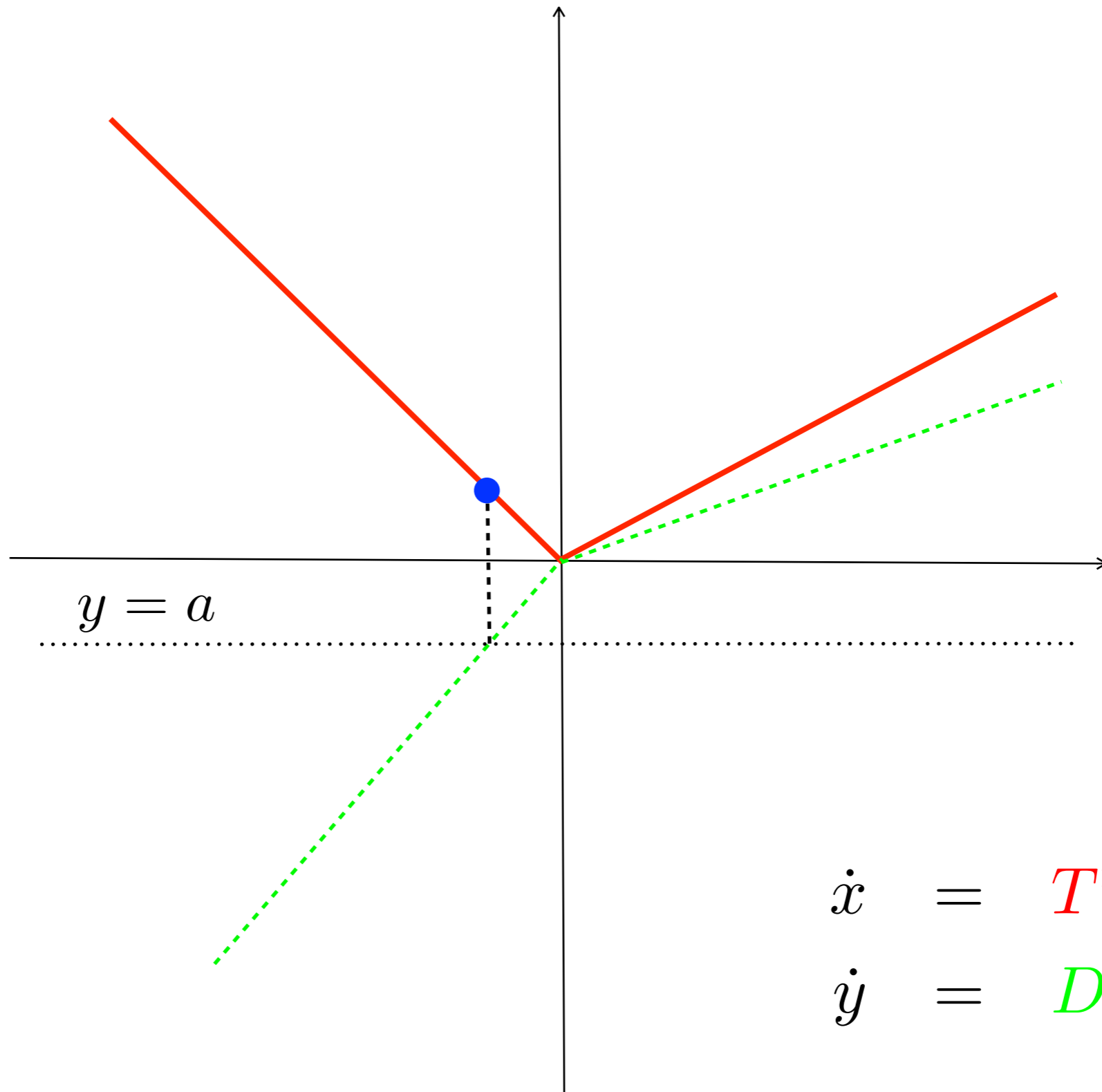
$$T^- \sigma^- + T^+ \sigma^+ + bh = 0.$$



Boundary Equilibrium Bifurcations (BEB's) in the continuous case

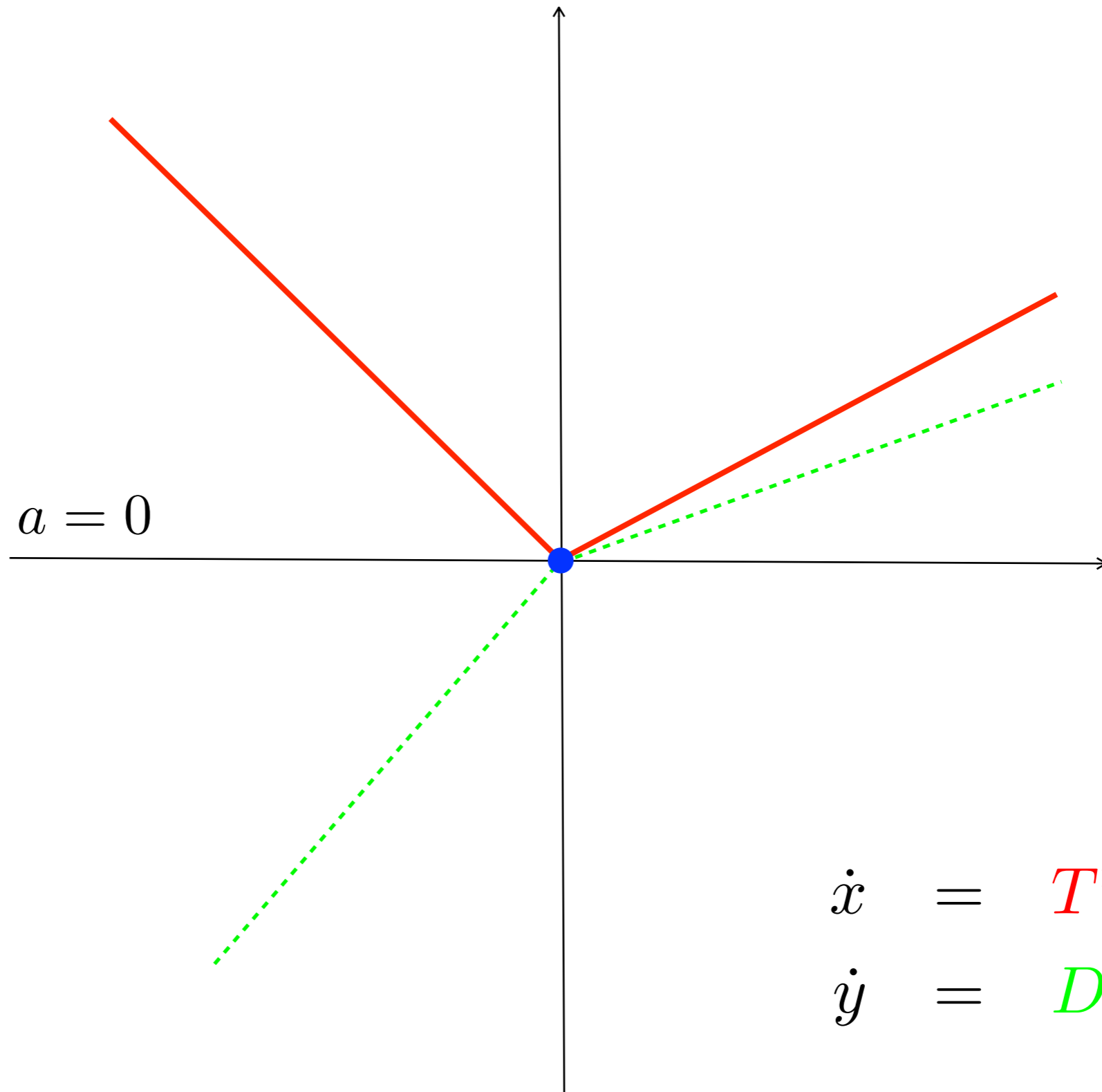


Boundary Equilibrium Bifurcations (BEB's) in the continuous case

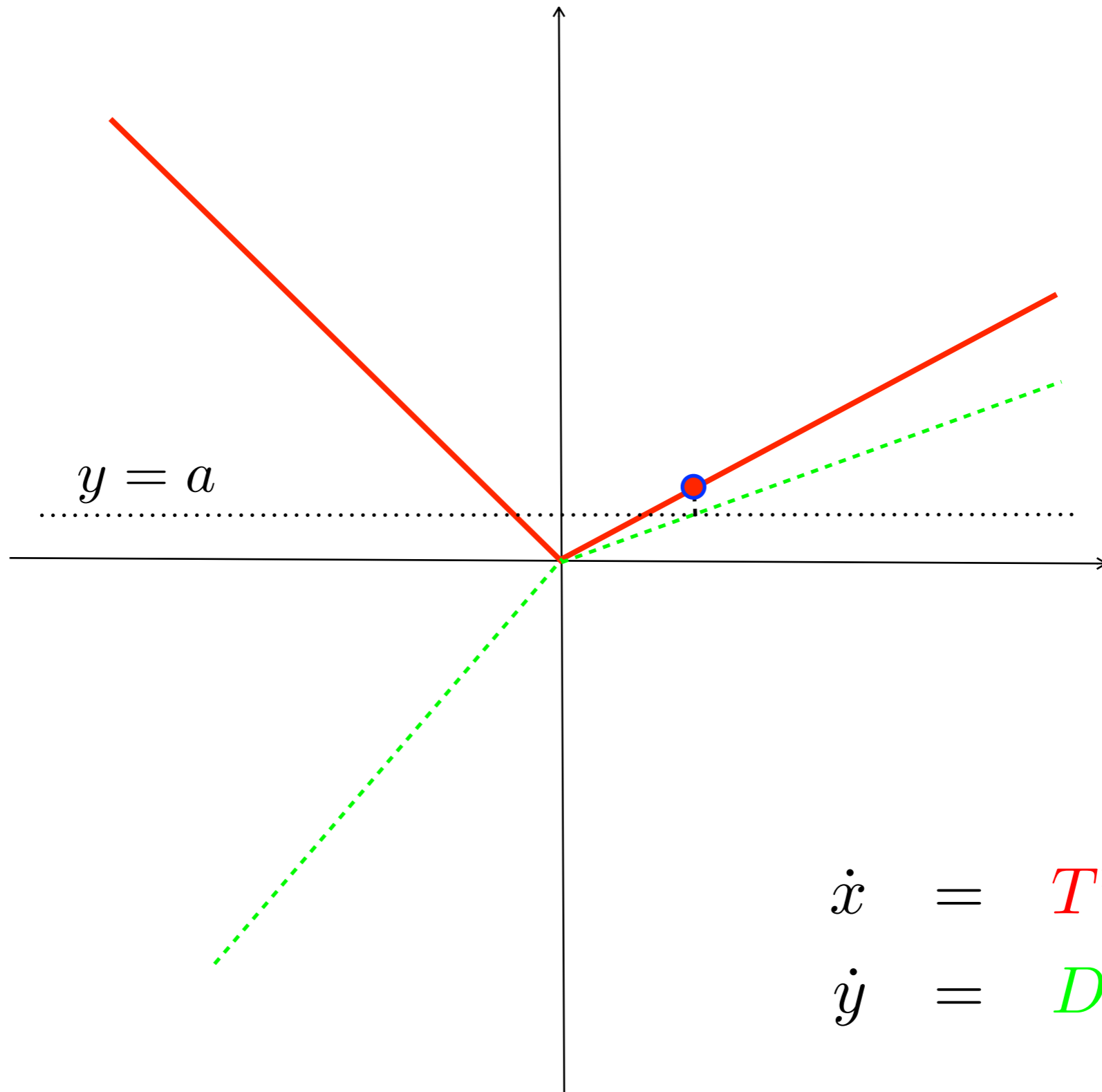


$$\begin{aligned}\dot{x} &= T^{\pm}x - y \\ \dot{y} &= D^{\pm}x - a\end{aligned}$$

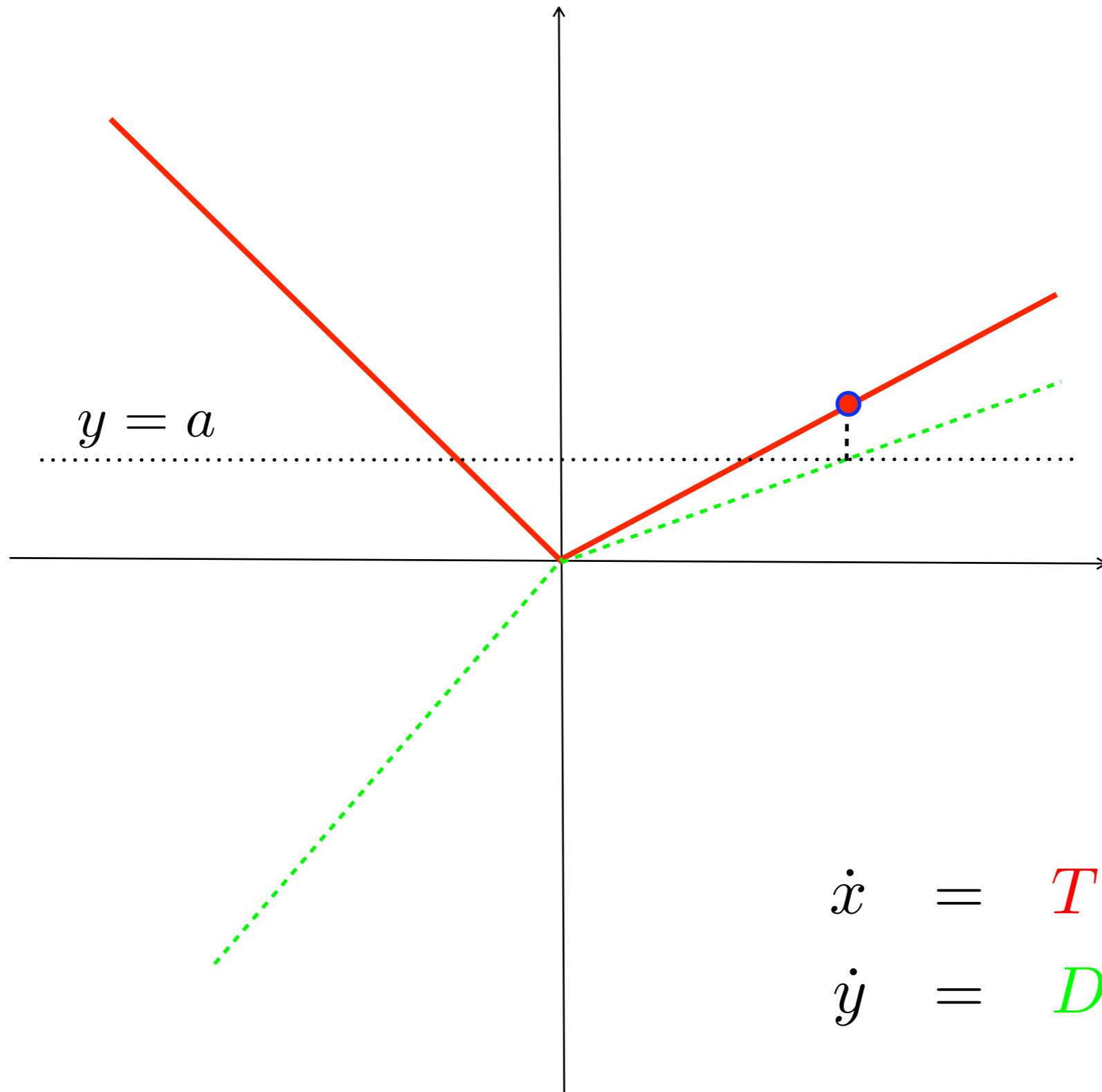
Boundary Equilibrium Bifurcations (BEB's) in the continuous case



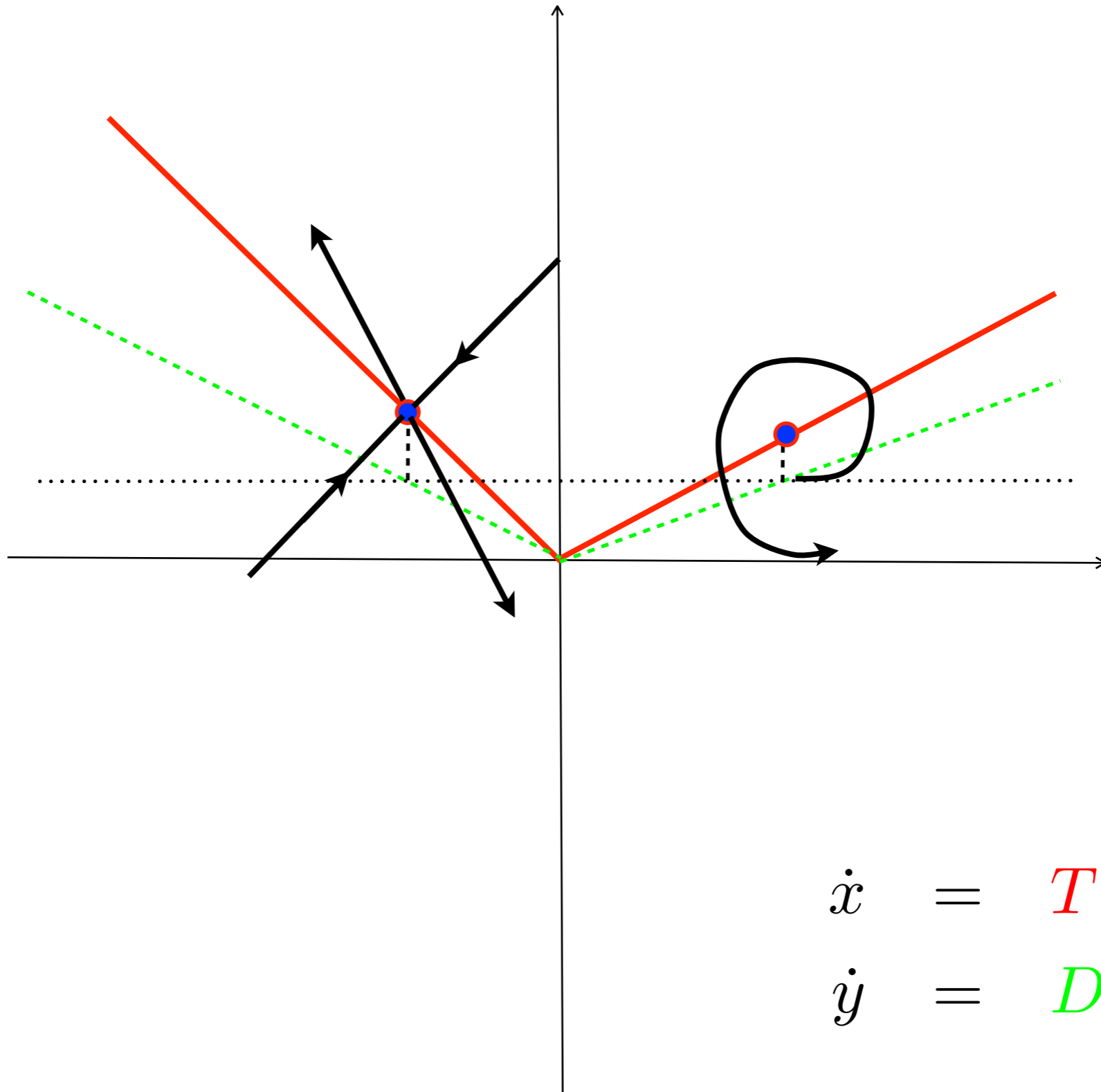
Boundary Equilibrium Bifurcations (BEB's) in the continuous case



Boundary Equilibrium Bifurcations (BEB's) in the continuous case

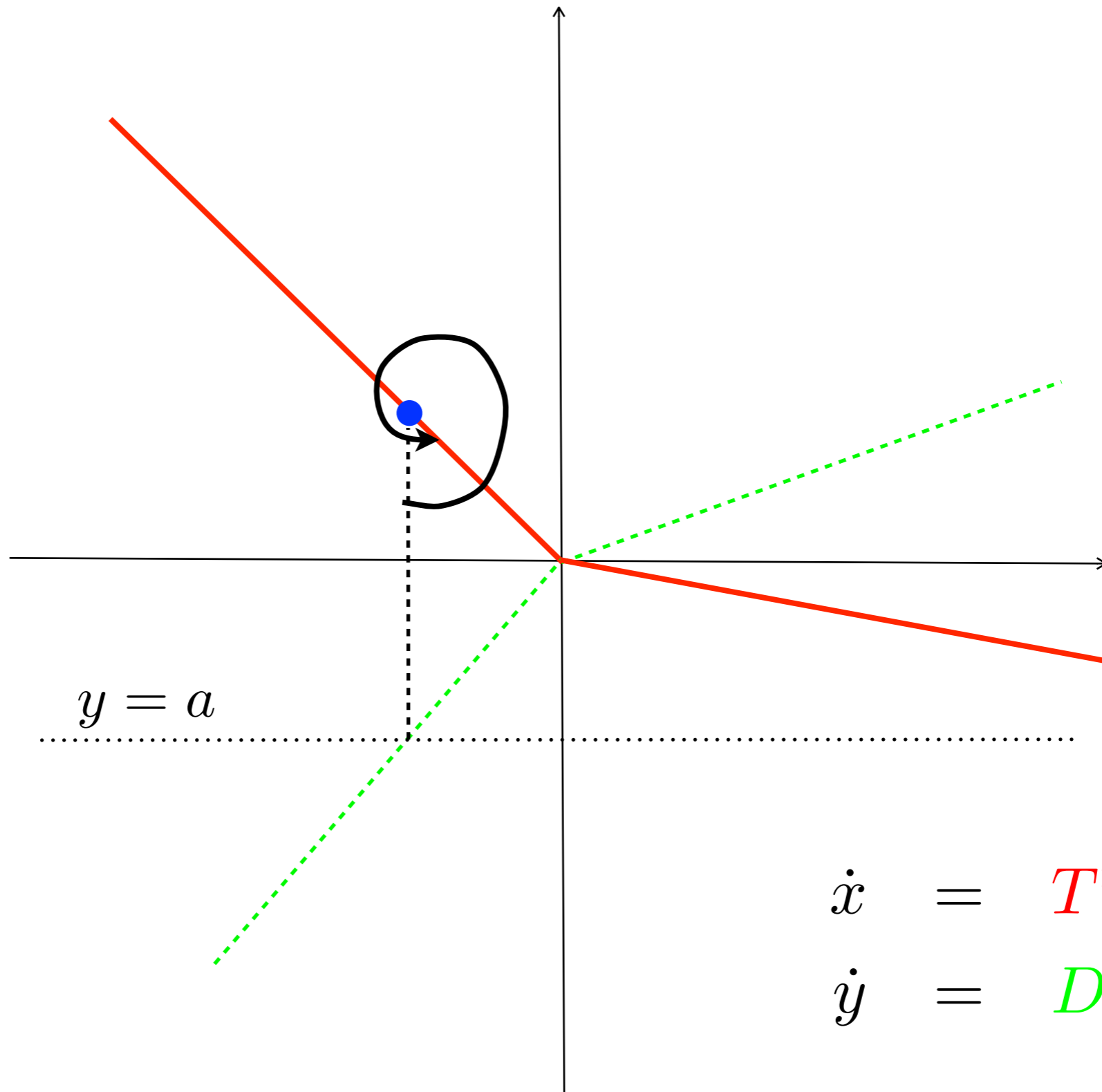


Saddle-Focus BEB's are possible!



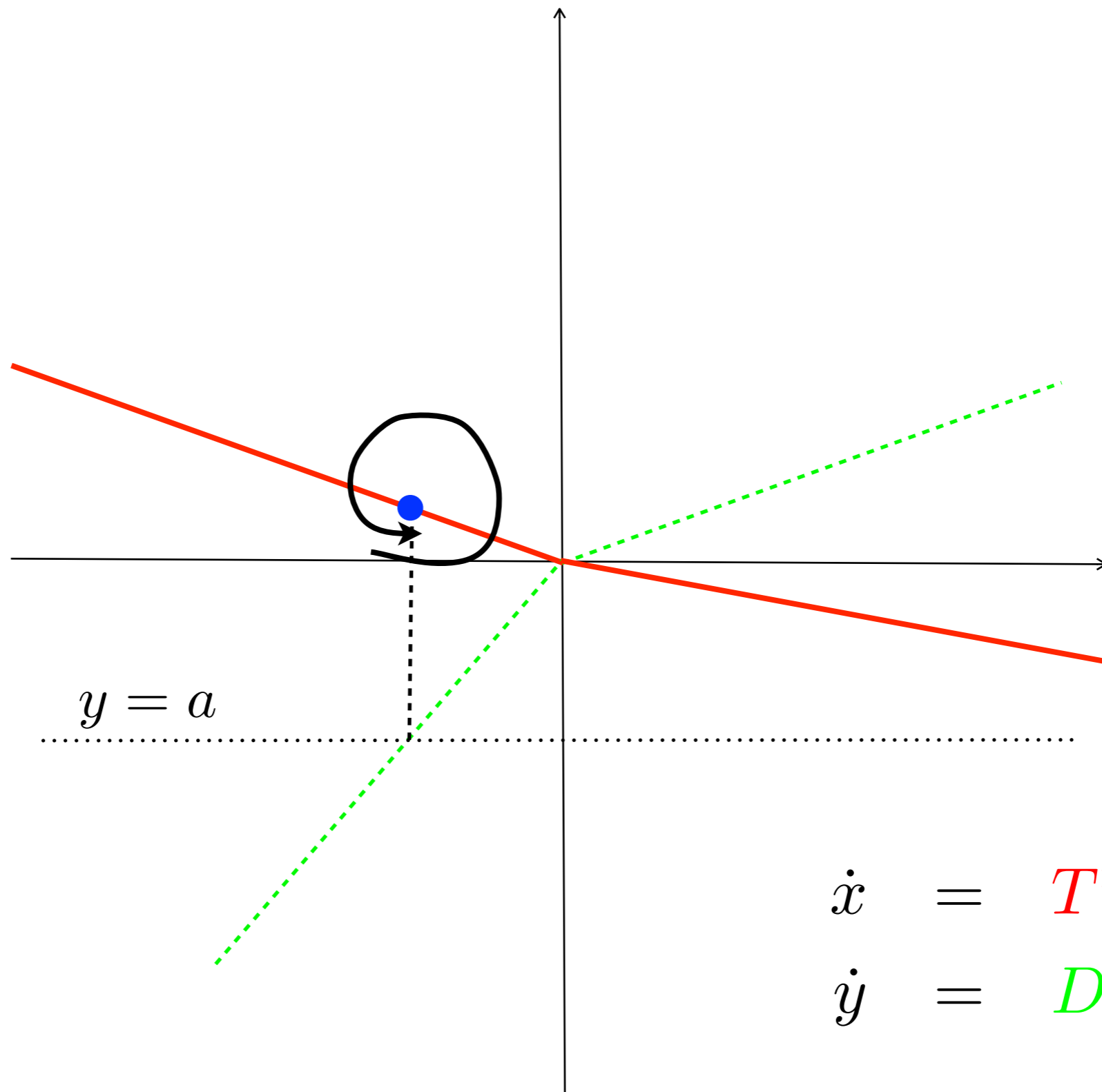
$$\begin{aligned}\dot{x} &= T^{\pm}x - y \\ \dot{y} &= D^{\pm}x - a\end{aligned}$$

Focus-Center-Limit Cycle Bifurcations



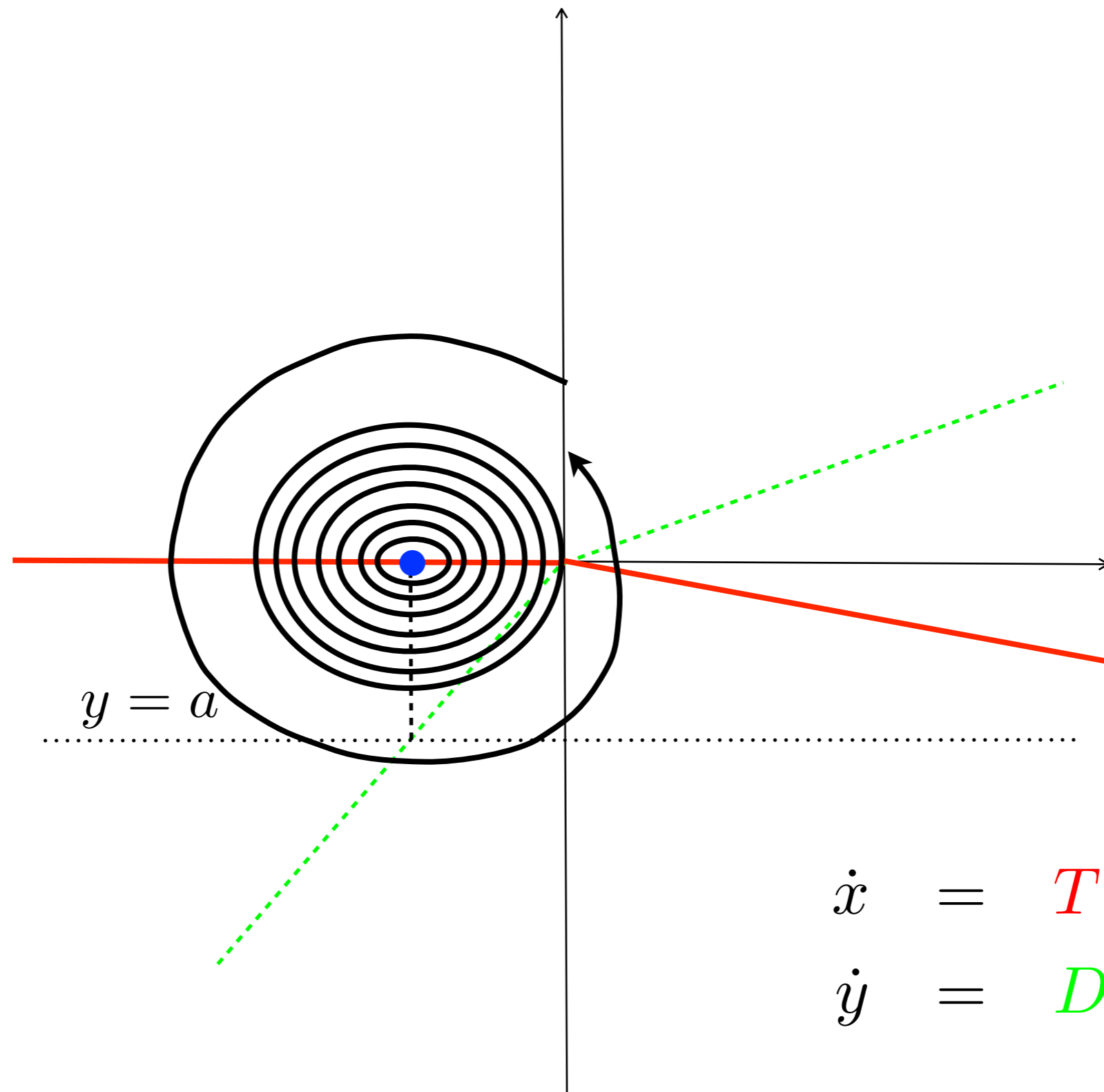
$$\begin{aligned}\dot{x} &= T^{\pm}x - y \\ \dot{y} &= D^{\pm}x - a\end{aligned}$$

Focus-Center-Limit Cycle Bifurcations



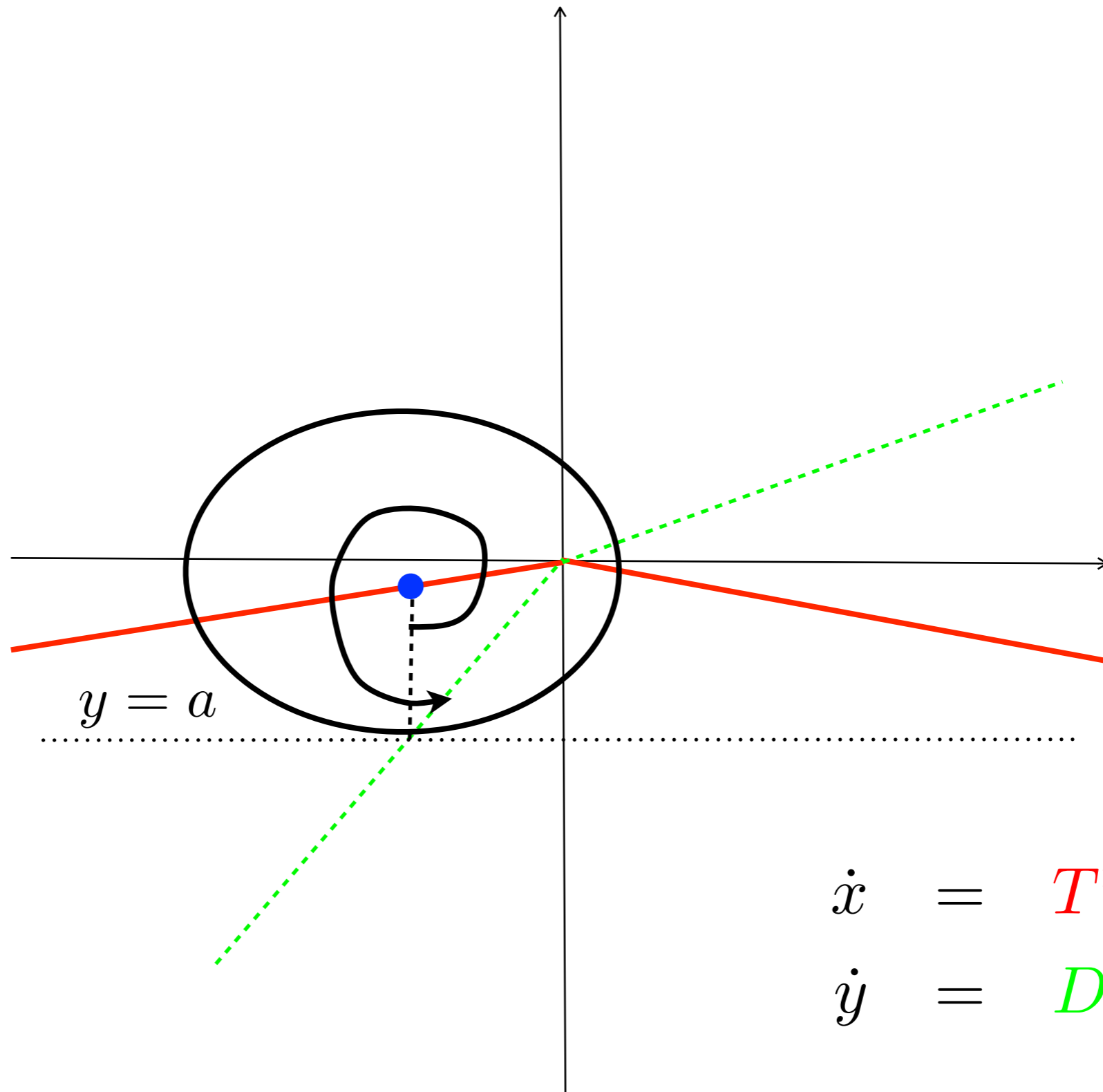
$$\begin{aligned}\dot{x} &= T^\pm x - y \\ \dot{y} &= D^\pm x - a\end{aligned}$$

Focus-Center-Limit Cycle Bifurcations



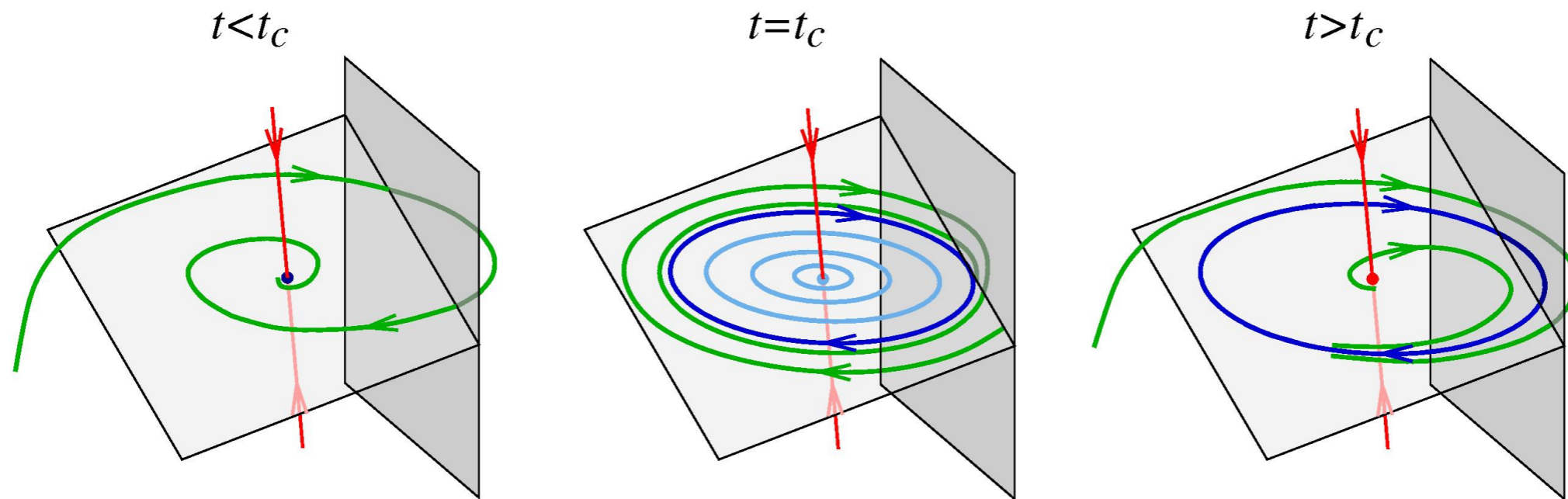
$$\begin{aligned}\dot{x} &= T^{\pm}x - y \\ \dot{y} &= D^{\pm}x - a\end{aligned}$$

Focus-Center-Limit Cycle Bifurcations



$$\begin{aligned}\dot{x} &= T^{\pm}x - y \\ \dot{y} &= D^{\pm}x - a\end{aligned}$$

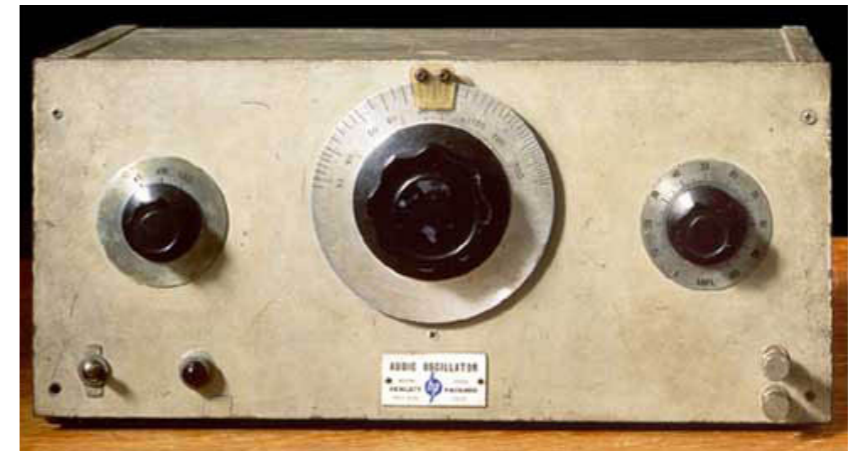
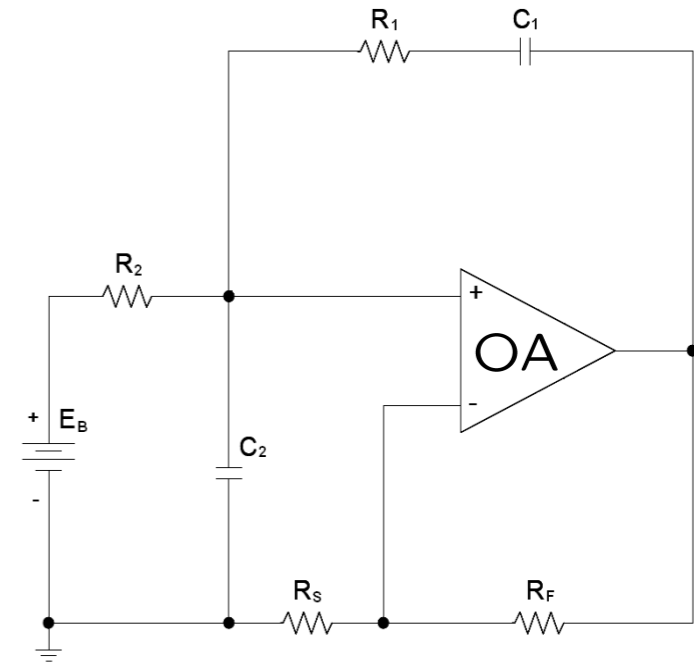
Focus-Center-Limit Cycle Bifurcations in 3D



- The trace t is chosen as the bifurcation parameter, and m, d are assumed to be constant. The center condition holds for the critical value $t_c = d/m$
- A complex eigenvalue pair crosses the imaginary axis like in the Hopf bifurcation of differentiable systems

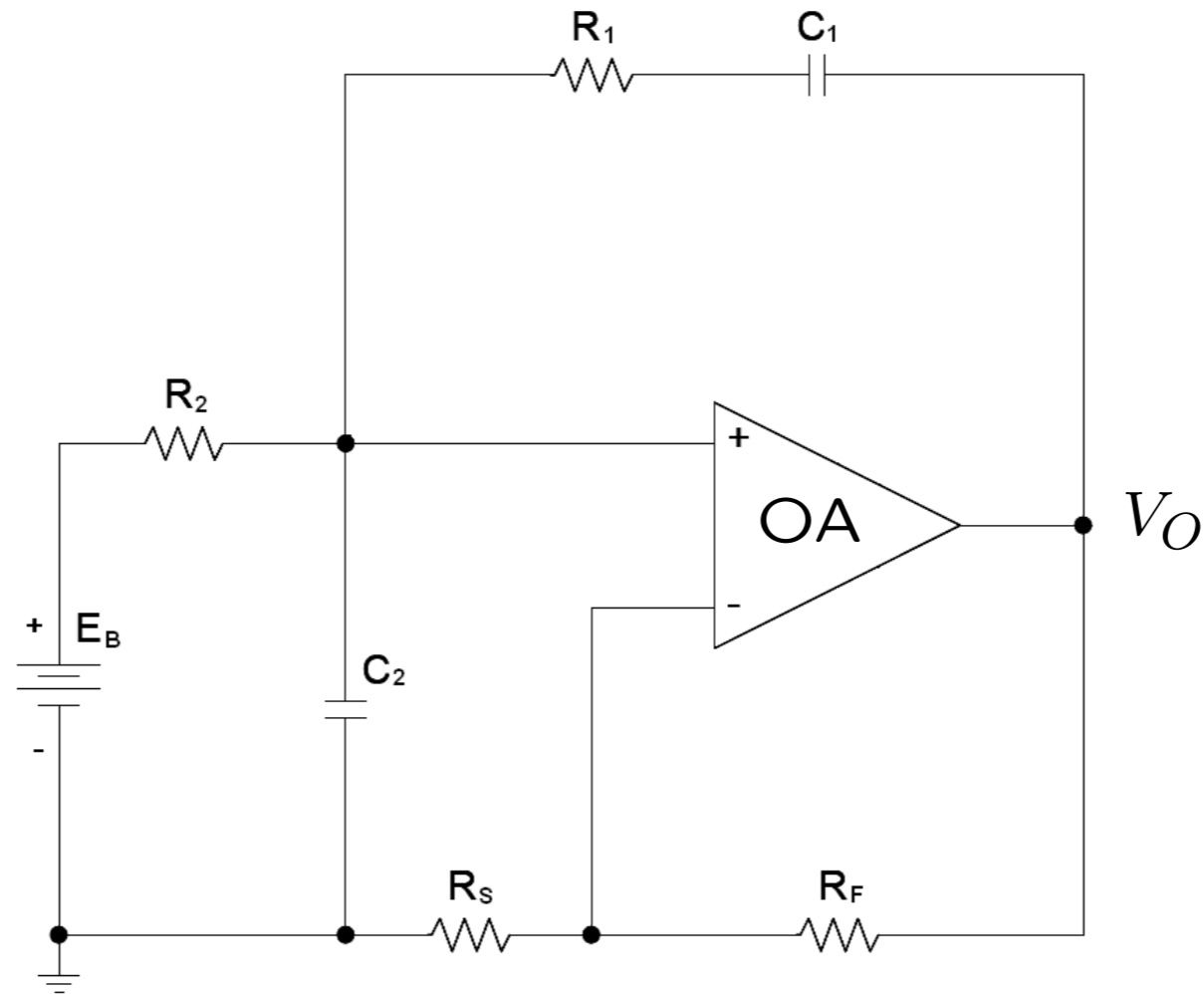
F-C-Limit Cycle Bifurcations in Symmetric 2D systems

- Max Wien (1891)
- Implemented as A200 oscillator by William Hewlett (1939) (the first commercial HP assembly product).
- The battery E_B is introduced to break the symmetry.



Kriegsmann, G.A. [1987] "The rapid bifurcation of the Wien bridge oscillator", *IEEE Trans. Circuits Syst.* 34, 1093–1096.

F-C-Limit Cycle Bifurcations in Symmetric 2D systems



Kirchhoff laws give

$$R_1 C_1 \dot{V}_{C_1} = -V_{C_1} - V_{C_2} + V_O, \quad C_1 \dot{V}_{C_1} - C_2 \dot{V}_{C_2} = \frac{V_{C_2} - E_B}{R_2} \quad (1)$$

where V_{C_1} y V_{C_2} are voltages across capacitors C_1 y C_2 , and $V_O = f(V_{C_2})$ is the output voltage of Op amp.

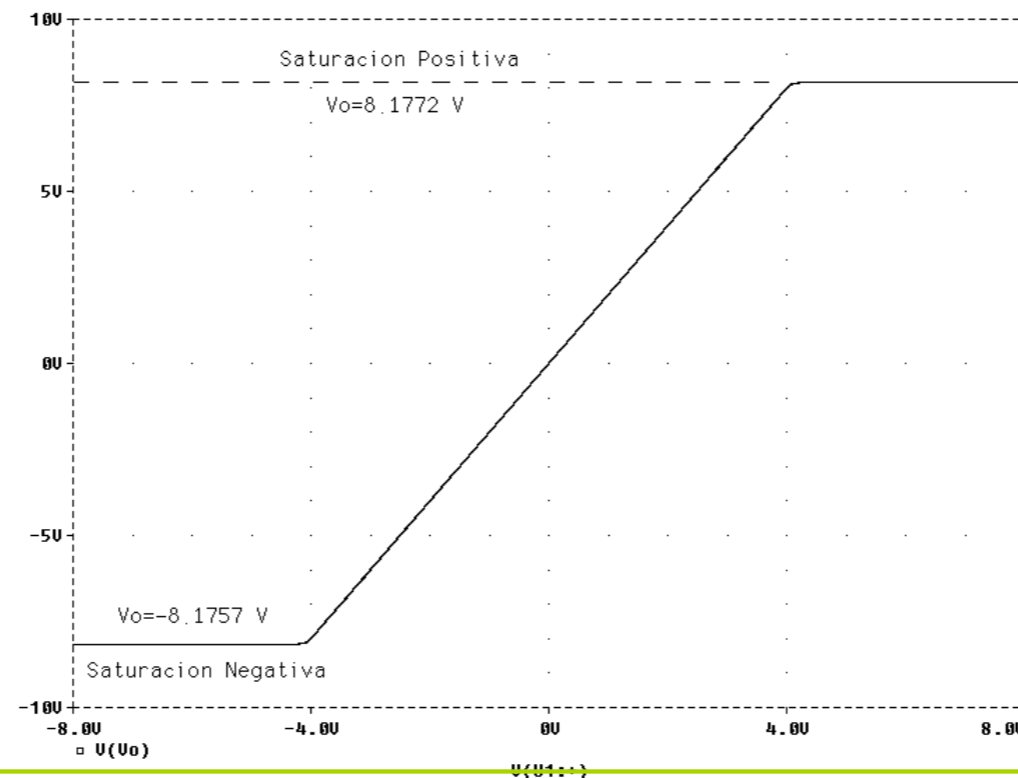
F-C-Limit Cycle Bifurcations in Symmetric 2D systems

Modelling the op amp characteristics, two options are normally chosen:

- A smooth function like $V_O = f(V_{C2}) = \frac{2E}{\pi} \arctan\left(\frac{\pi\alpha}{2E} V_{C2}\right)$,
- The piecewise linear function

$$V_O = f(V_{C2}) = \begin{cases} E \operatorname{sign}(\alpha V_{C2} - E), & \text{si } |\alpha V_{C2}| > E, \\ \alpha V_{C2}, & \text{si } |\alpha V_{C2}| \leq E, \end{cases}$$

where $\alpha = 1 + \frac{R_f}{R_s}$ is the resulting OA gain and E is the saturation voltage.



Output voltage V_O versus V_{C2} for the OA LF411 commercial model

F-C-Limit Cycle Bifurcations in Symmetric 2D systems

- As it will be seen, the piecewise linear approach is much better. Under generic assumptions and after some linear algebra, above equations can be written in Liénard form,

$$\dot{\mathbf{x}} = \begin{bmatrix} t & -1 \\ d & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} T - t \\ D - d \end{bmatrix} \text{sat}(x) + \begin{bmatrix} 0 \\ -D\bar{x} \end{bmatrix},$$

where \bar{x} stands for the equilibrium abscissa at the central region,

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^2,$$

t, T are the traces, d, D , are the determinants in the linear zones.

$\begin{aligned} \dot{x} &= tx - y - (T - t), \\ \dot{y} &= dx - D(1 + \bar{x}) + d \\ x &< -1 \end{aligned}$	$\begin{aligned} \dot{x} &= Tx - y, \\ \dot{y} &= Dx - D\bar{x} \\ -1 &\leq x \leq 1 \end{aligned}$	$\begin{aligned} \dot{x} &= tx - y + T - t, \\ \dot{y} &= dx + D(1 - \bar{x}) - d \\ x &> 1 \end{aligned}$
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F-C-Limit Cycle Bifurcations in Symmetric 2D systems

Effectively, it suffices to change variables in (1) as follows

$$x = \frac{V_{C_2}}{E}, \quad y = \frac{1}{R_1 C_2} \frac{V_{C_1}}{E} - \frac{1}{R_1 C_1} \frac{V_{C_2}}{E} - \frac{\alpha}{R_2 C_2} \frac{E_B}{E},$$

and write

$$\frac{1}{E} f\left(\frac{E}{\alpha} x\right) = \text{sat}(x) := \begin{cases} \text{sgn}(x) & \text{if } |x| > 1, \\ x & \text{if } |x| \leq 1. \end{cases}$$

Thus we arrive at the piecewise linear differential system with three linear regions

$$\dot{\mathbf{x}} = \begin{pmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_1 C_2} + \frac{1}{R_2 C_2}\right) & -1 \\ \frac{1}{R_1 C_1 R_2 C_2} & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{\alpha}{R_1 C_2} \\ 0 \end{pmatrix} \text{sat}(x) + \begin{pmatrix} 0 \\ -\frac{\alpha}{R_1 C_1 R_2 C_2} \frac{E_B}{E} \end{pmatrix}. \quad (2)$$

- Here determinants are positive and equal but traces are different, namely

$$D = d = \frac{1}{R_1 C_1 R_2 C_2}, \quad t = -\left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} + \frac{1}{R_1 C_1}\right) < 0,$$

$$T = \frac{\alpha - 1}{R_1 C_2} - \frac{1}{R_1 C_1} - \frac{1}{R_2 C_2} = \frac{1}{R_1 C_2} \left(\alpha - \underbrace{\left(1 + \frac{R_1}{R_2} + \frac{C_2}{C_1}\right)}_{\alpha_c} \right).$$

- There is only one equilibrium point at $\bar{x} = E_B \alpha / E$. For $E_B = 0$ the equilibrium is at the origin and the system is symmetric.

F-C-Limit Cycle Bifurcations in Symmetric 2D systems

Theorem Every observable $S3CPL_2$ system with $D > 0$, $T^2 < 4D$ and $t \neq 0$, undergoes for $T = 0$ a F-C-LC bifurcation.

The bifurcating limit cycle exists for $Tt < 0$ and T suf. small, and it is stable provided that $t < 0$.

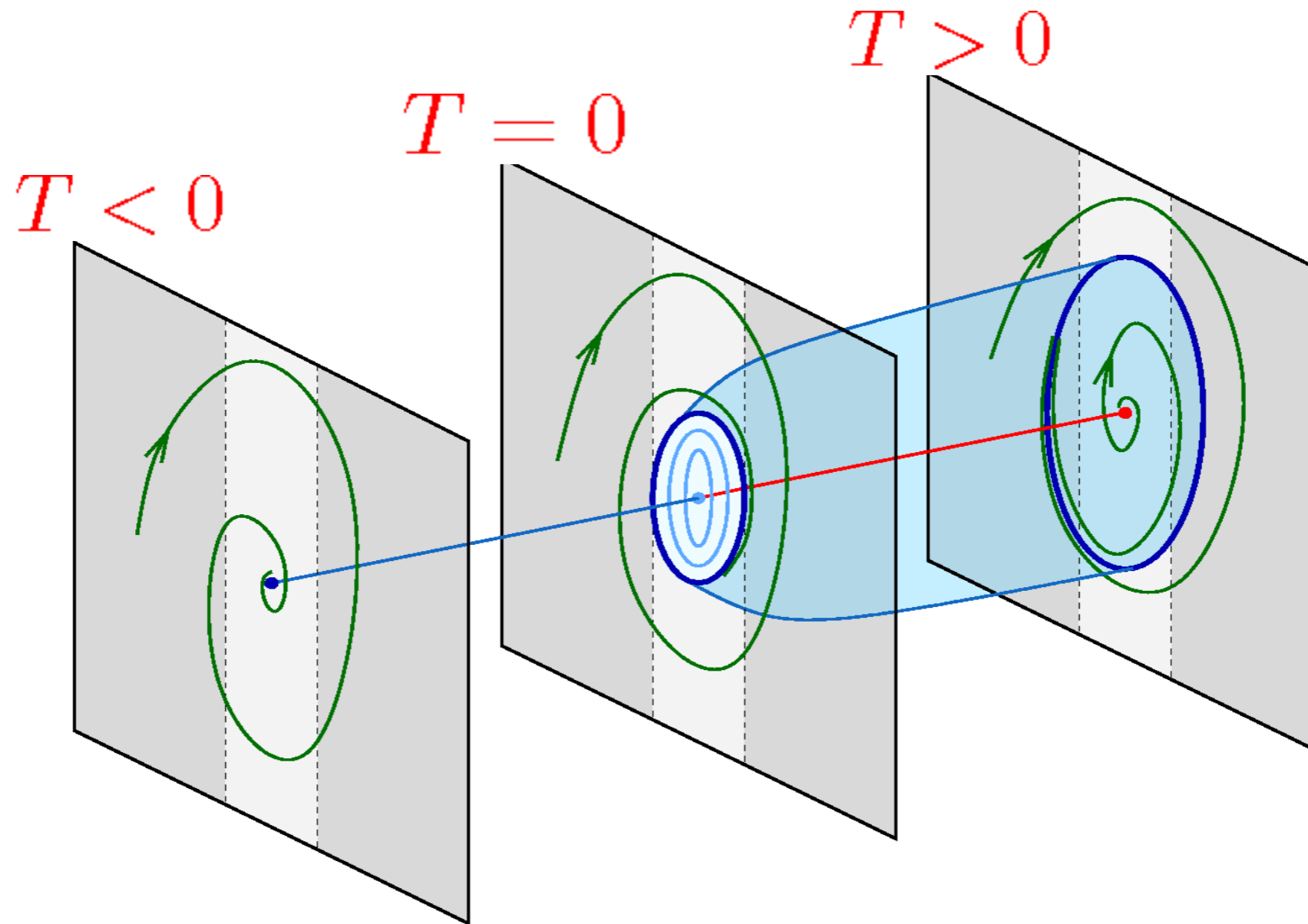
The amplitude a , period P and logarithm of the characteristic multiplier ρ are analytic functions at 0 in the variable $T^{1/3}$, namely

$$\begin{aligned} a &= 1 + \frac{(6\pi)^{\frac{2}{3}}}{8t^{\frac{2}{3}}} T^{\frac{2}{3}} + \frac{(6\pi^4)^{\frac{1}{3}}(120D - 2t^2 - 21d)}{960Dt^{\frac{4}{3}}} T^{\frac{4}{3}} + O(T^{\frac{5}{3}}), \\ P &= \frac{2\pi}{\sqrt{D}} + \frac{\pi(d - D)}{D^{\frac{3}{2}}t} T - \frac{(6^2\pi^5)^{\frac{1}{3}}((d - D)^2 + t^2D)}{10D^{\frac{5}{2}}t^{\frac{5}{3}}} T^{\frac{5}{3}} + O(T^2), \\ \rho &= -2(6\pi)^{\frac{1}{3}} t^{\frac{2}{3}} T^{\frac{1}{3}} + \frac{\pi}{15} (12d + 15 - t^2) T + O(T^{\frac{4}{3}}). \end{aligned}$$

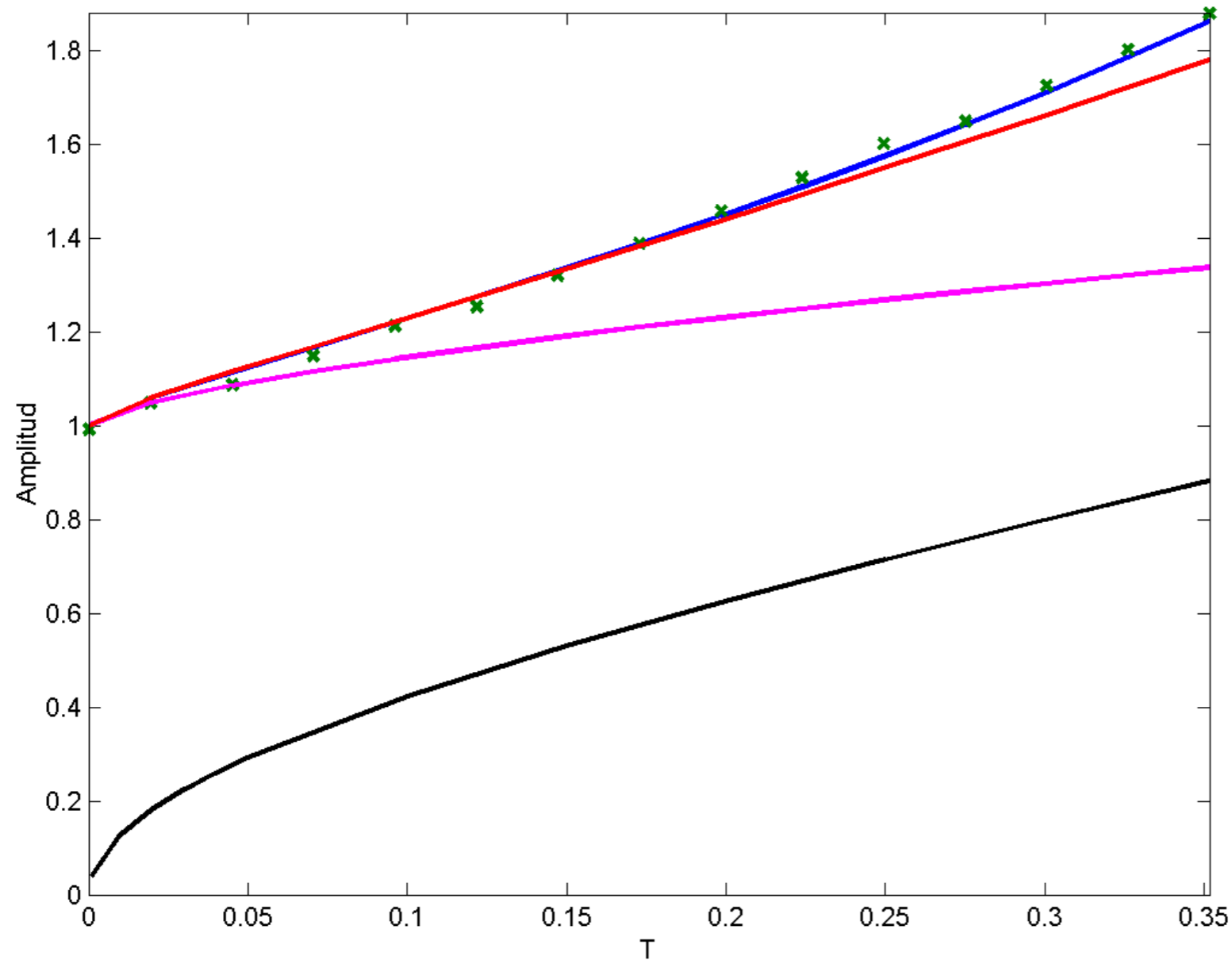
Freire E.; Ponce E.; Ros J. "Limit Cycle Bifurcation from Center in Symmetric Piecewise-Linear Systems". International Journal of Bifurcation and Chaos, Vol. 9, No. 5, 895-907 (1999).

- There are discrepancies with the classical Hopf bifurcation

F-C-Limit Cycle Bifurcations in Symmetric 2D systems



F-C-Limit Cycle Bifurcations in Symmetric 2D systems



- x experimental measures
- numer. sol. of closing eq.
- numer. sol. smooth model
- two terms of our series
- three terms of our series

Some unexpected results

- Algebraically computable PWL nodal oscillators
- Three limit cycles in planar PWL systems with 2 zones
- The continuous matching of 3D PWL stable systems can be unstable

Algebraically computable PWL nodal oscillators

Algebraically computable PWL nodal oscillators

Consider the family of piecewise linear differential systems

$$\dot{\mathbf{x}} = A\mathbf{x} + \varphi(c^T \mathbf{x})b,$$

where $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$, A is a 2×2 matrix, $b, c \in \mathbb{R}^2$ and the nonlinearity φ is a symmetric piecewise linear continuous function

$$\varphi(\sigma) = \begin{cases} m_a \sigma - (m_b - m_a) \delta, & \sigma \leq -\delta, \\ m_b \sigma, & |\sigma| < \delta, \\ m_a \sigma + (m_b - m_a) \delta, & \sigma \geq \delta, \end{cases}$$

with $m_a \neq m_b$, $\delta > 0$. Assume that there exist $\mu > 0$ and $\eta \in \mathbb{R}$, such that the different linear parts satisfy

$$\begin{aligned} \text{Spec} (A + m_a b c^T) &= \{-\mu, -2\mu\}, \\ \text{Spec} (A + m_b b c^T) &= \{\eta, 2\eta\}, \end{aligned}$$

and that the system is “observable”, that is

$$\det \begin{pmatrix} c^T \\ c^T A \end{pmatrix} \neq 0.$$

Algebraically computable PWL nodal oscillators

Then the system is topologically equivalent to the Liénard system

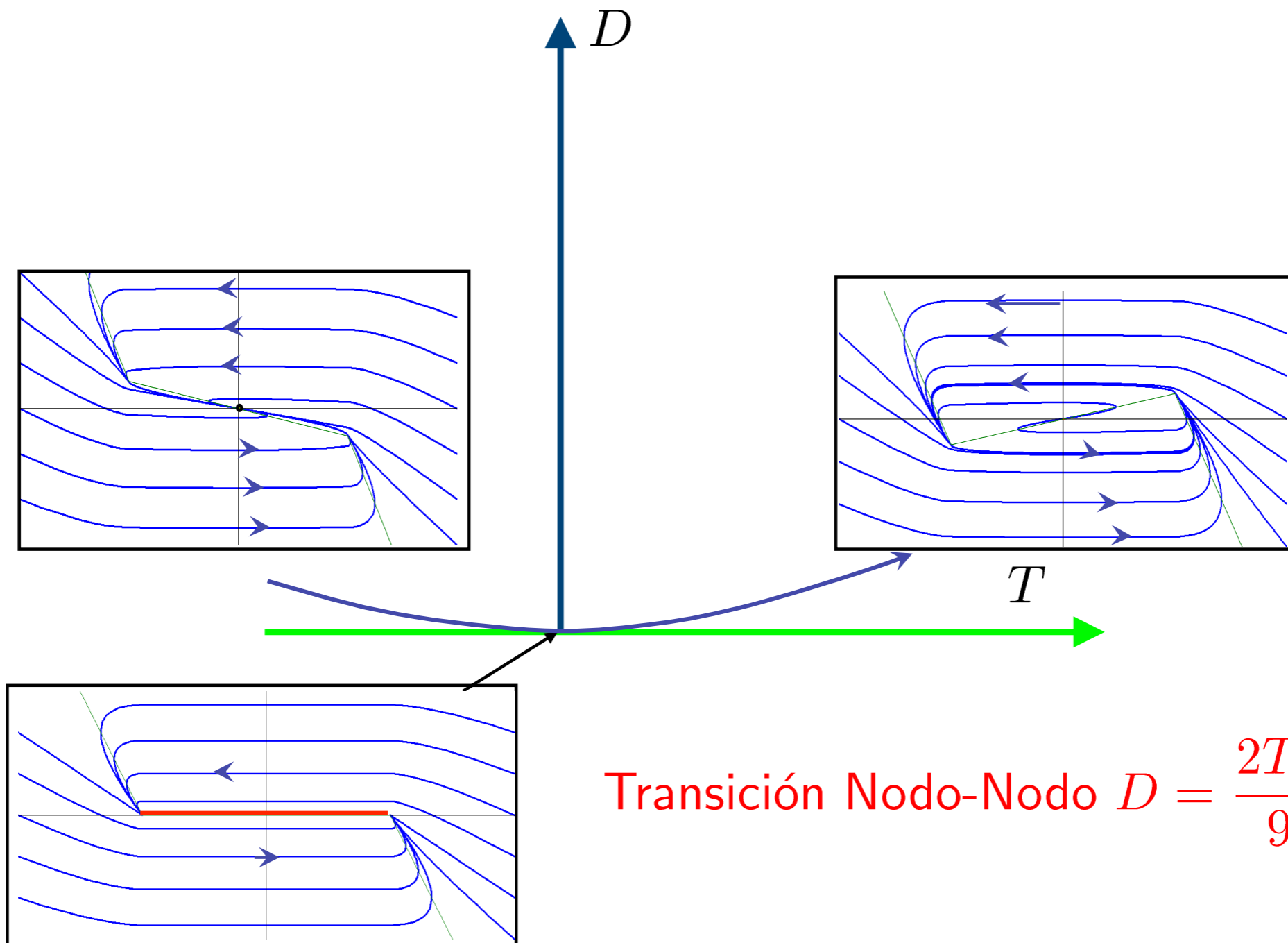
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3(\alpha + 1) \\ 2(\alpha^2 - 1) \end{pmatrix} \text{sat}(x),$$

where $\alpha = \frac{\eta}{\mu}$ and “sat” stands for the normalized saturation function

$$\text{sat}(x) = \begin{cases} -1, & \text{if } x < -1, \\ x, & \text{if } |x| \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$$

Algebraically computable PWL nodal oscillators

$$T = 3\alpha, \quad D = 2\alpha^2, \quad t = -3, \quad d = 2$$



Algebraically computable PWL nodal oscillators

Theorem For the piecewise linear differential system, the following statements hold.

- (a) If $\alpha < 0$ then the origin is the only equilibrium point, in particular it is a stable node, being the global attractor for all the orbits of the system.
- (b) For $\alpha = 0$ there exists a continuum of equilibrium points, namely all the points of the segment $\Sigma = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, y = 0\}$. This segment is the global attractor for the system. It is formed by unstable points, but the endpoints of the segment are the ω -limit set for $\mathbb{R}^2 \setminus \Sigma$.
- (c) For $\alpha > 0$ the only equilibrium point is the origin, which is an unstable node. Furthermore there exists one periodic orbit which is a stable limit cycle, being symmetric with respect to the origin and the ω -limit set for all orbits except the origin.

Algebraically computable PWL nodal oscillators

All the points of this limit cycle can be described in an algebraic way in terms of the parameter α . In particular, the limit cycle intersections $(1, y_0)$ and $(1, Y_0)$ with the line $x = 1$ can be algebraically determined as follows. For each $\alpha > 0$ there exists only a value $v \in (\sqrt{2} - 1, 1)$ such that

$$\alpha = \alpha(v) = \frac{(1 + 2v - v^2)(v^2 + 2v - 1)}{(1 - v)^2(1 + 4v + v^2)}, \quad (1)$$

and

$$y_0 = -\frac{\alpha(1 - v + 2v^2)}{v(1 - v)}, \quad Y_0 = \frac{\alpha(2 - v + v^2)}{1 - v}. \quad (2)$$

Furthermore, the period of the limit cycle is

$$P = -2 \log \left(\frac{v^2 + 2v - 1}{1 + 2v - v^2} v^{\frac{\alpha+1}{\alpha}} \right), \quad (3)$$

and its characteristic multiplier ν satisfies

$$\nu = \left(\frac{v^2 + 2v - 1}{1 + 2v - v^2} \right)^6 < 1,$$

Algebraically computable PWL nodal oscillators

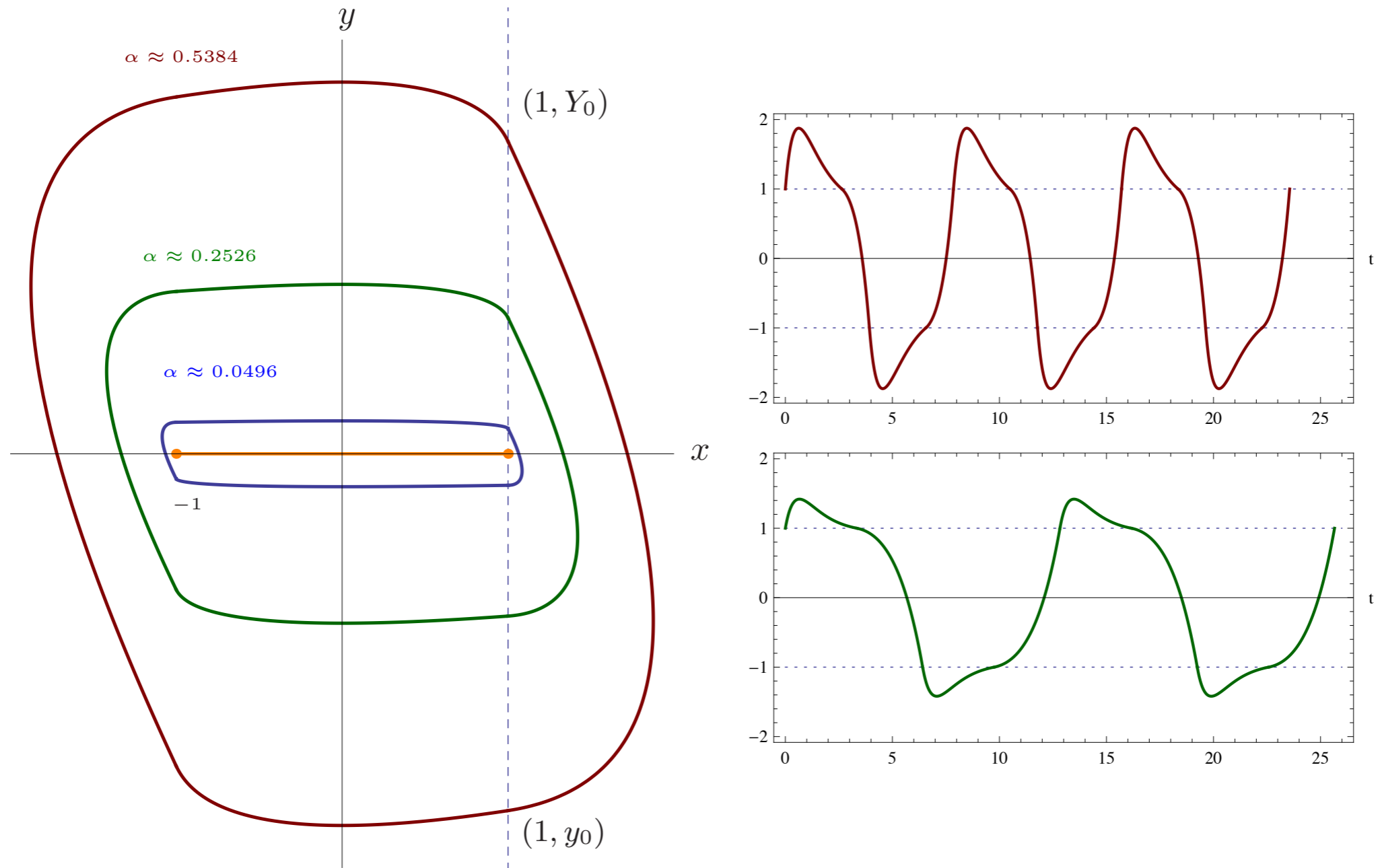


Figure 1: The segment of equilibrium points for $\alpha = 0$ and the limit cycle for several values of $\alpha > 0$ (left). The waveforms for $x(t)$ when $\alpha \approx 0.2526$ ($v = 0.46$, two cycles), and $\alpha \approx 0.5384$ ($v = 0.5$, three cycles) (right).

Three limit cycles in planar PWL systems with 2 zones

The Huan-Yang example

The planar non-smooth piecewise linear differential system with two zones separated by a straight line corresponding to Example 5.1 of Huan and Yang is

$$\dot{\mathbf{x}} = \begin{cases} A^- \mathbf{x} & \text{if } x < 1, \\ A^+ \mathbf{x} & \text{if } x \geq 1, \end{cases}$$

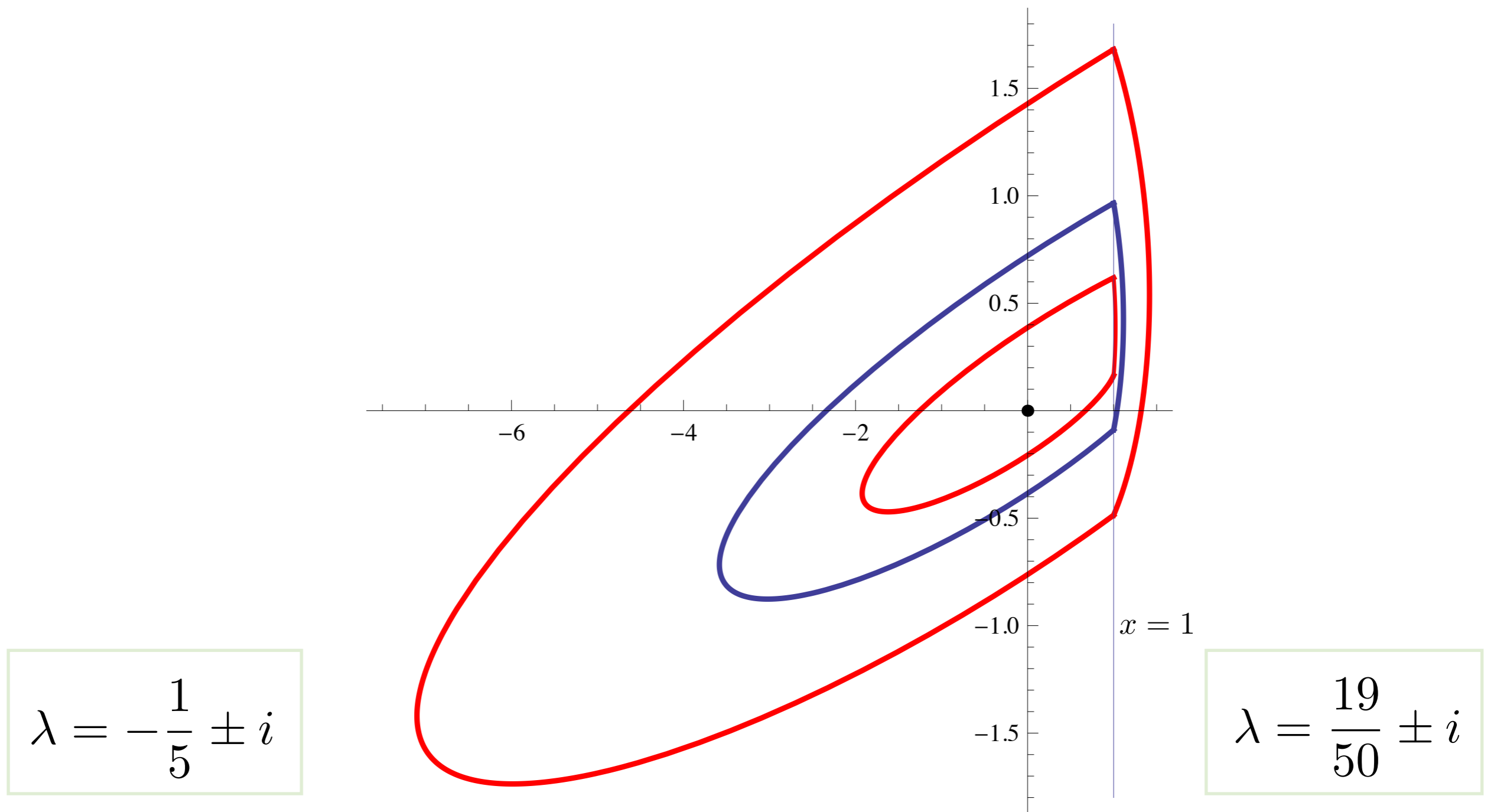
where $\mathbf{x} = (x, y)^T$ with

$$A^- = \begin{pmatrix} 1 & -5 \\ \frac{377}{1000} & -\frac{13}{10} \end{pmatrix}, \quad \text{and} \quad A^+ = \begin{pmatrix} \frac{19}{500} & -\frac{1}{10} \\ \frac{1}{10} & \frac{19}{500} \end{pmatrix}.$$

Theorem (J. Llibre & E.P.) The above planar non-smooth piecewise linear differential system with two zones has 3 limit cycles surrounding its unique equilibrium point located at the origin.

S.-M. HUAN AND X.-S. YANG, *On the number of limit cycles in general planar piecewise linear systems*, *Discrete and Continuous Dynamical Systems-A* **32** (2012) pp. 2147–2164.

J. LLIBRE AND E. P., *Three Nested Limit Cycles In Discontinuous Piecewise Linear Differential Systems With Two Zones*, *Dynamics of Continuous, Discrete and Impulsive Systems-B* **19** (2012) pp. 325–335.



$$A^- = \begin{pmatrix} \frac{4}{3} & -\frac{20}{3} \\ \frac{377}{750} & -\frac{26}{15} \end{pmatrix}, \quad \text{and} \quad A^+ = \begin{pmatrix} \frac{19}{50} & -1 \\ 1 & \frac{19}{50} \end{pmatrix}.$$

(after rescaling time, differently in each side)

The discontinuous canonical form in the focus-focus case

Assume $T^\pm = 2\alpha^\pm$, $D^\pm = (\alpha^\pm)^2 + (\omega^\pm)^2$ with $\omega^\pm > 0$ in the canonical form, so that the corresponding eigenvalues are $\lambda^\pm = \alpha^\pm \pm i\omega^\pm$, and introduce the parameters

$$\gamma_R = \frac{\alpha^+}{\omega^+}, \quad \gamma_L = \frac{\alpha^-}{\omega^-}, \quad a_R = \frac{a^+}{\omega^+}, \quad a_L = \frac{a^-}{\omega^-}.$$

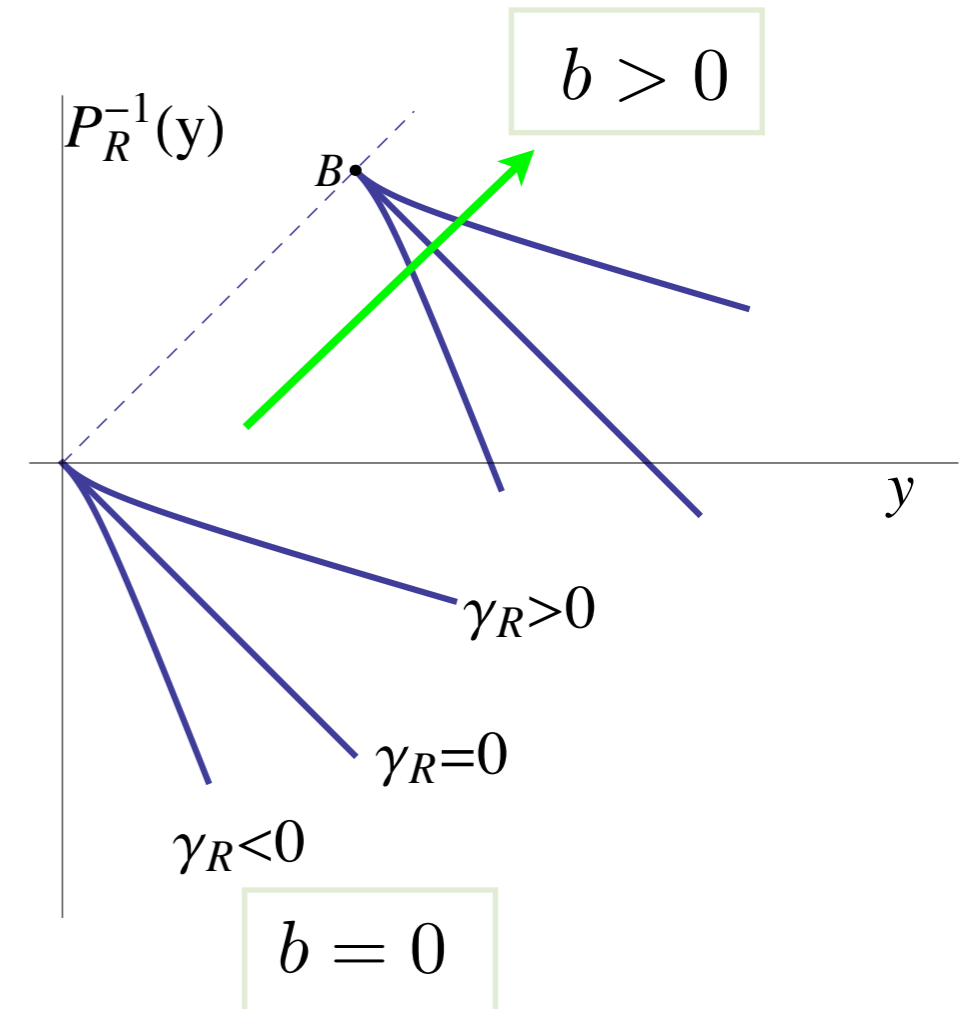
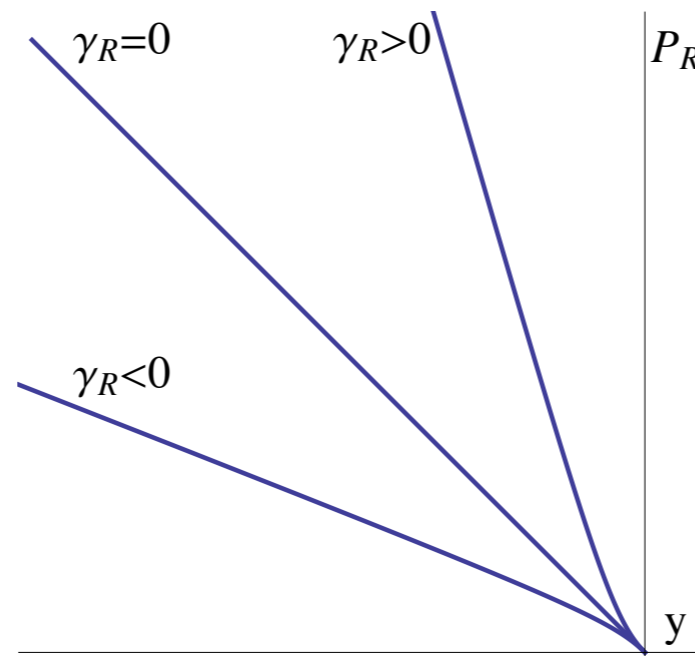
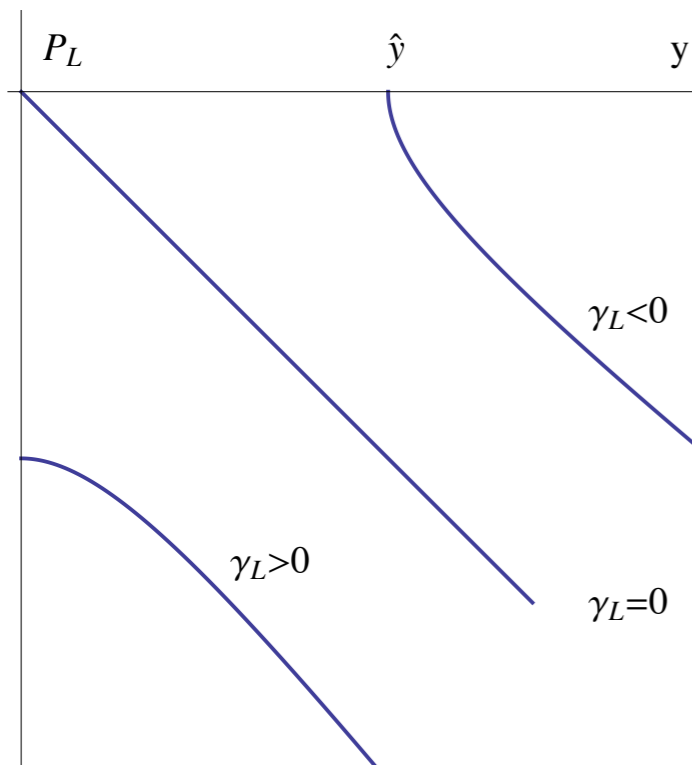
Then the previous canonical form can be written in the form

$$\dot{\mathbf{x}} = \begin{pmatrix} 2\gamma_L & -1 \\ 1 + \gamma_L^2 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ a_L \end{pmatrix} \text{ if } \mathbf{x} \in S^-,$$
$$\dot{\mathbf{x}} = \begin{pmatrix} 2\gamma_R & -1 \\ 1 + \gamma_R^2 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} -b \\ a_R \end{pmatrix} \text{ if } \mathbf{x} \in S^+.$$

It suffices to do a new non-smooth change of variables

$$(x, y, t) \rightarrow \left(\frac{x}{\omega(x)}, y, \frac{t}{\omega(x)} \right), \text{ where } \omega(x) = \begin{cases} \omega^- & \text{if } x < 0, \\ \omega^+ & \text{if } x > 0. \end{cases}$$

The half-return maps and their dependence on parameters

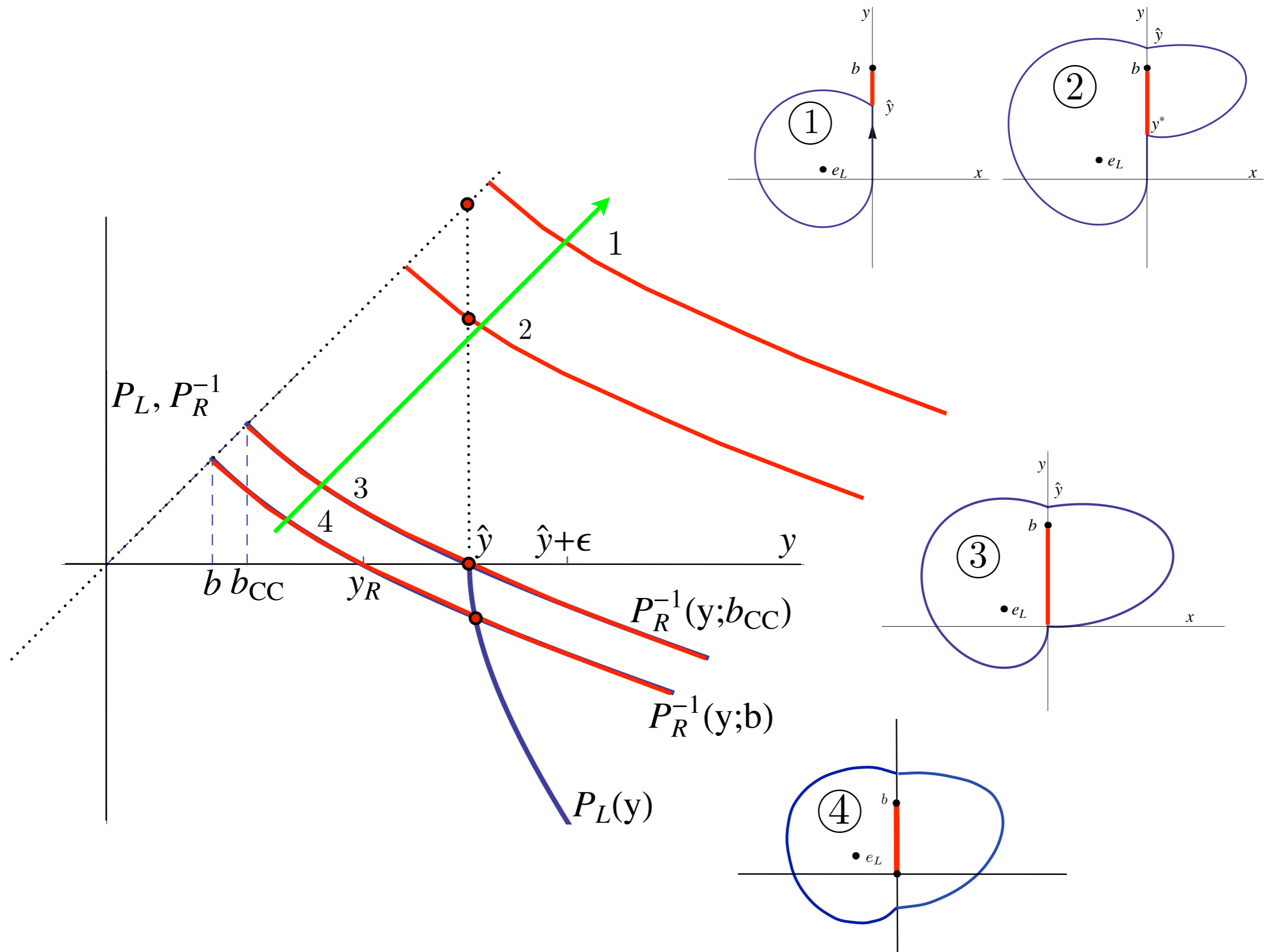


Asymptotes:

$$A_L(y) = -e^{\gamma_L \pi} y + 2x_L \gamma_L (1 + e^{\gamma_L \pi})$$

$$A_{R^{-1}}(y) = -e^{-\gamma_R \pi} y + (b + 2x_R \gamma_R)(1 + e^{-\gamma_R \pi})$$

The b-bifurcation through the crossing critical cycle



The crossing critical cycle curve in the parameter plane (γ_R, b)

We assume $x_L < 0$, $\gamma_L < 0$ and $x_R < 0$ fixed, and look for possible bifurcations leading to one crossing limit cycle by moving parameters b and/or γ_R .

Proposition Assume that $x_L < 0$, $x_R < 0$ and $\gamma_L < 0$. Then there exists one smooth function $b = b_{CC}(\gamma_R)$ with $0 < b_{CC}(\gamma_R) < \hat{y}$ and $b_{CC}(0) = \hat{y}/2$, defined for every value of γ_R such that for $b = b_{CC}(\gamma_R)$ the system has one unstable crossing critical cycle.

In addition there exists $\varepsilon > 0$ such that for $b_{CC}(\gamma_R) - \varepsilon < b < b_{CC}(\gamma_R)$ there exists one unstable crossing periodic orbit which bifurcates from the crossing critical cycle.

Results that can be deduced from stability of the point at infinity

Theorem (stable equilibrium and extremal values of b)

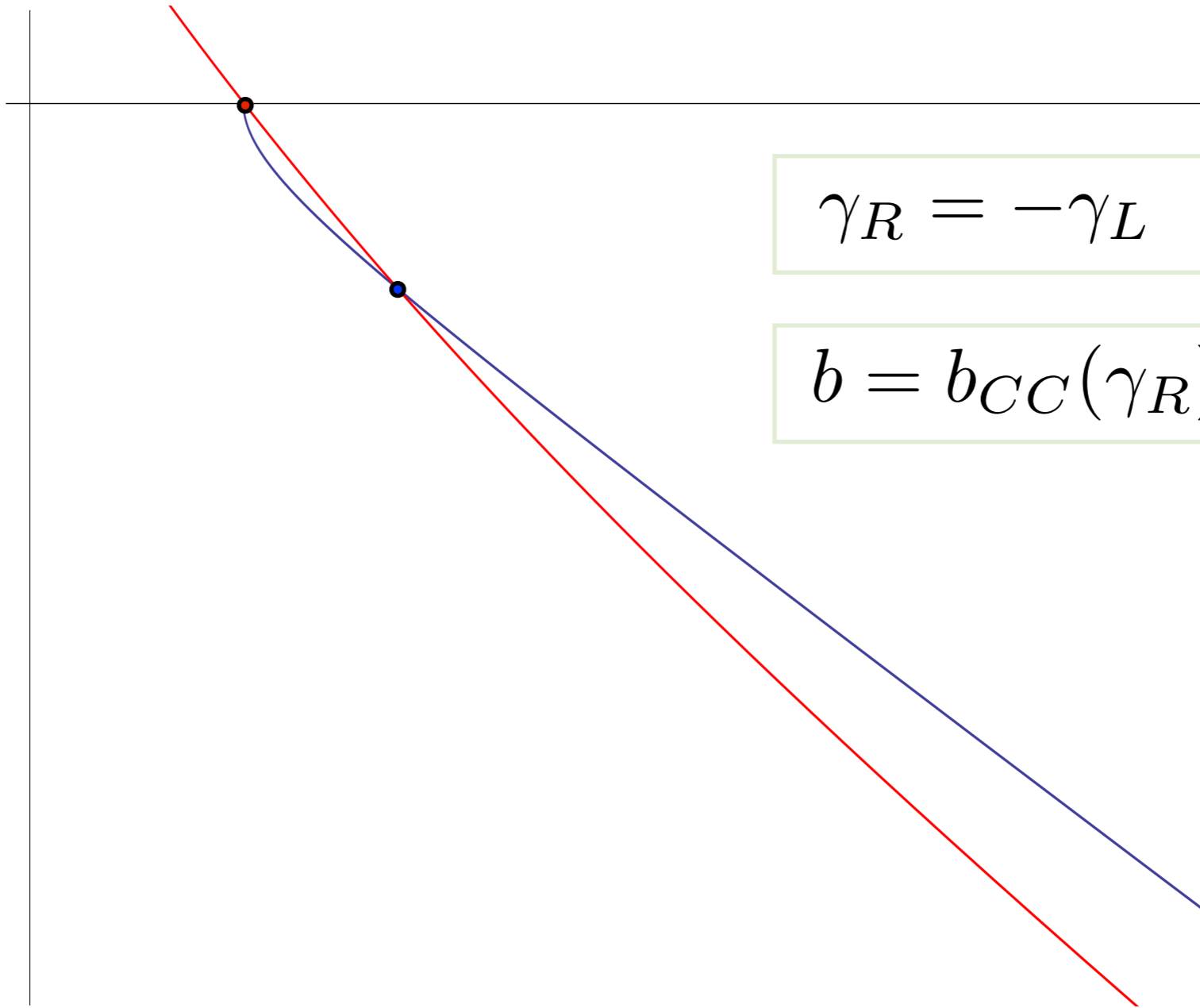
Assuming $x_L < 0$, $x_R < 0$, $\gamma_L < 0$ and $\gamma_R > 0$, and defining $b_\infty = 2(x_L + x_R)\gamma_L$, the following statements hold.

- (a) If $\gamma_L + \gamma_R < 0$ and $b \geq b_{CC}$, then there is at least one stable crossing periodic orbit.
- (b) If $\gamma_L + \gamma_R \leq 0$ and $b < 2x_L\gamma_L$, then there are no crossing periodic orbits.
- (c) If $\gamma_L + \gamma_R > 0$ and $b < b_{CC}$, then there is at least one unstable crossing periodic orbit.
- (d) If $\gamma_L + \gamma_R \geq 0$, then there exists a constant $M > 0$ such that for all $b > M$ there are no crossing periodic orbits.
- (e) If $b < b_\infty$, then there exist $\varepsilon_1 > 0$ such that for $-\gamma_L < \gamma_R < -\gamma_L + \varepsilon_1$, there is at least one unstable crossing periodic orbit and when $b > b_\infty$, then there exist $\varepsilon_2 > 0$ such that for $-\varepsilon_2 - \gamma_L < \gamma_R < -\gamma_L$, there is at least one stable crossing periodic orbit.

Getting our aim by combining local and global results...

Theorem (stable equilibrium, b near b_{CC}) Assuming that $x_L < 0$, $x_R < 0$, $\gamma_L < 0$ and $\gamma_R > 0$, the following statements hold.

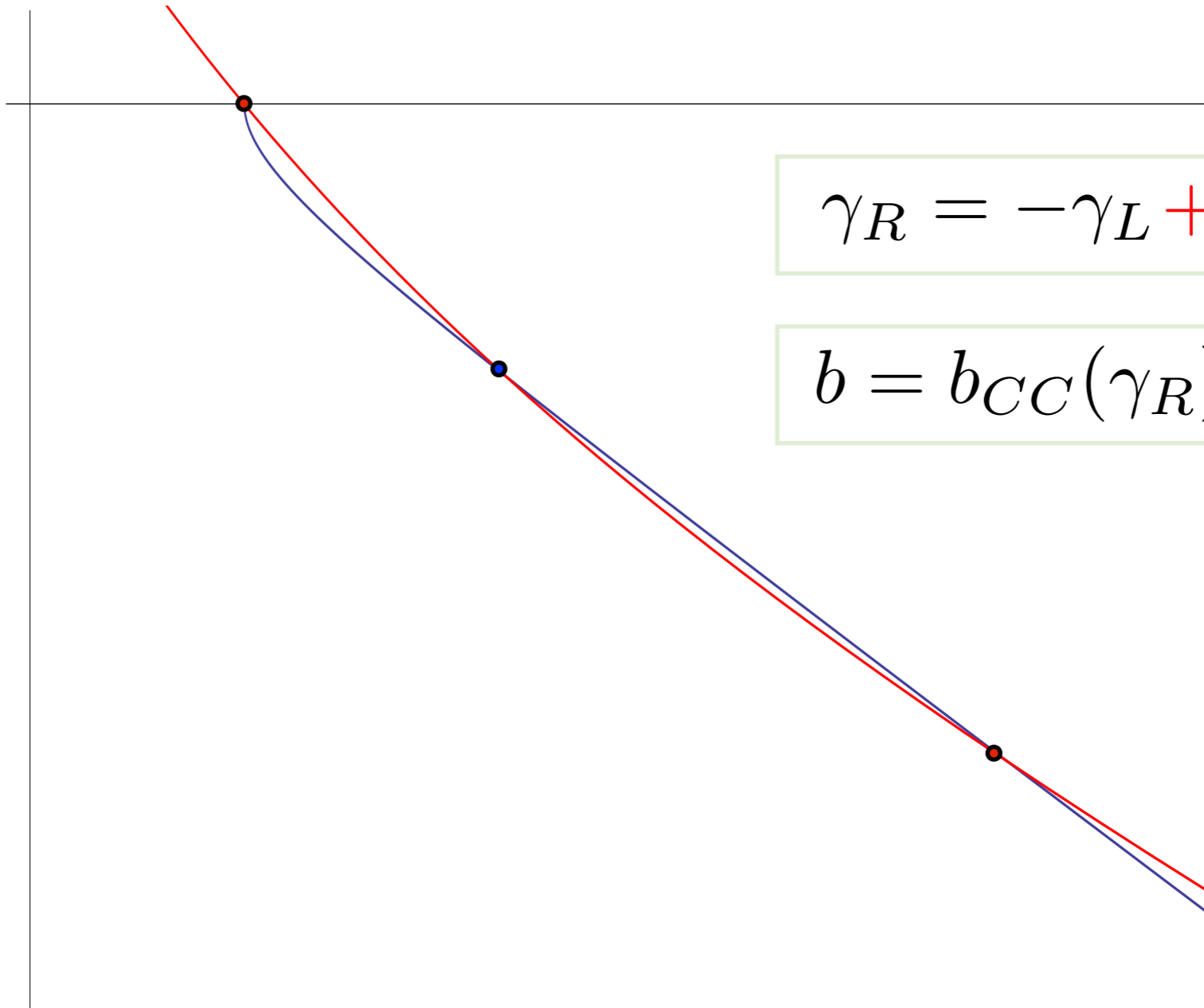
- (a) If $\gamma_L + \gamma_R < 0$ there exists $\varepsilon > 0$ such that for $b_{CC} - \varepsilon < b < b_{CC}$ the system has at least two crossing periodic orbits with opposite stabilities.
- (b) Provided that $\hat{y} < b_\infty$, the following statements also hold.
 - (i) Assume $\gamma_R = -\gamma_L$. Then, there exists $\varepsilon_0 > 0$ such that for $b_{CC} \leq b < b_{CC} + \varepsilon_0$ the system has at least a stable crossing periodic orbit. In addition, there exists $\varepsilon_1 > 0$ such that for $b_{CC} - \varepsilon_1 < b < b_{CC}$ the system has at least two crossing periodic orbits with opposite stabilities.
 - (ii) There exists $\varepsilon_2 > 0$ such that for $-\gamma_L < \gamma_R < -\gamma_L + \varepsilon_2$ and $b = b_{CC}(\gamma_R)$ the system has at least two crossing periodic orbits with opposite stabilities. Furthermore, for $-\gamma_L < \gamma_R < -\gamma_L + \varepsilon_2$ there exists $\varepsilon_3(\gamma_R) > 0$ such that for $b = b_{CC}(\gamma_R) - \varepsilon_3(\gamma_R)$ the system has at least **three nested crossing periodic orbits** being stable the intermediate one and unstable the two other.



$$\gamma_R = -\gamma_L$$

$$b = b_{CC}(\gamma_R)$$

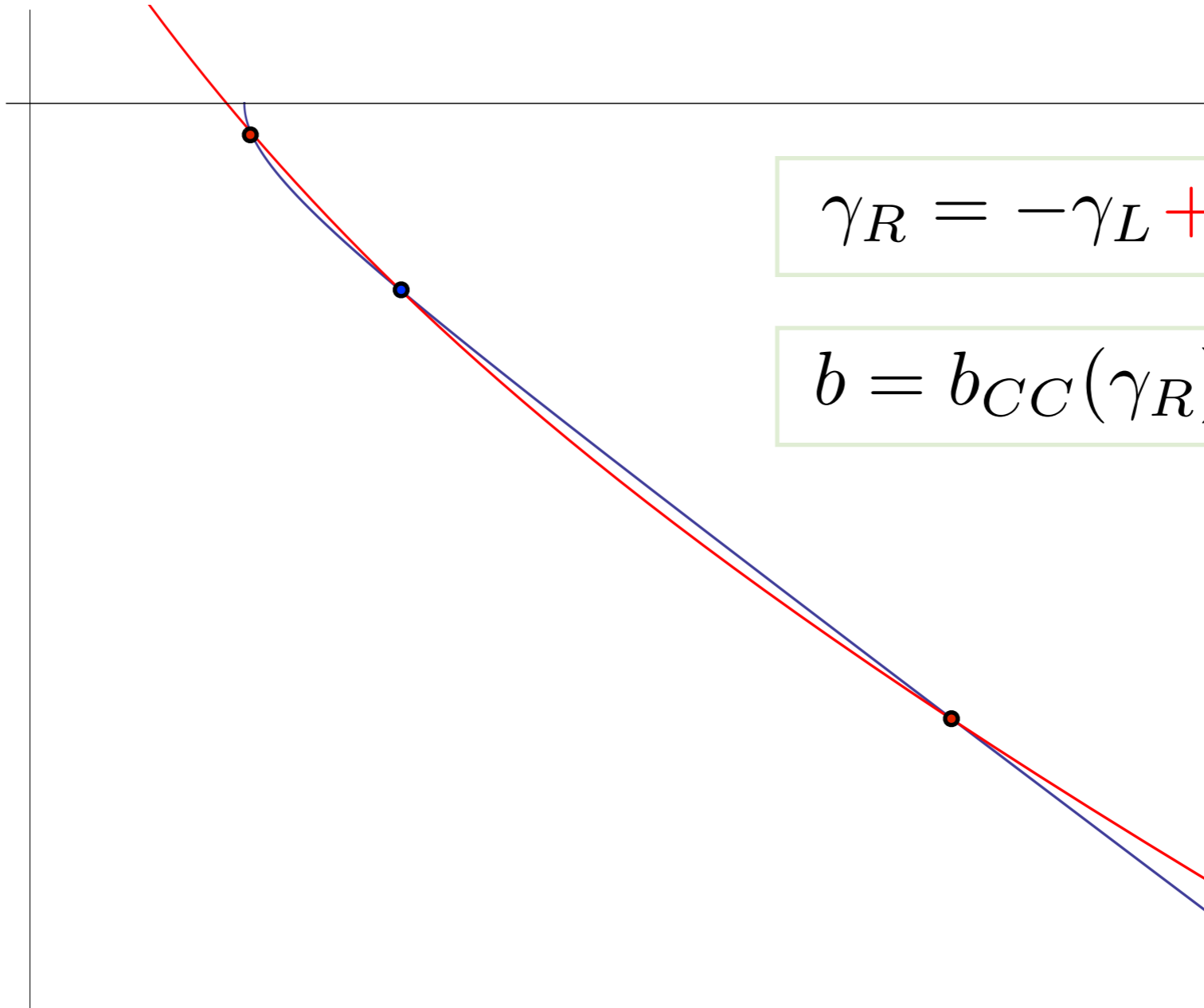
Hunting the three crossing limit cycles



$$\gamma_R = -\gamma_L + \varepsilon$$

$$b = b_{CC}(\gamma_R)$$

Hunting the three crossing limit cycles

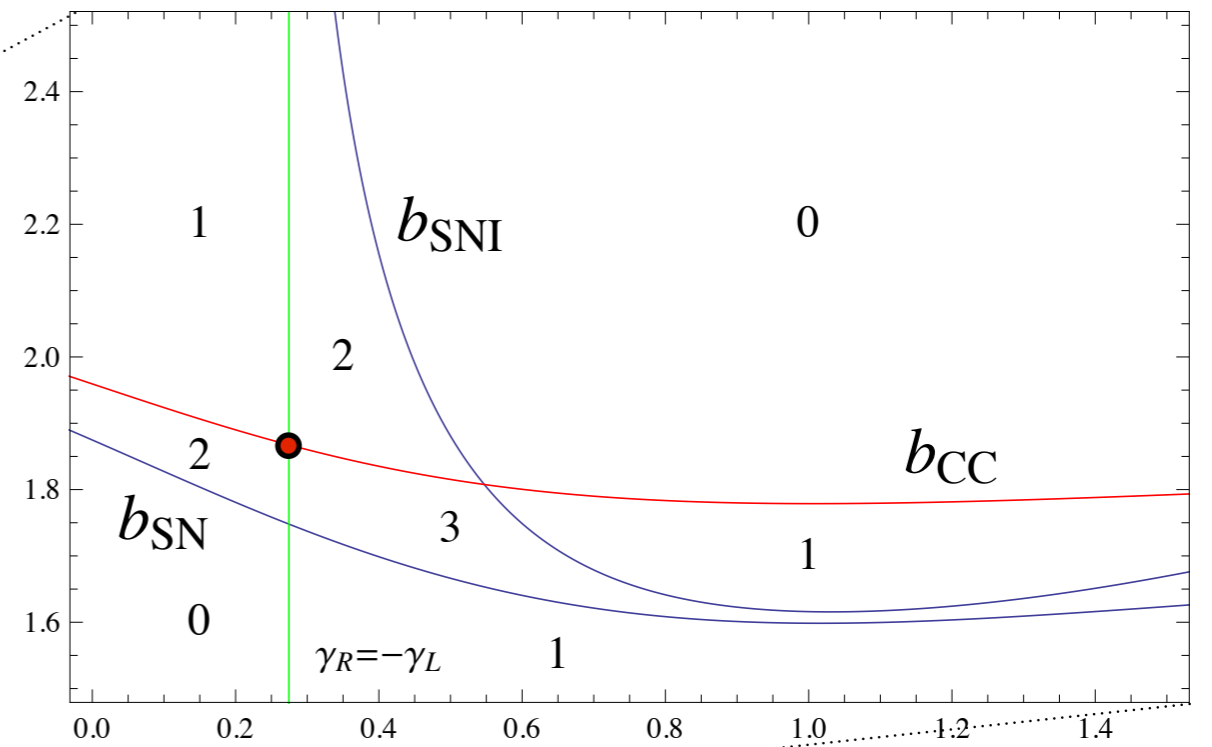
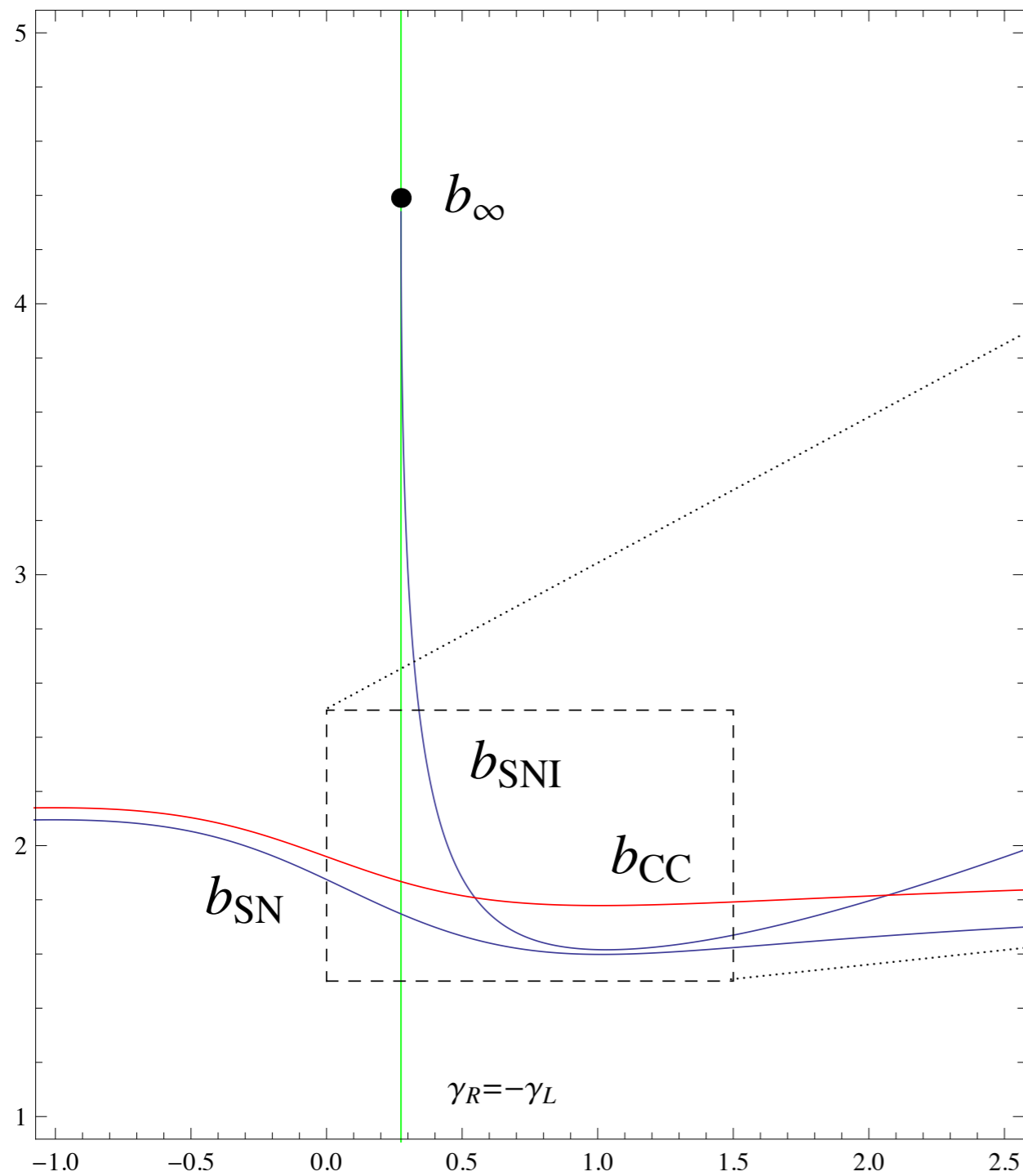


$$\gamma_R = -\gamma_L + \varepsilon$$

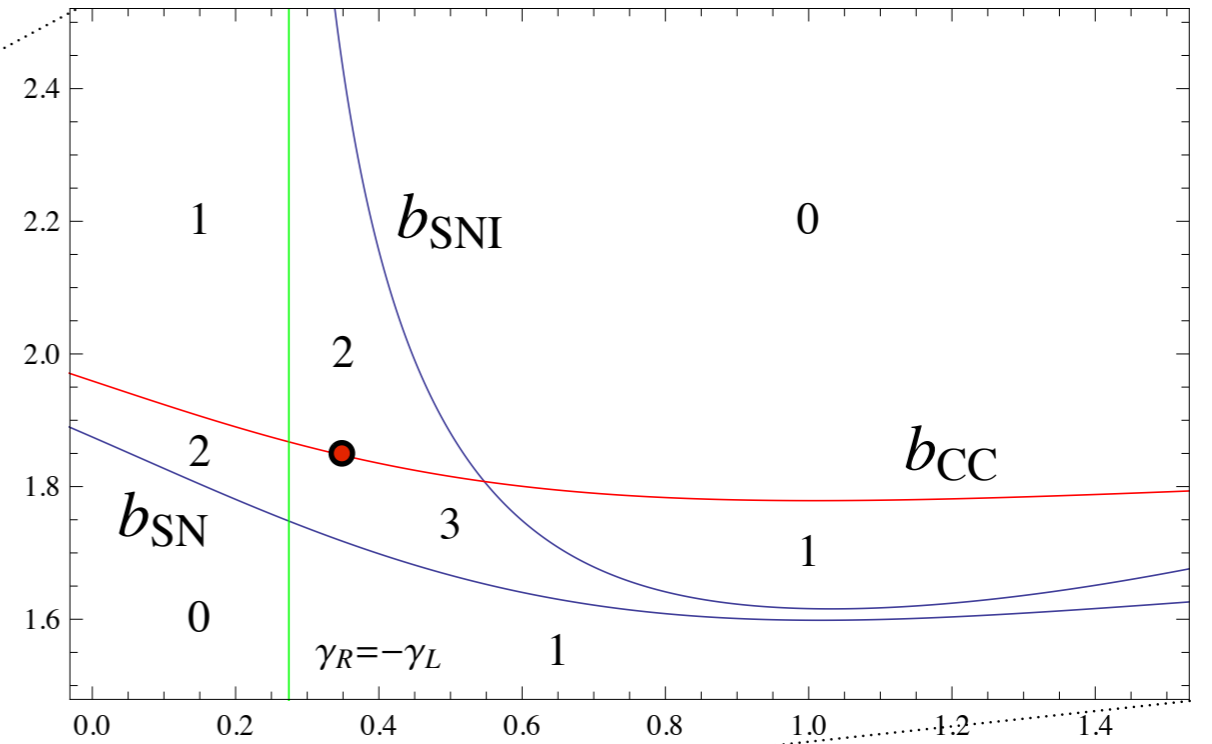
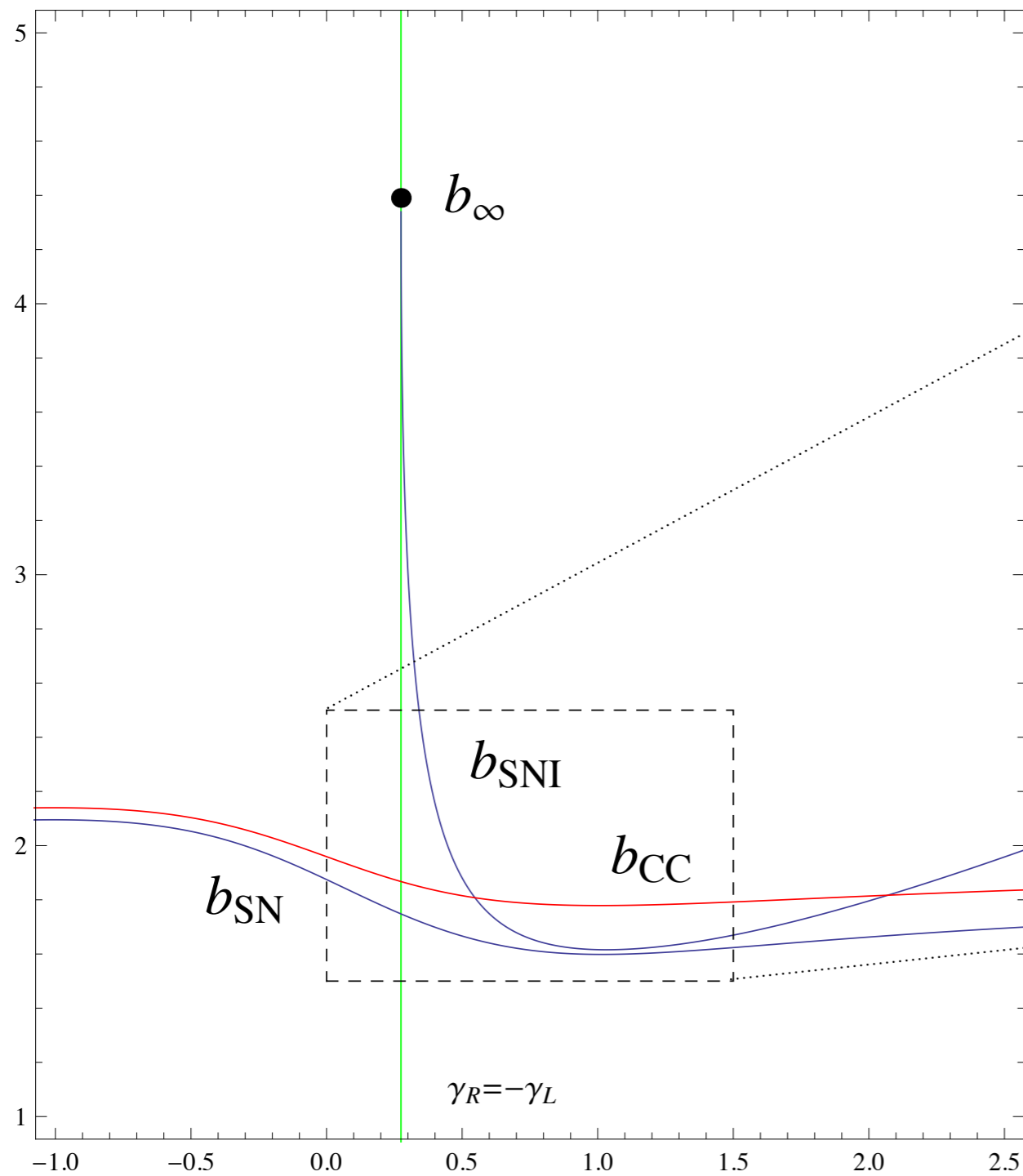
$$b = b_{CC}(\gamma_R) - \bar{\varepsilon}$$

Hunting the three crossing limit cycles

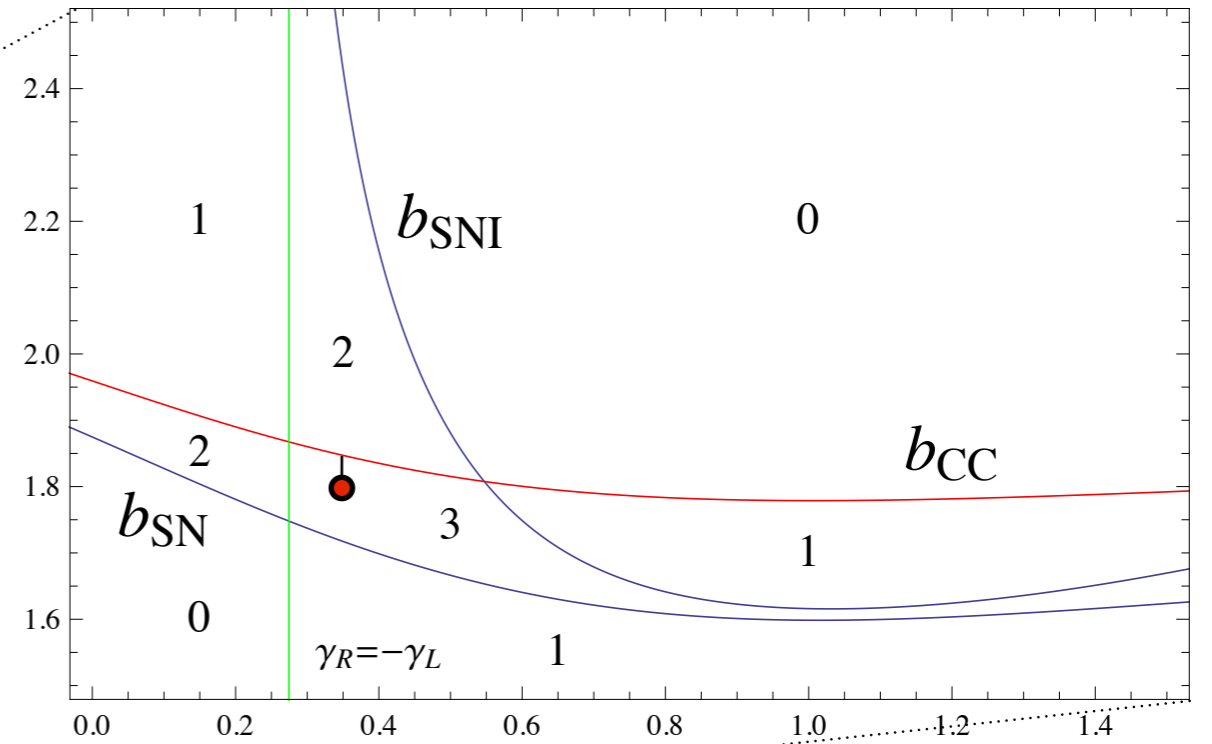
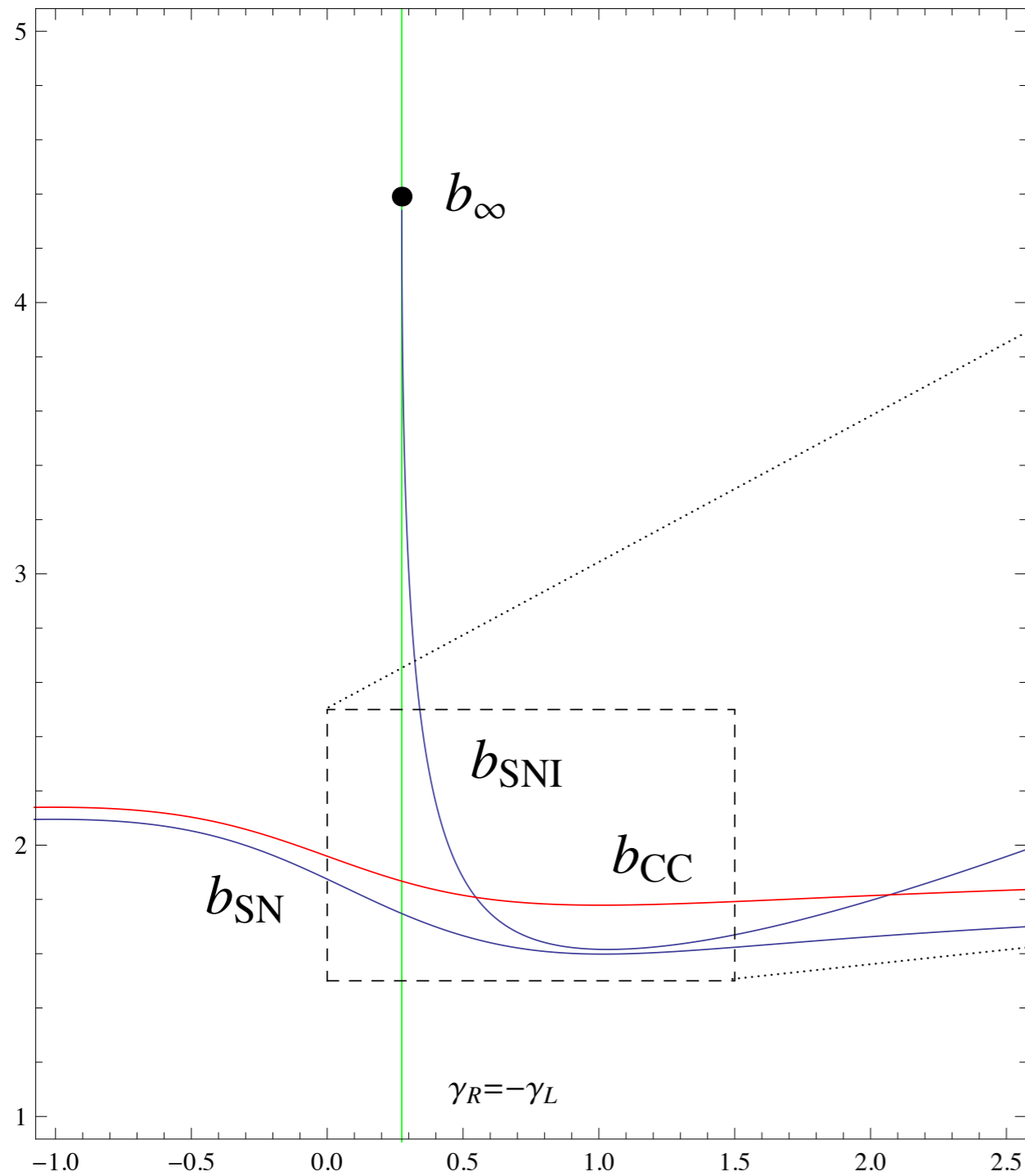
Hunting the three crossing limit cycles in a real case



Hunting the three crossing limit cycles in a real case



Hunting the three crossing limit cycles in a real case



The continuous matching of 3D PWL stable systems can be unstable

The continuous matching of 3D PWL stable systems can be unstable

- We consider semi-homogeneous CPWL systems in \mathbb{R}^3

$$\dot{\mathbf{x}} = \begin{cases} A^+ \mathbf{x}, & \text{if } x \geq 0, \\ A^- \mathbf{x}, & \text{if } x < 0, \end{cases}$$

where $\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$, the dot denotes derivatives respect to the time s , and

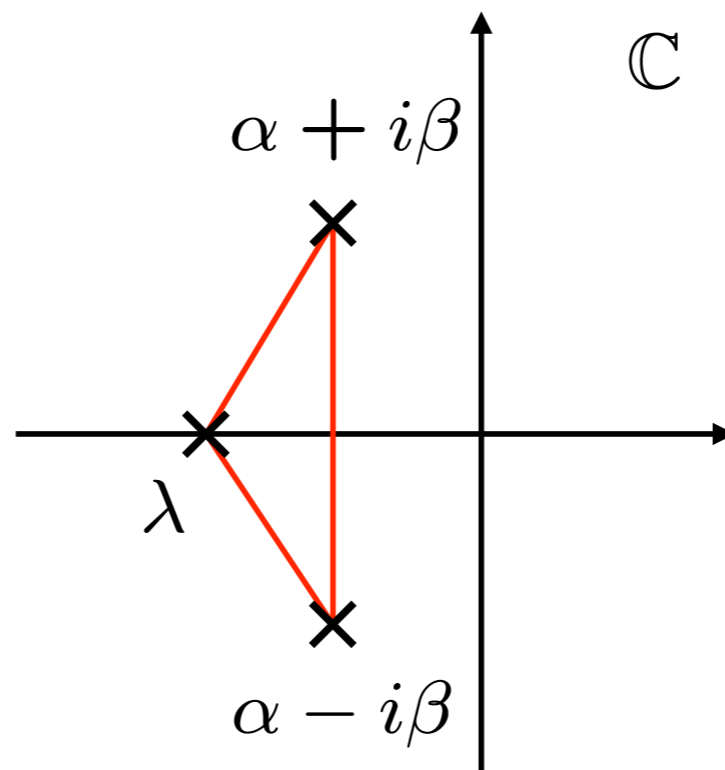
$$A^+ = \begin{pmatrix} t^+ & -1 & 0 \\ m^+ & 0 & -1 \\ d^+ & 0 & 0 \end{pmatrix}, \quad A^- = \begin{pmatrix} t^- & -1 & 0 \\ m^- & 0 & -1 \\ d^- & 0 & 0 \end{pmatrix}.$$

are already in the generalized Liénard form.

- Here, the stability of the origin is not a trivial issue, even when A^+ and A^- are Hurwitz matrices.

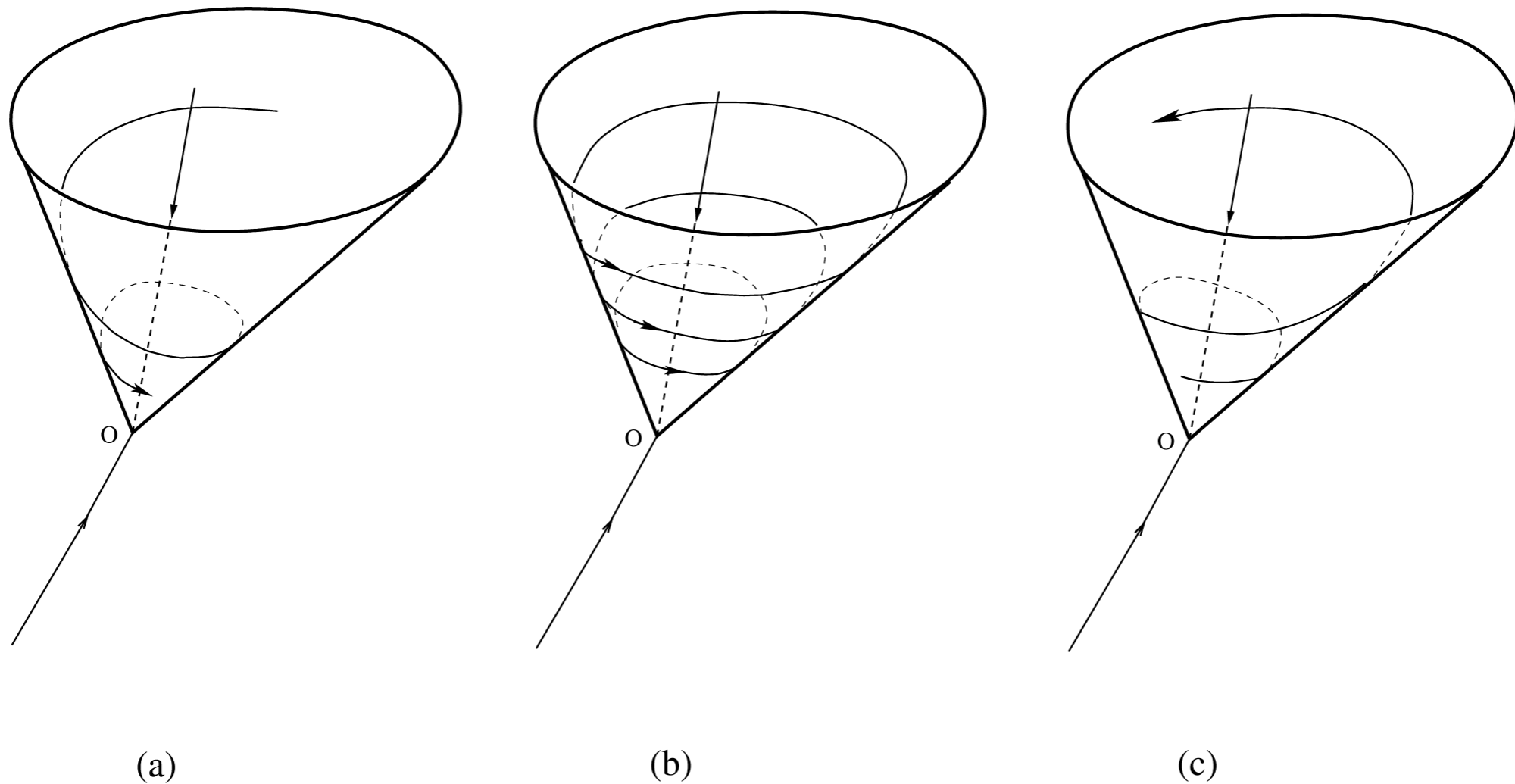
- We concentrate our attention in the case where the eigenvalues are λ^\pm , $\alpha^\pm + i\beta^\pm$ and $\alpha^\pm - i\beta^\pm$ with $\beta^\pm > 0$.
- The two following parameters (to be assumed positive) turn out to be crucial:

$$\gamma^+ = \frac{\alpha^+ - \lambda^+}{\beta^+} \quad \text{and} \quad \gamma^- = \frac{\alpha^- - \lambda^-}{\beta^-}.$$

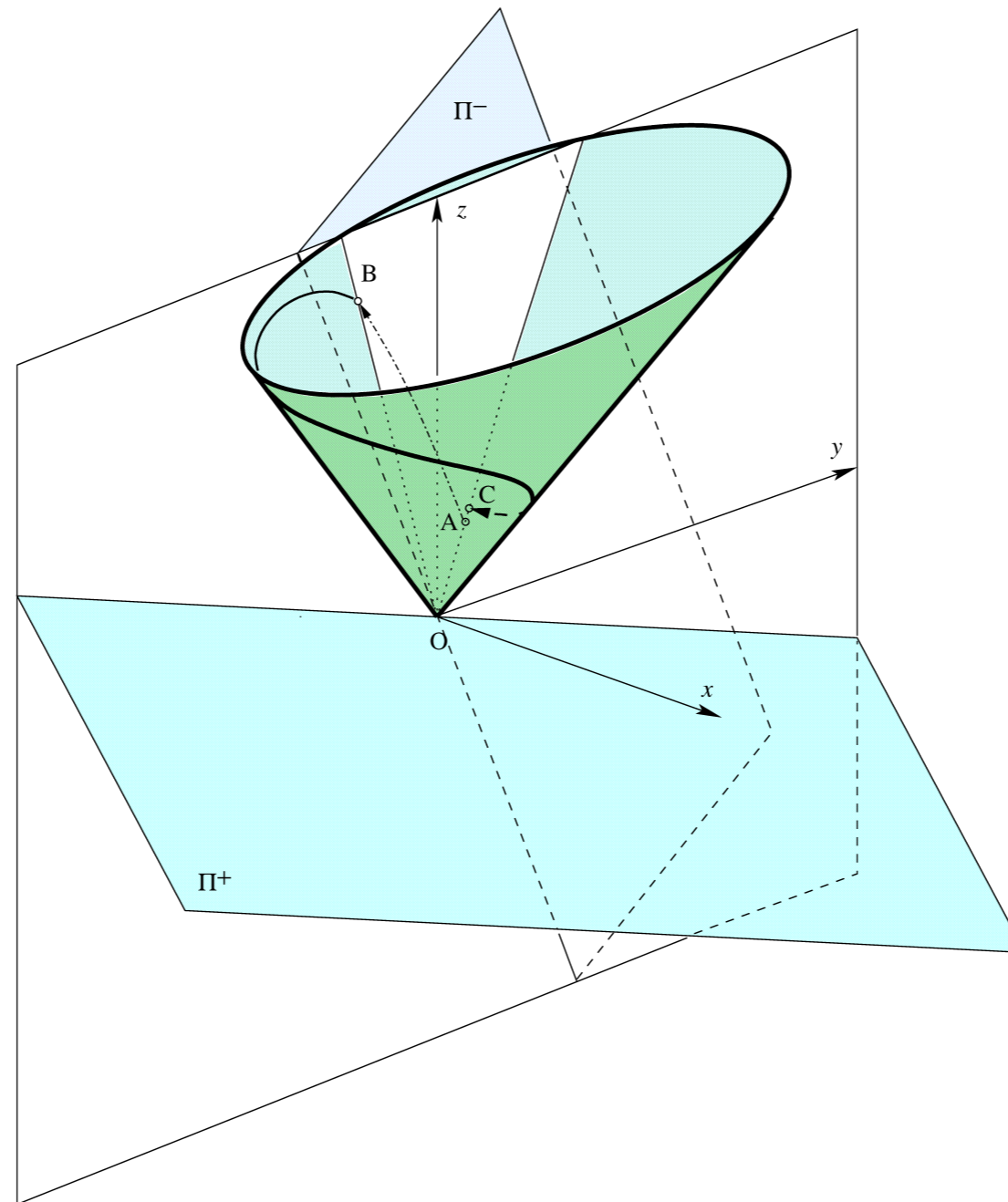


Theorem. If both matrices are Hurwitz with complex eigenvalues such that $\gamma^+ > 0$ and $\gamma^- > 0$, then the following statements hold.

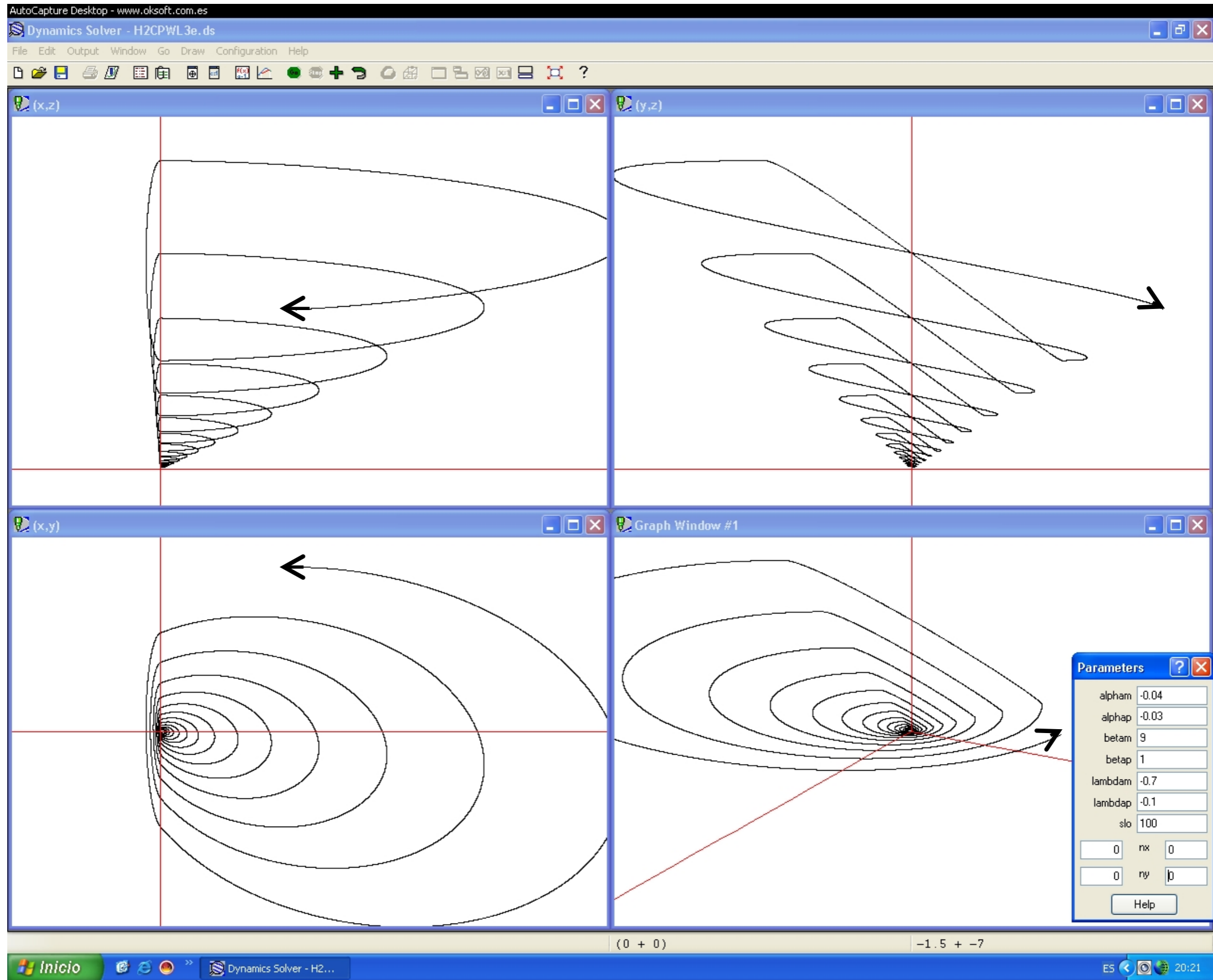
- (i) The origin has a one-dimensional stable manifold and a two-dimensional invariant manifold which is an attractive two-zonal cone. Generically, both manifolds are non-smooth.
- (ii) The dynamics on the cone is either of stable focus type, or a center, or of unstable focus type, and there exist specific systems for the three cases.



Geometry of the unstable case



Sketch of an orbit on the invariant cone which, spiraling towards the origin, starts from point A , approaches the half-plane Π^- reaching B and, within region $x > 0$, tends to the half-plane Π^+ up to C . The point C can be farther from the origin than point A .



Conclusions

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...que veinte años no es nada!

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...que veinte años no es nada!

Alfredo La Pera, from lyrics of the tango 'Volver' by Carlos Gardel