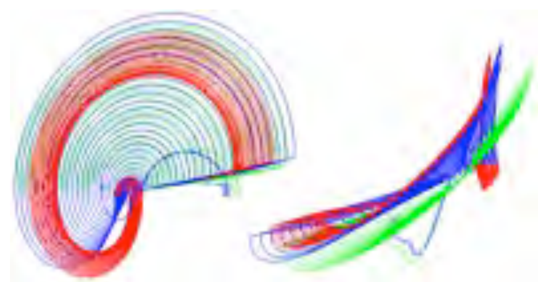
Ddays Benicassim 25 October 2012

Mike R. Jeffrey (University of Bristol)

A doorway to explosions and uncertainty in higher dimensions - unfolding the singularities of nonsmooth dynamics



“C'est même des hypothèses simples qu'il faut le plus se défier, parce que ce sont celles qui ont le plus de chances de passer inaperçues.”

Thermodynamique: Leçons professées pendant le premier semestre 1888-1889 (1892), Preface

It is the simple hypotheses of which one must be most wary; because these are the ones that have the most chances of passing unnoticed.

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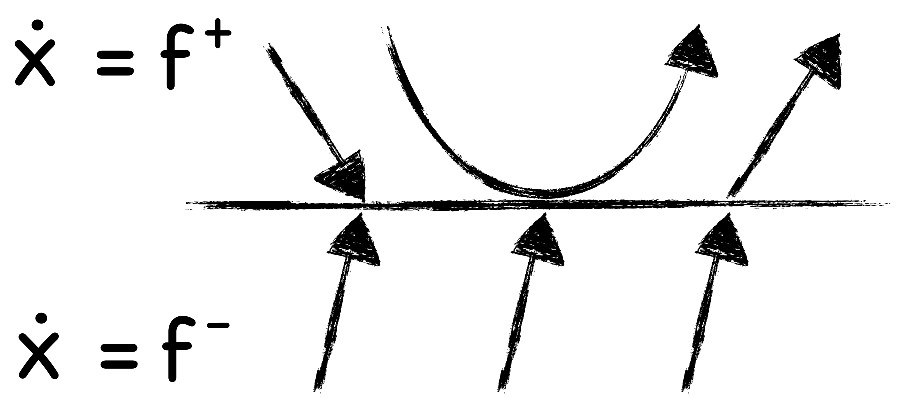
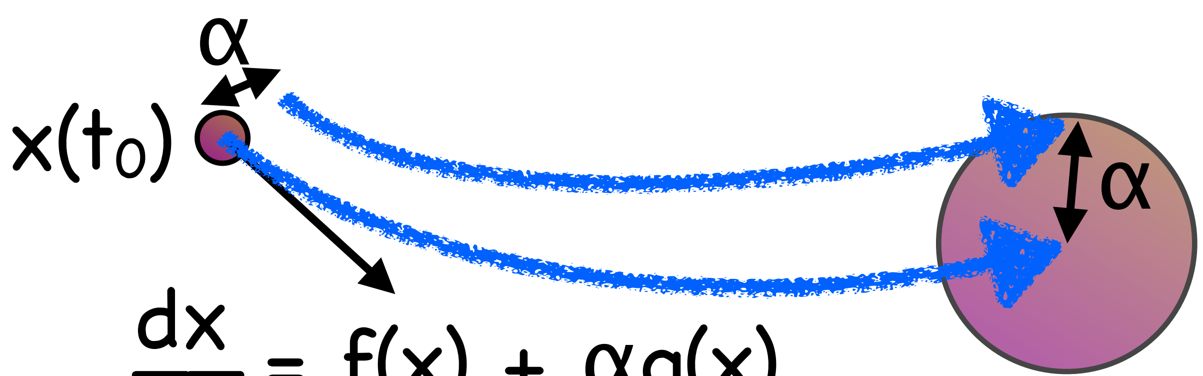


"Traditional type" problems
(exciting open questions & big challenges)

- routes to chaos
- global connections / Melnikov theory
- unfolding bifurcations

Why do discontinuities happen? – Sensitivity to initial conditions

(There is a traditional way to introduce nonsmooth dynamics, this is not it...)

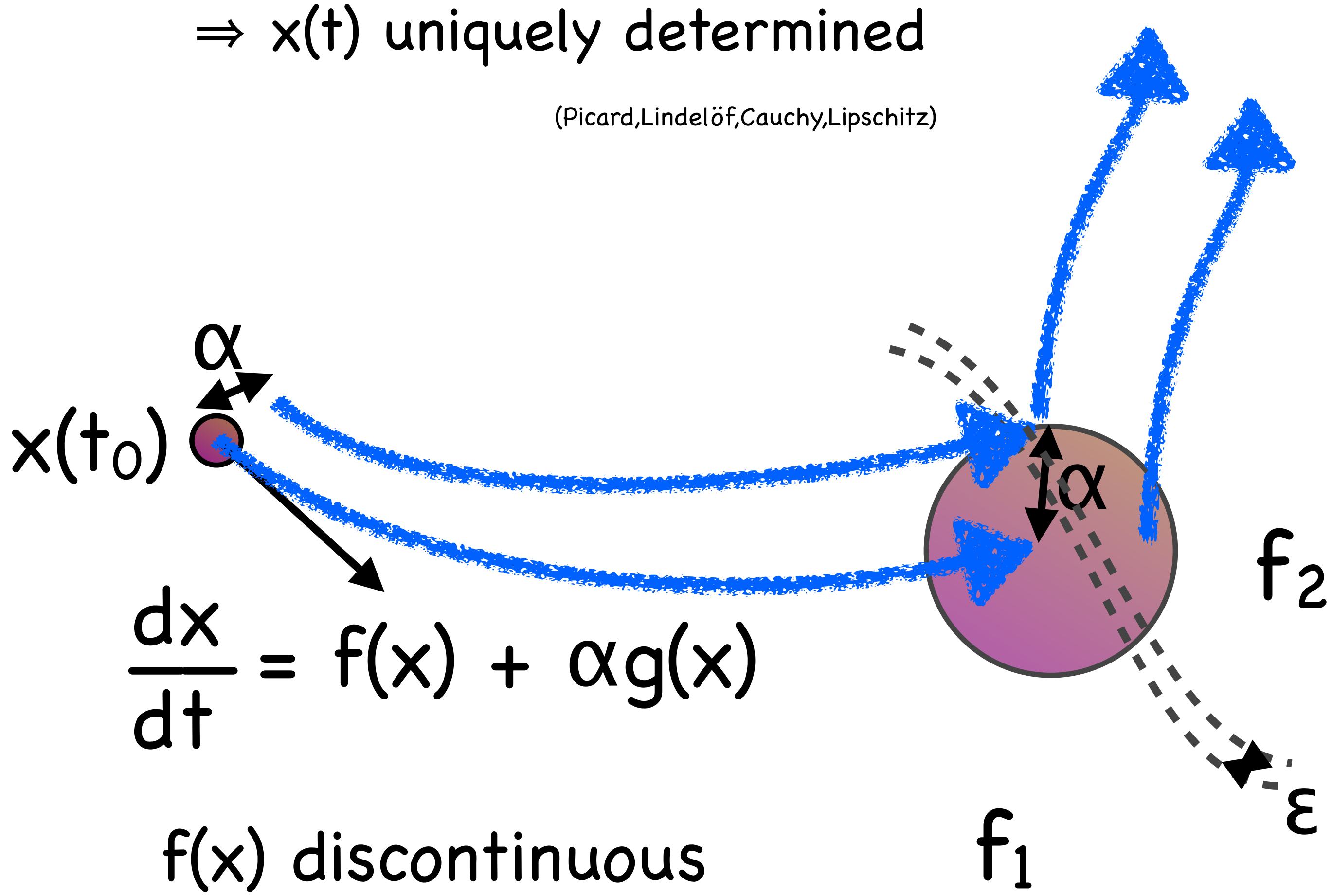


$$\frac{dx}{dt} = f(x) + \alpha g(x)$$

f(x) continuous

$\Rightarrow$ x(t) uniquely determined

(Picard,Lindelöf,Cauchy,Lipschitz)



$$\frac{dx}{dt} = f(x) + \alpha g(x)$$

f(x) discontinuous

$$f(x) = \begin{cases} f_1(x) & \text{if } H(x) > \varepsilon \\ f_2(x) & \text{if } H(x) < \varepsilon \end{cases}$$

Why do discontinuities happen? - Sensitivity to initial conditions

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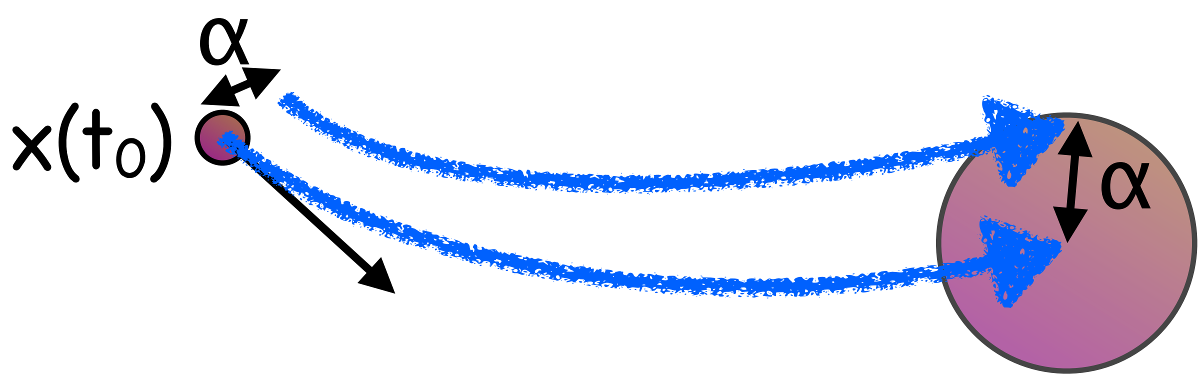
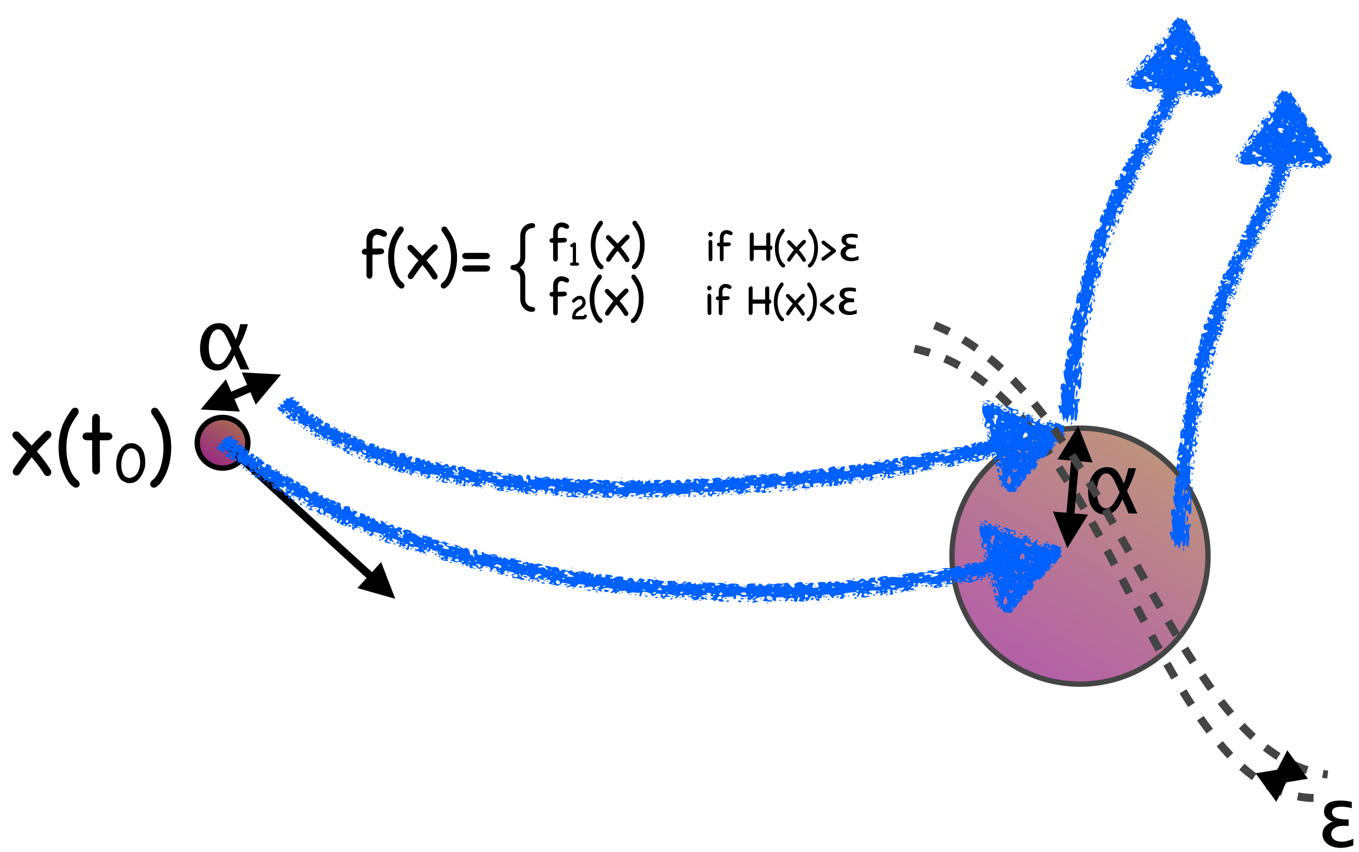


$$f(x)= \begin{cases} f_1(x) & \text{if } H(x)>\varepsilon \\ f_2(x) & \text{if } H(x)<\varepsilon \end{cases}$$

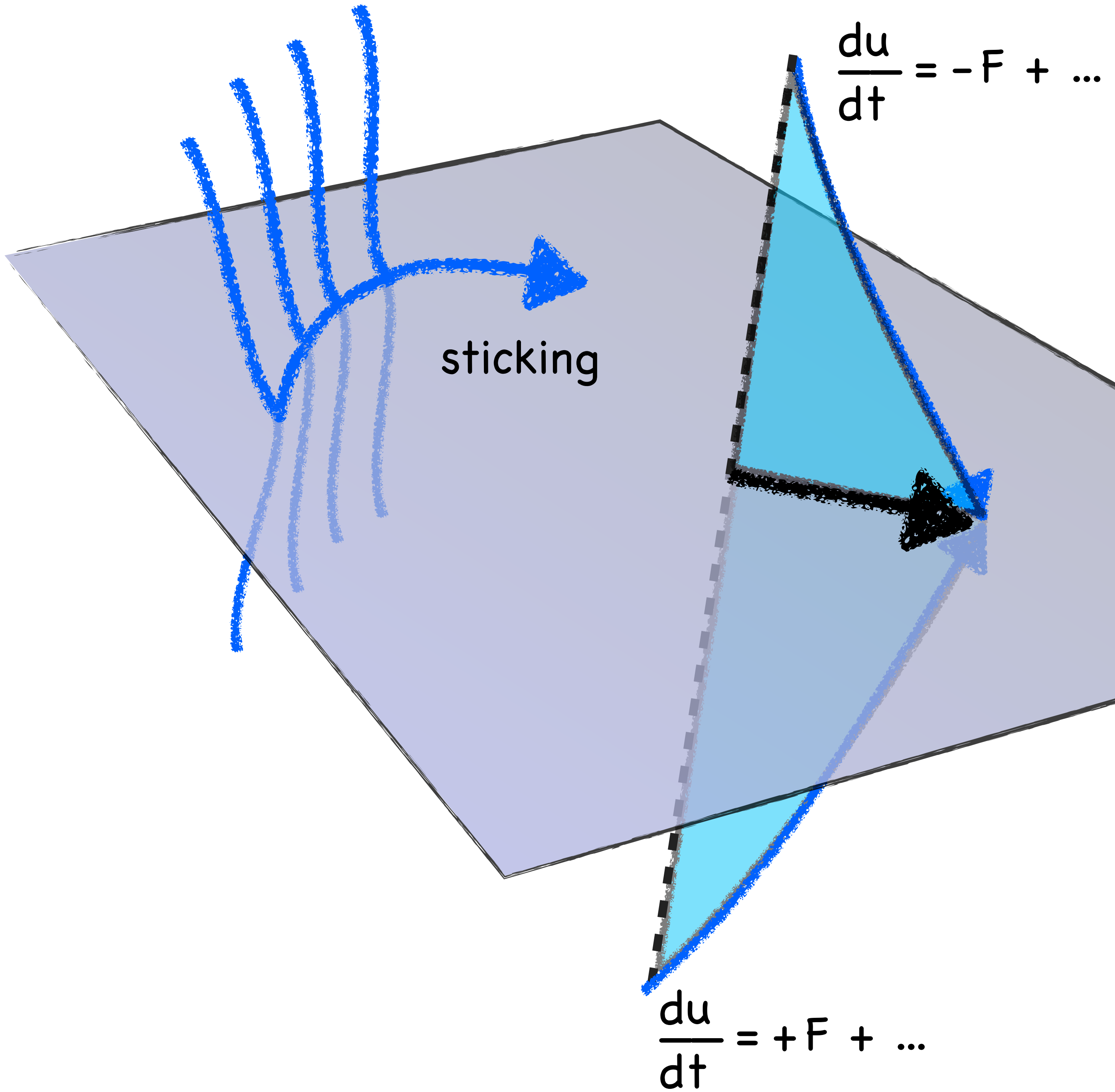


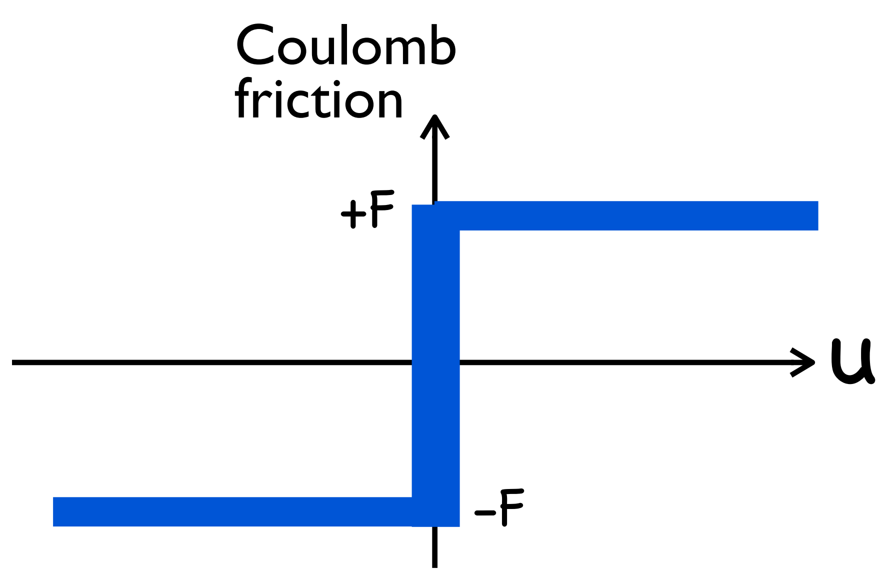
We have discontinuity for the same reason we have chaos: sensitivity to initial conditions

Why do discontinuities matter? - Loss of uniqueness & determinism

$$\frac{dx}{dt} = u$$

$$\frac{du}{dt} = -\ F\text{sign}(u) + \ldots$$





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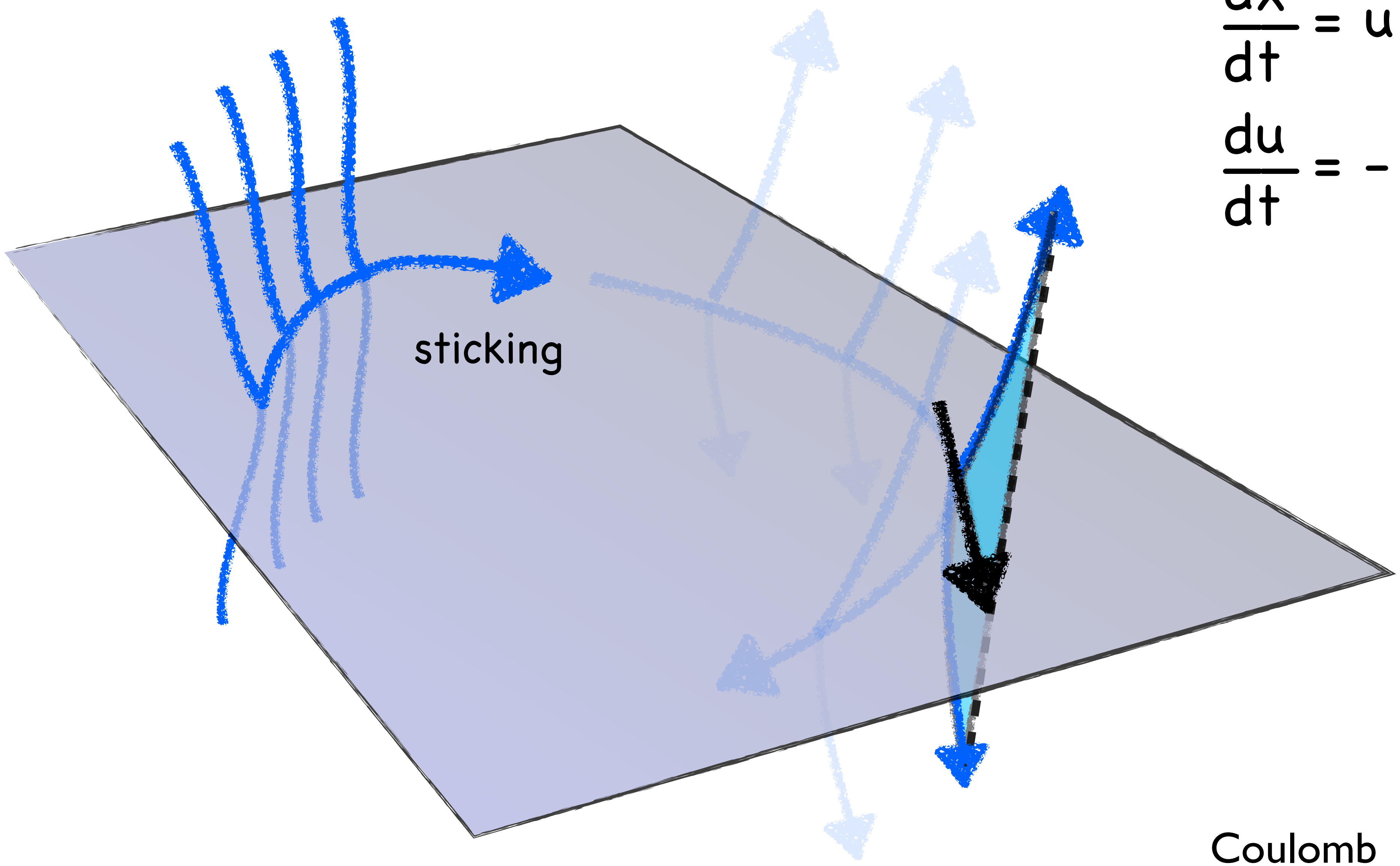

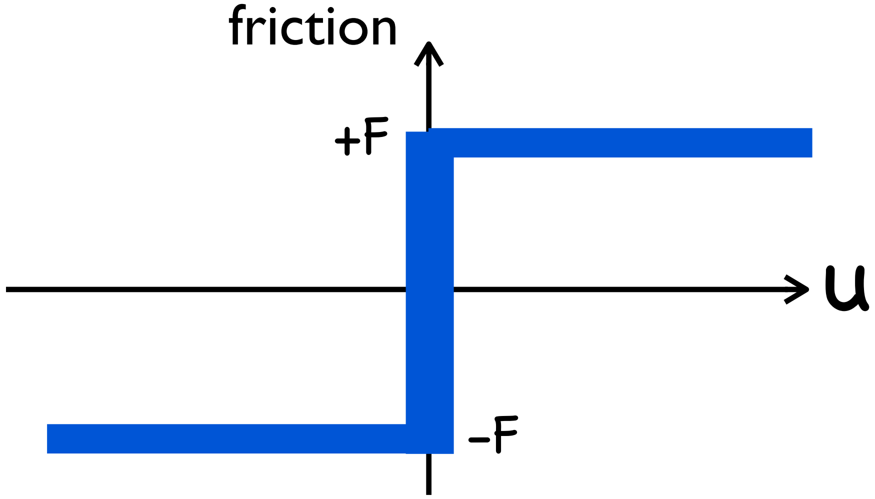

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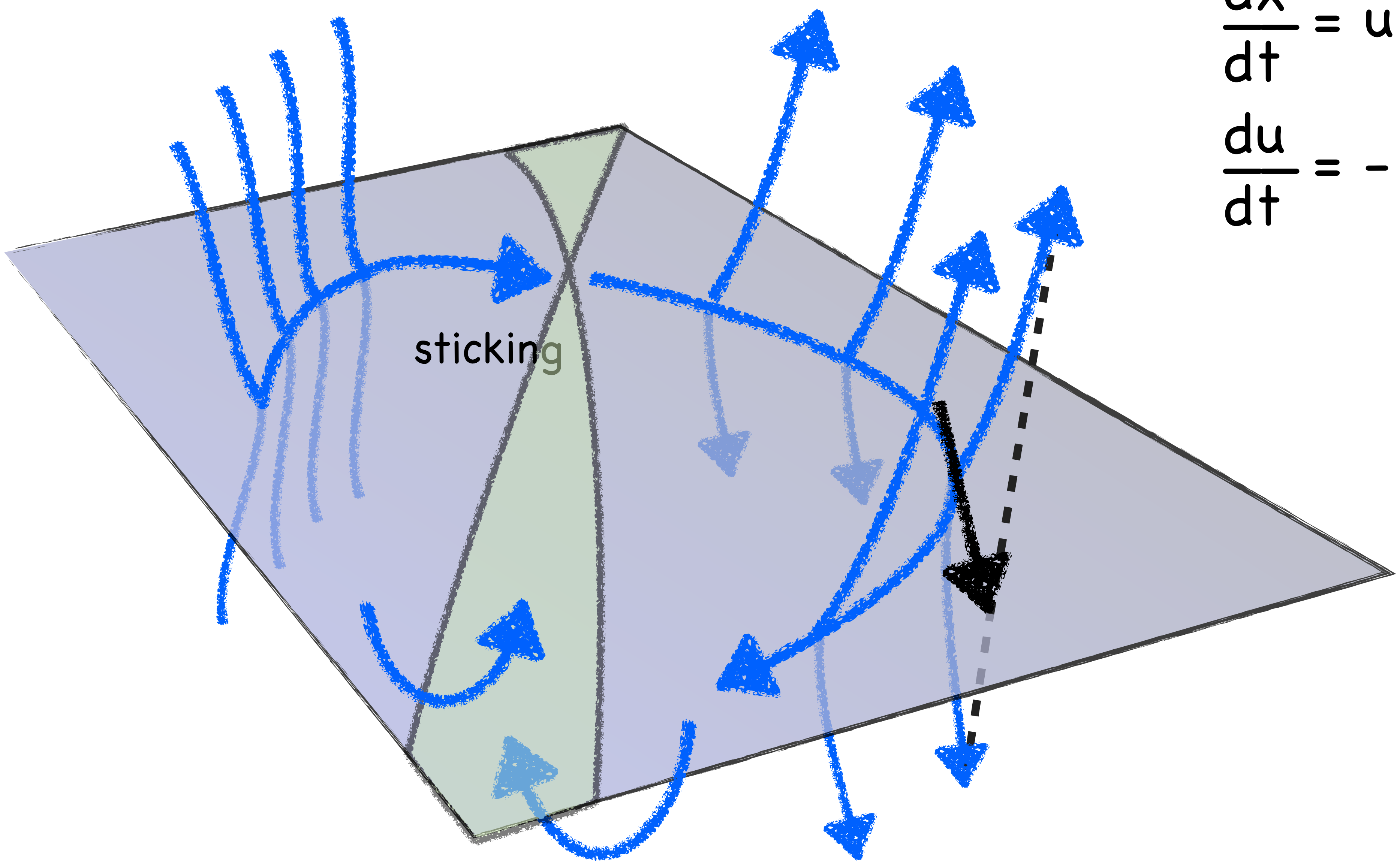


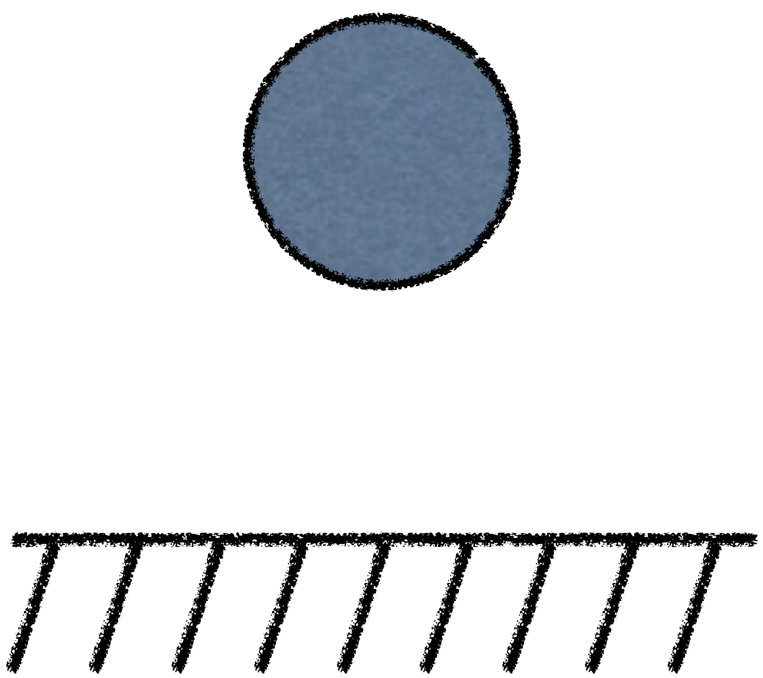
<u>Multiplicity of solutions:</u>

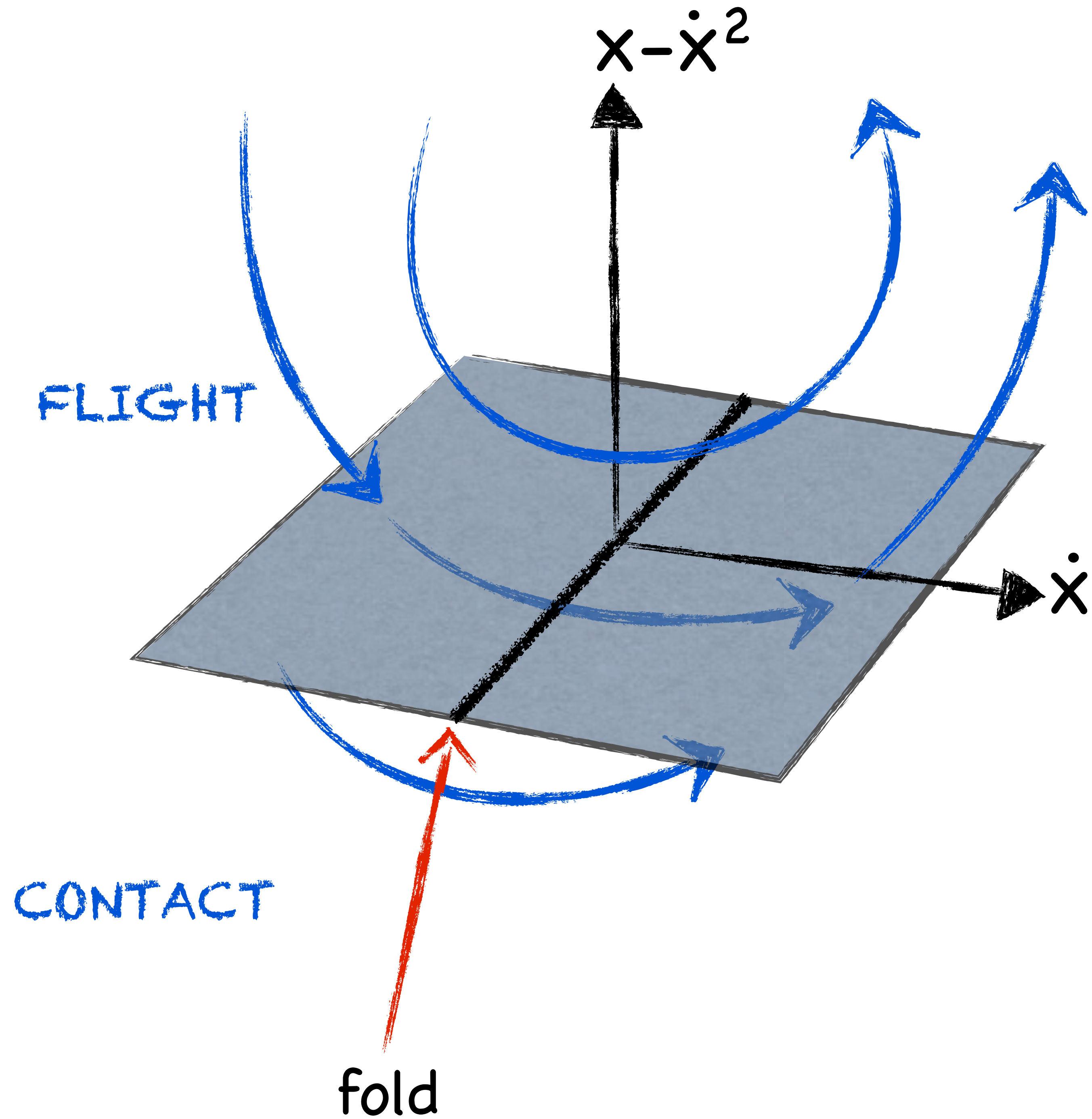
Given initial condition only,
solutions is determined <u>up to a set.</u>

Given 2 boundary conditions,
a solution is well-determined.

Search for the two-fold singularity

IMPACT



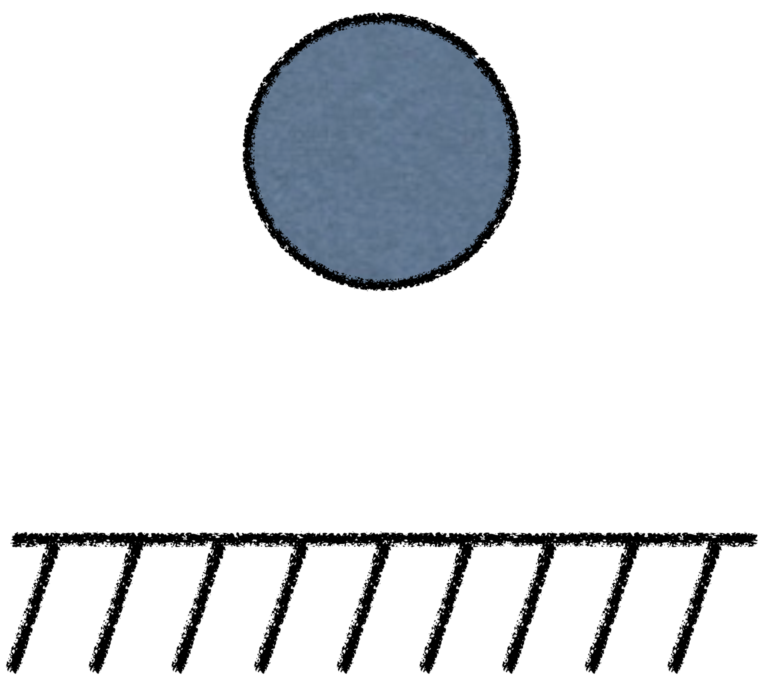


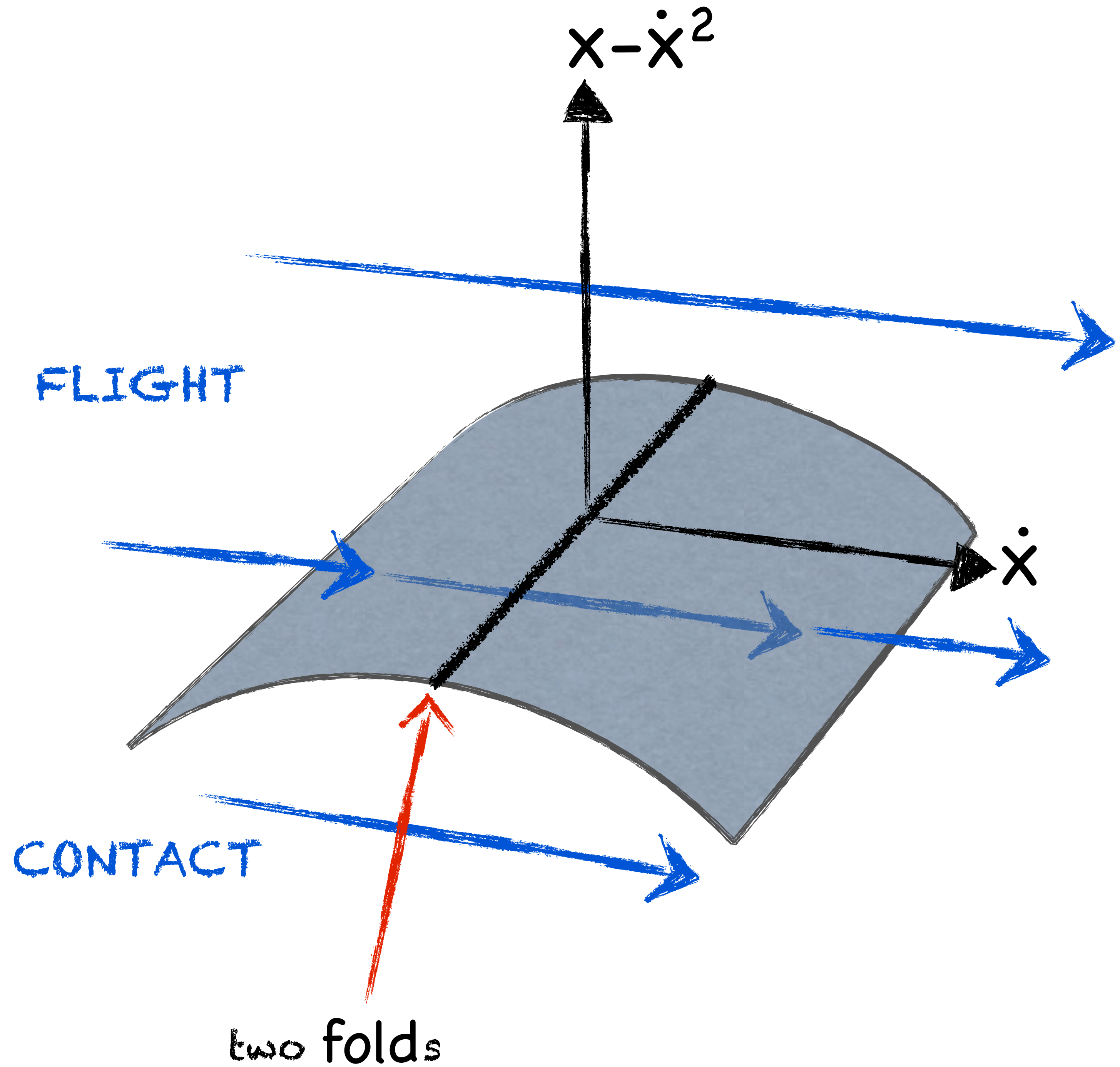
$\ddot{x} = f(x,\dot{x},t)$ FLIGHT

$\ddot{x} = g(x,\dot{x},\varepsilon t)$ IMPACT

Search for the two-fold singularity

IMPACT



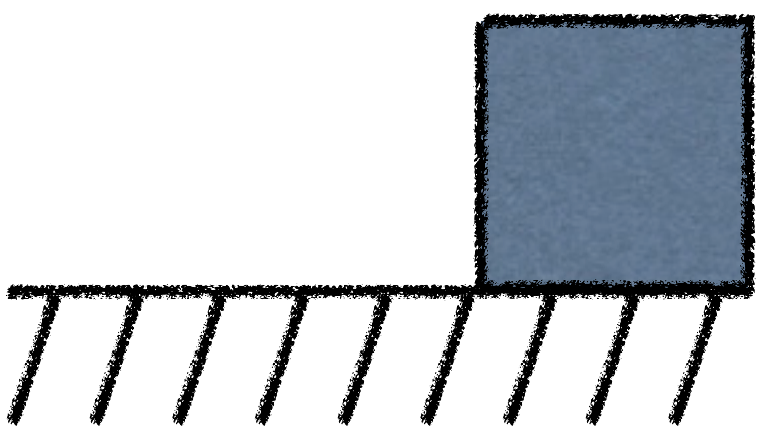


$$\ddot{x} = f(x, \dot{x}, t) \quad \text{FLIGHT}$$

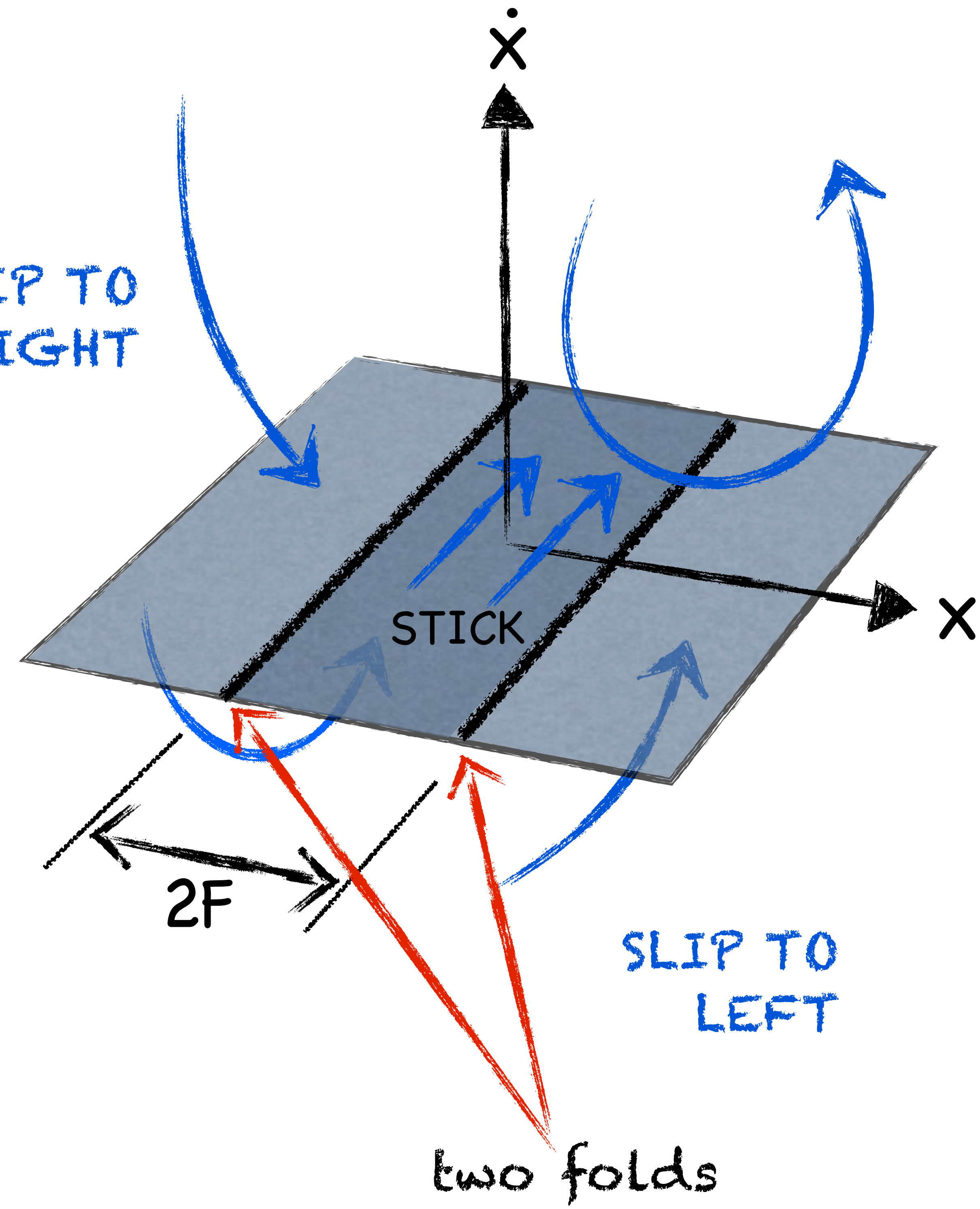
$$\ddot{x} = g(x, \dot{x}, \varepsilon t) \quad \text{IMPACT}$$

Search for the two-fold singularity

FRICTION



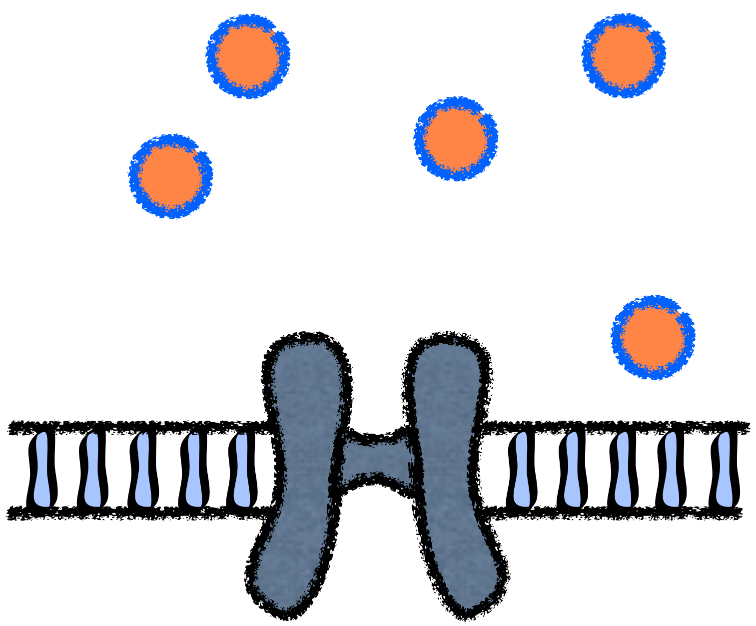
$$\ddot{x} = f(x,\dot{x},t) + F\mathrm{sign}(\dot{x})$$

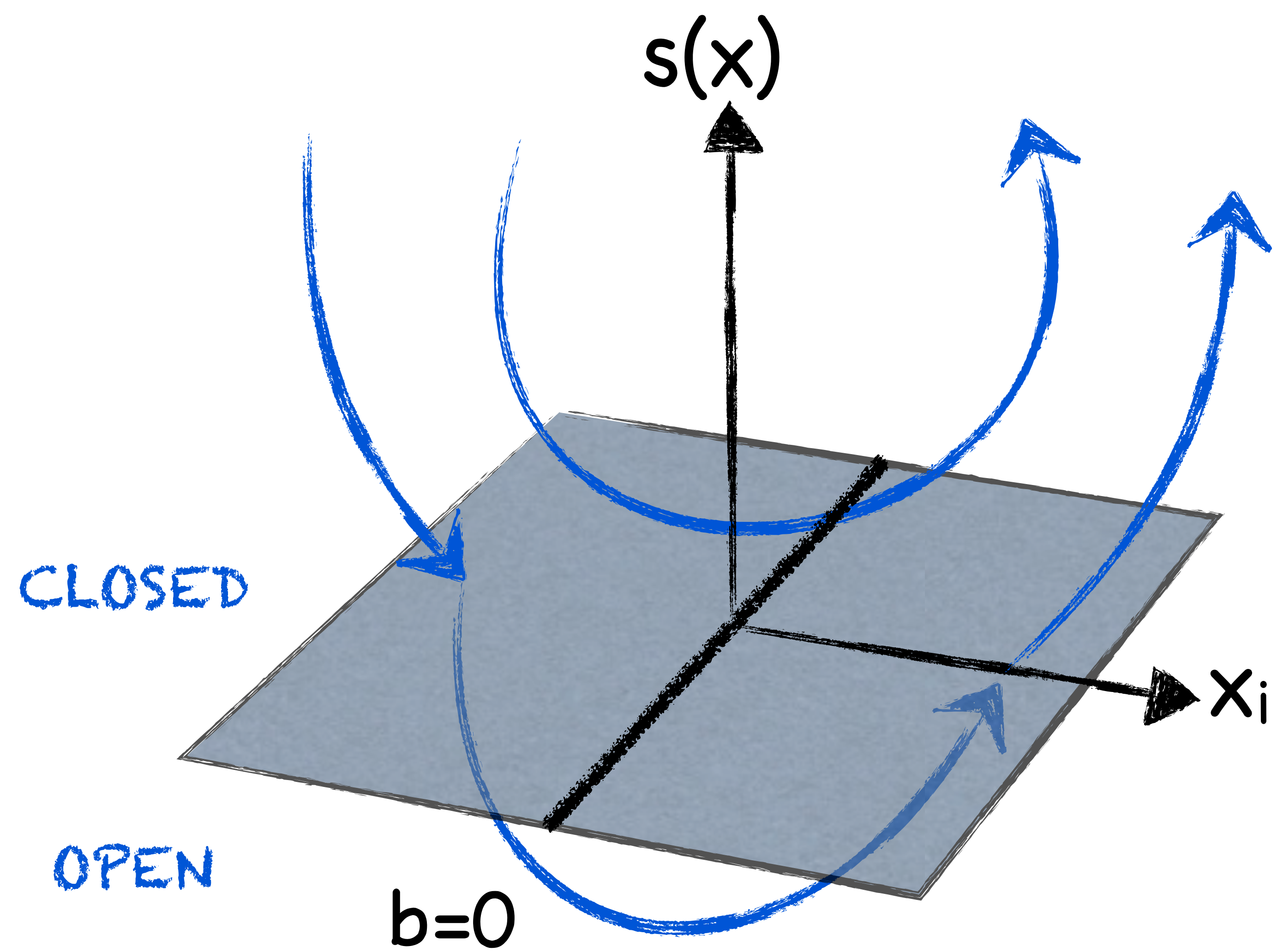


Searching for two-folds: the doorway to nondeterminism

SWITCHES

(relays, neurons, decisions...)





$\dot{x} = f(x)$ CLOSED

$\dot{x} = g(x)$ OPEN

$$\dot{x} = Ax + a + (Bx + b)u(x)$$

$x \in \mathbb{R}^n$

open $u=1$ if $s>0$

closed $u=0$ if $s<0$

Teixeira singularities in 3D switched feedback control systems Sys & Cont Lett **59** 10, 615-622 (2009) Colombo, di Bernardo, Fossas, Jeffrey

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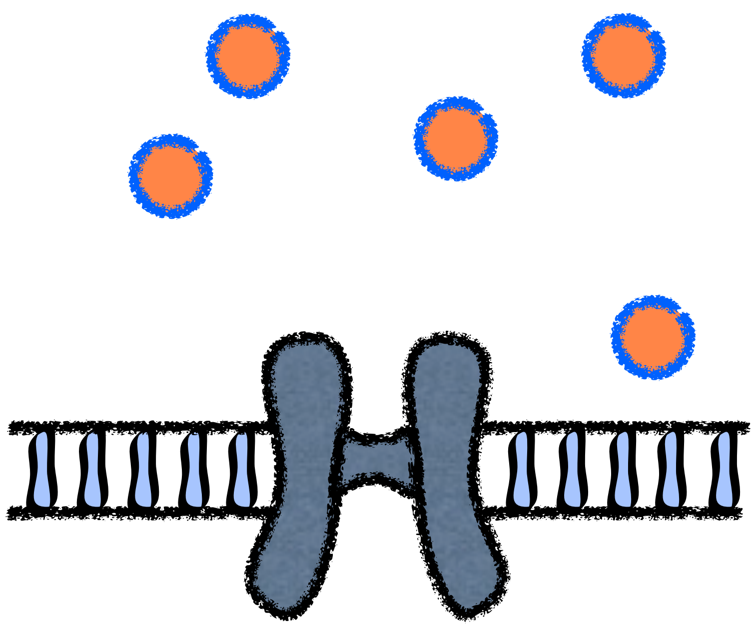

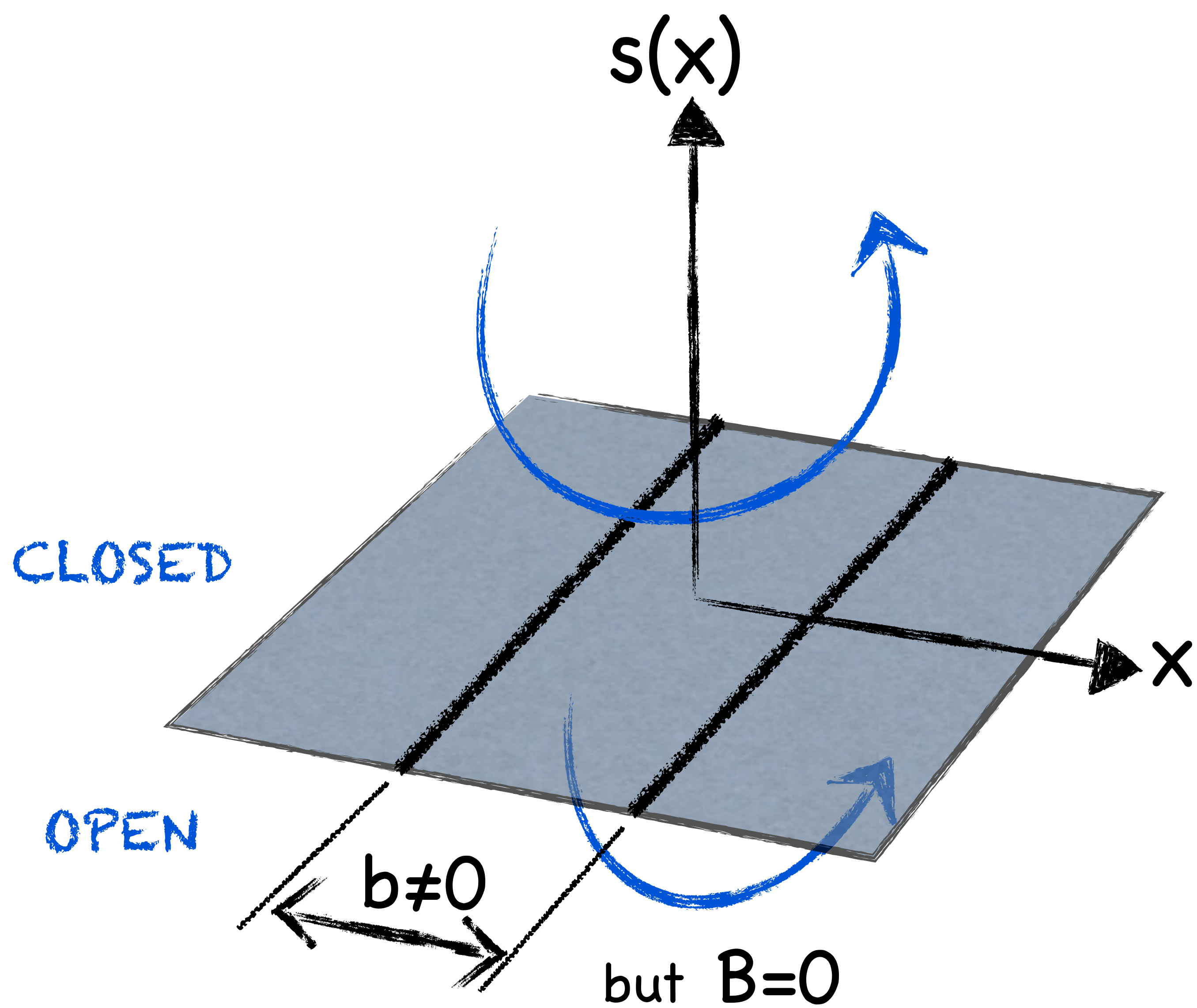

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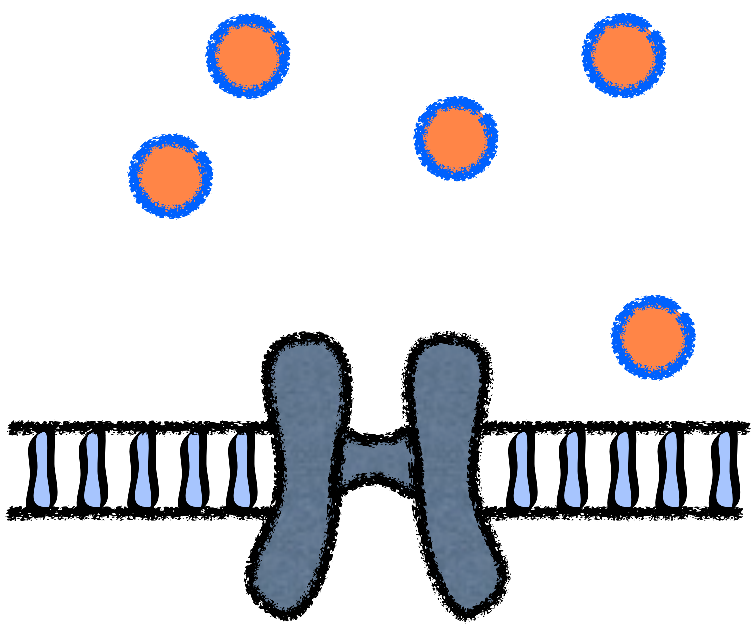

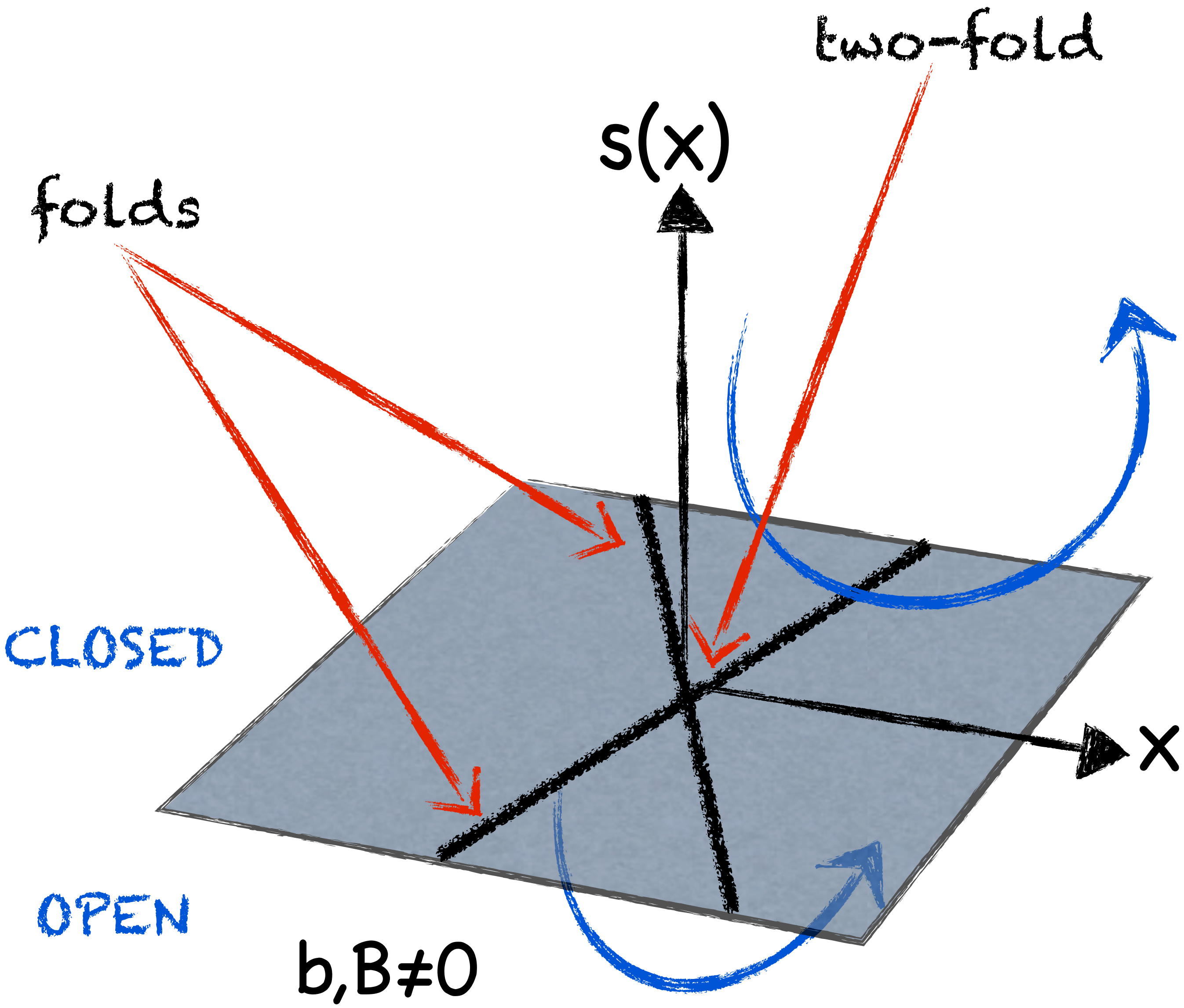

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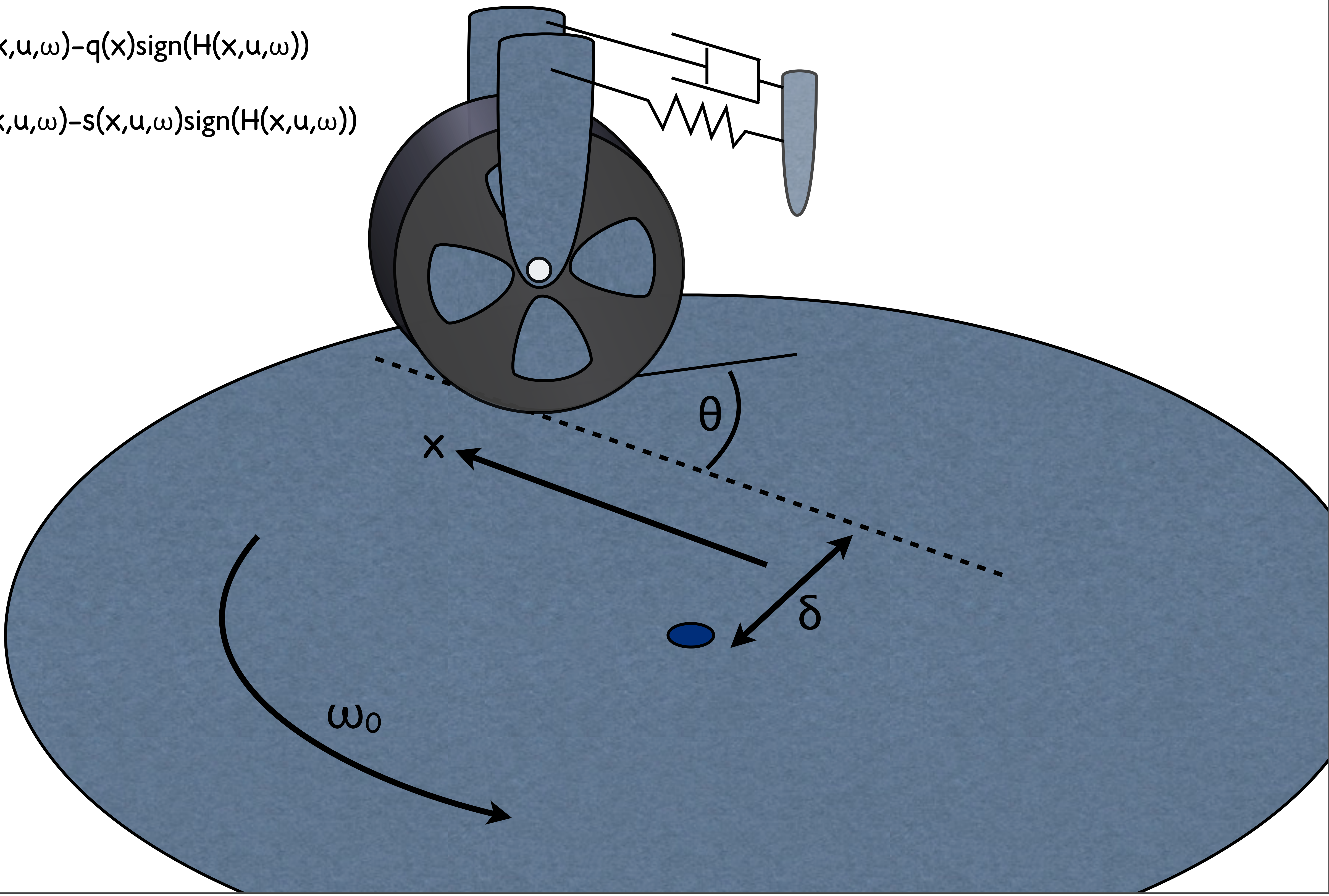
Non-deterministic wheel rattle

with Robert Szalai (Bristol)

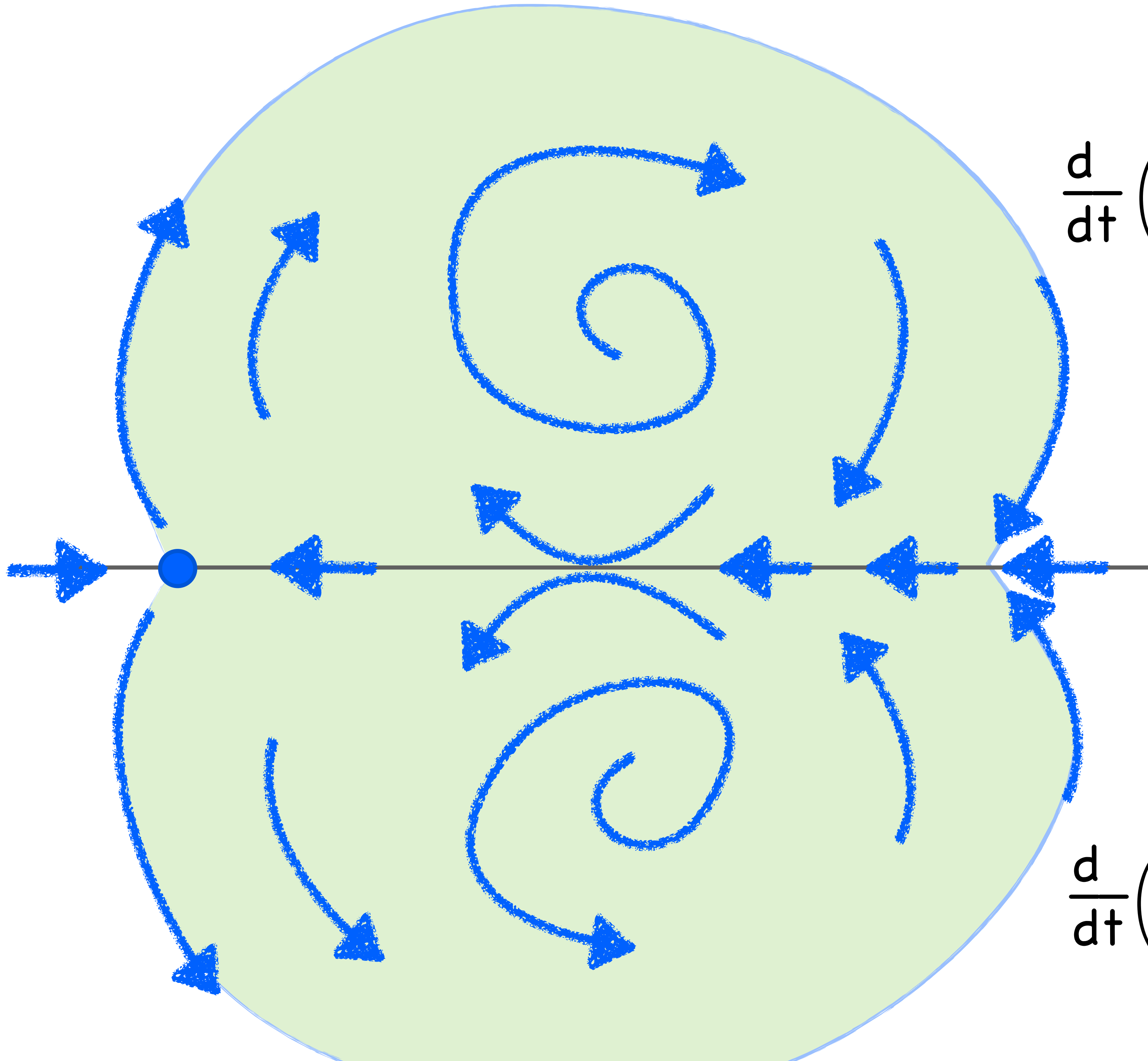
$$\frac{dx}{dt} = u$$

$$\frac{du}{dt} = p(x,u,\omega) - q(x)\,\mathrm{sign}(H(x,u,\omega))$$

$$\frac{d\omega}{dt} = r(x,u,\omega) - s(x,u,\omega)\,\mathrm{sign}(H(x,u,\omega))$$



Non-deterministic chaos



$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}a & 1\\-1 & a\end{pmatrix}\begin{pmatrix}x_1\\x_2-1\end{pmatrix}$$

$$+\ \text{h.o.t.}$$

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}a & -1\\1 & a\end{pmatrix}\begin{pmatrix}x_1\\x_2+1\end{pmatrix}$$

$$+\ \text{h.o.t.}$$

A flow ϕ is chaotic on a compact invariant set X if ϕ is transitive and exhibits sensitive dependence on X.

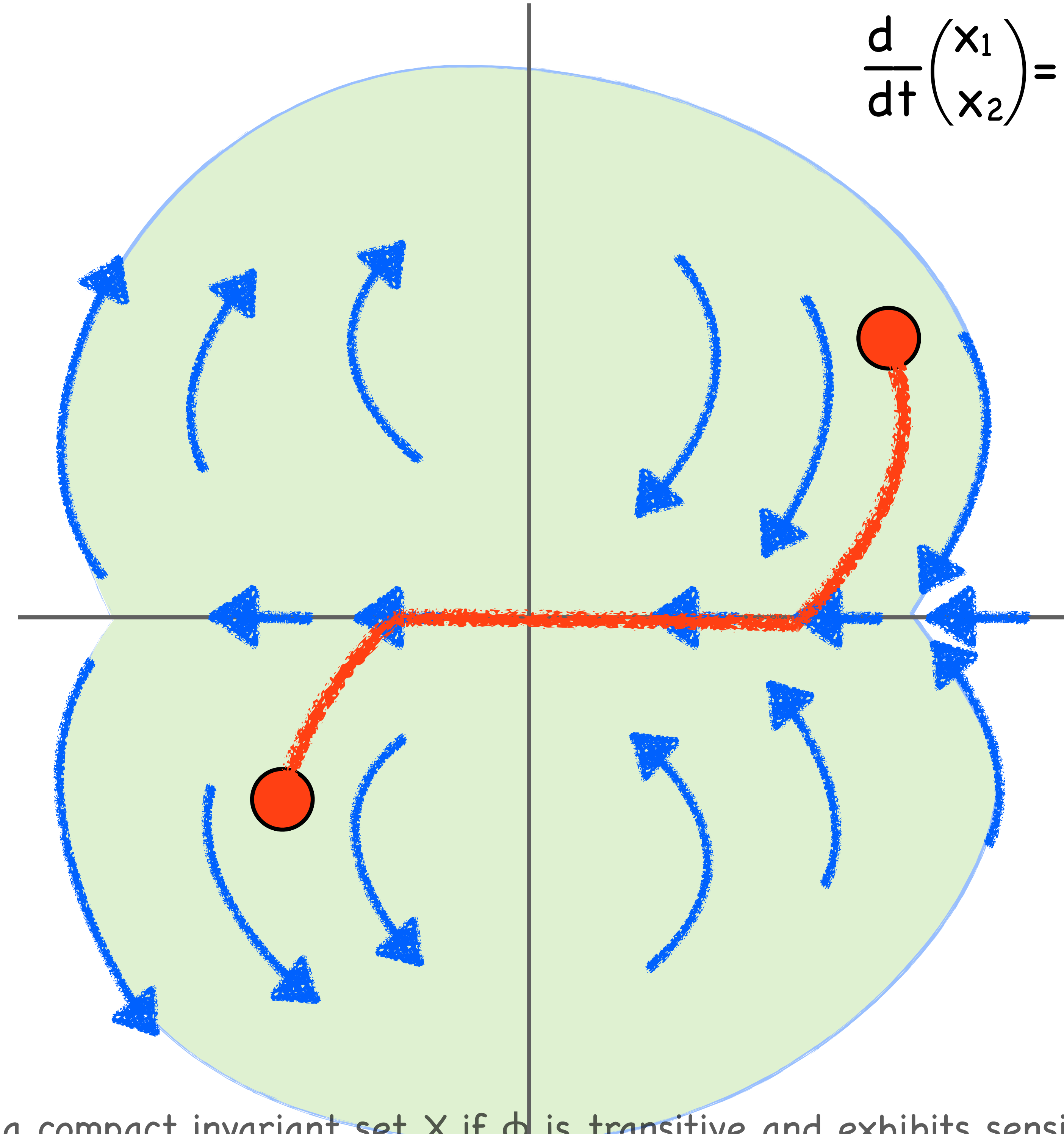
Transitivity: Flow ϕ is topologically transitive on invariant set X if for every pair of nonempty open sets U and V in X $\exists$ $t>0$ such that $\phi_t(U)\cap V\neq\emptyset$.

Sensitivity: Flow ϕ exhibits sensitivity to initial conditions on X if $\exists$ $r>0$ such that $\forall x\in X$, any $y\in X$ and some $t\geq 0$, we have diameter$[\phi_t(x)\cup\phi_t(y)] > r$.

Non-deterministic chaos, & the two-fold singularity in piecewise-smooth flows SIADS. **10** 423-451 (2011) Colombo & Jeffrey

Non-deterministic chaos

$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}c_1-1+|x_2|+a\,\mathrm{sign}(x_1)\\c_2-b\,\mathrm{sign}(x_1x_2)\end{pmatrix}$$



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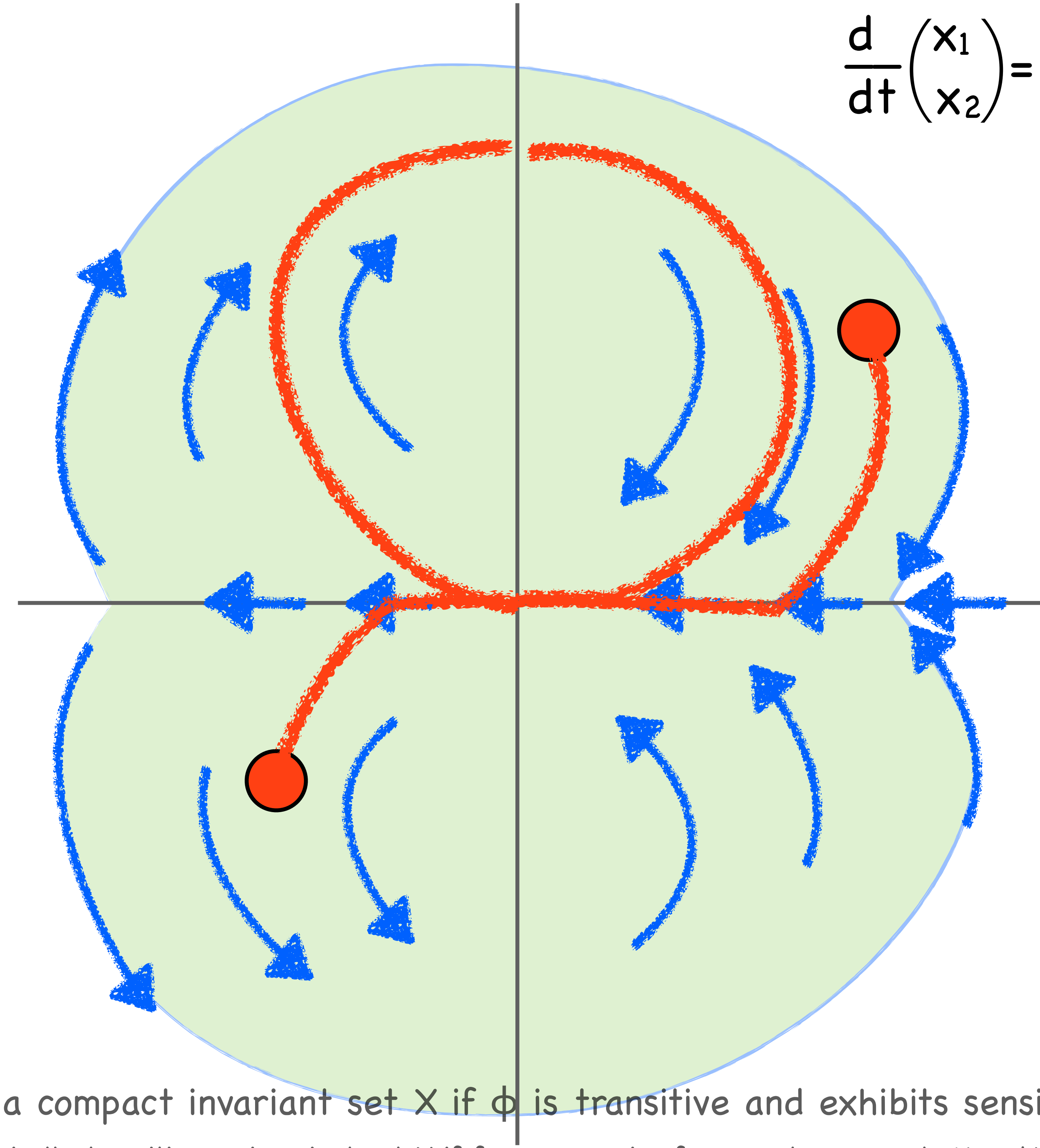

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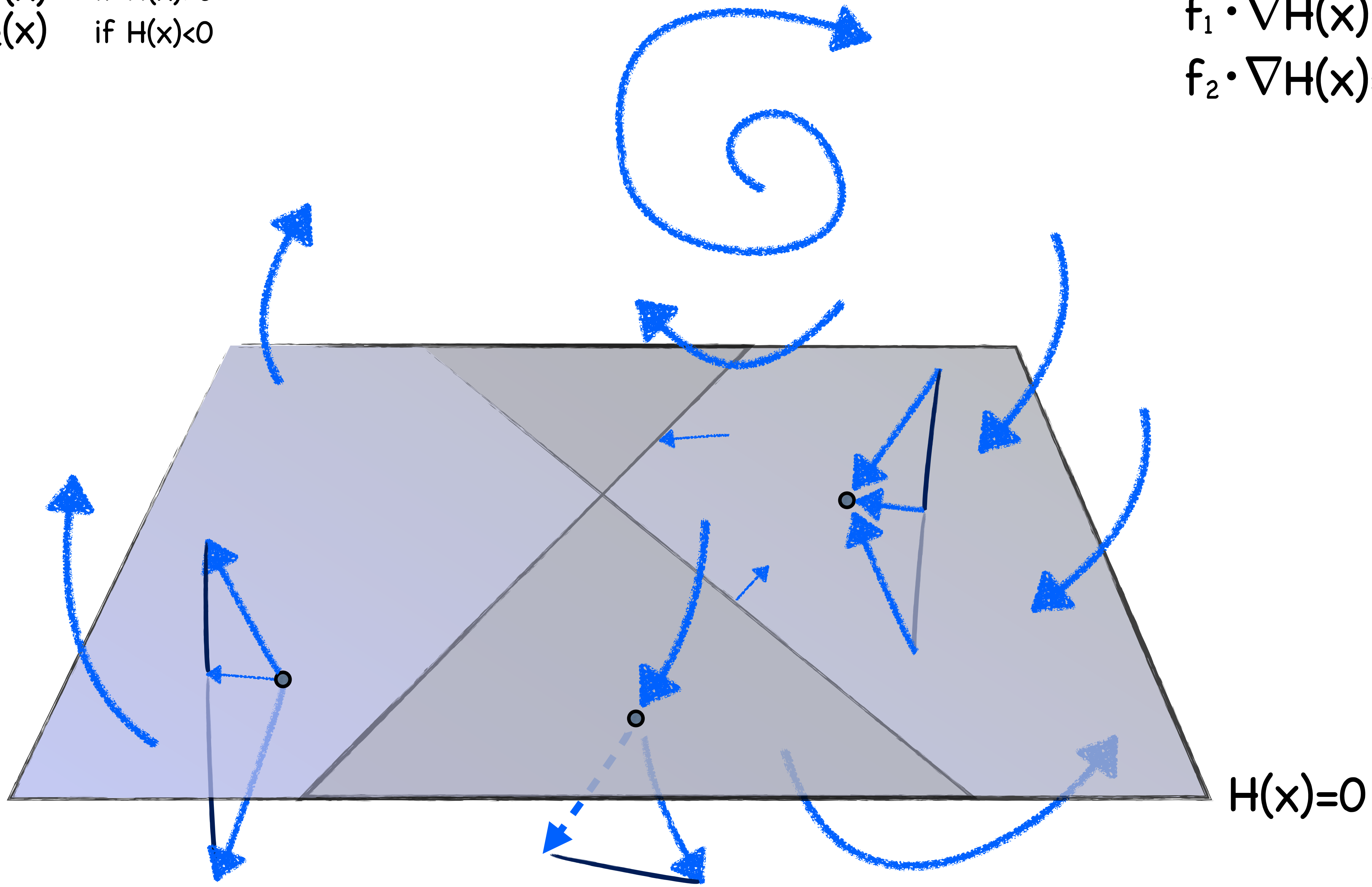
The two-fold singularity

$$\frac{dx}{dt} = \begin{cases} f_1(x) & \text{if } H(x)>0 \\ f_2(x) & \text{if } H(x)<0 \end{cases}$$

Two-fold singularity

$$f_1 \cdot \nabla H(x)=0$$

$$f_2 \cdot \nabla H(x)=0$$



Non-determinism in the limit of nonsmooth dynamics PRL. **016** 254103 (2011) Jeffrey

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The two-fold singularity

$$\frac{dx}{dt} = \begin{cases} f_1(x) & \text{if } H(x)>0 \\ f_2(x) & \text{if } H(x)<0 \end{cases}$$

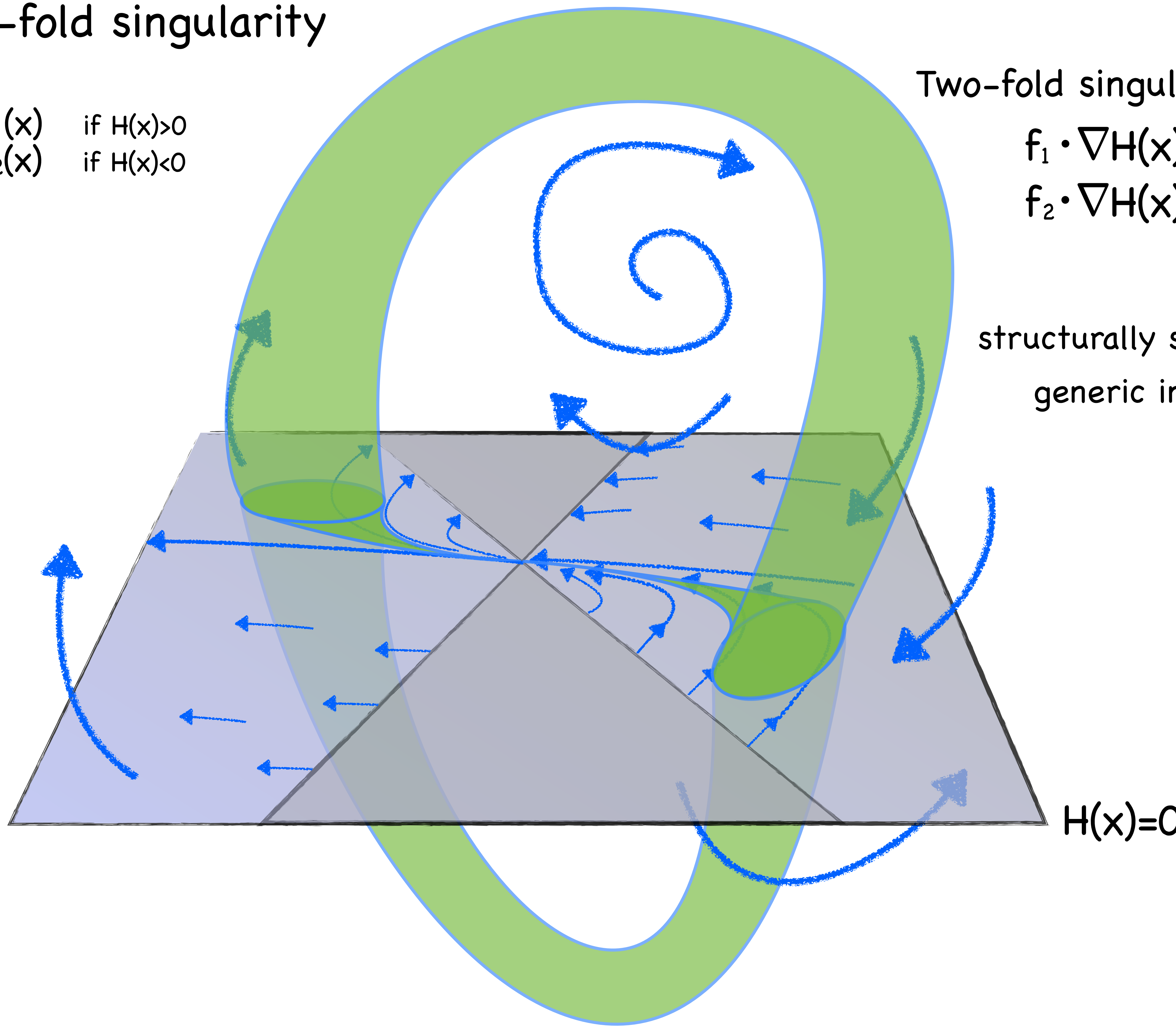
Two-fold singularity

$f_1 \cdot \nabla H(x)=0$

$f_2 \cdot \nabla H(x)=0$

structurally stable

generic in $\mathbb{R}^{n>2}$



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Sensitivity in the nonsmooth limit

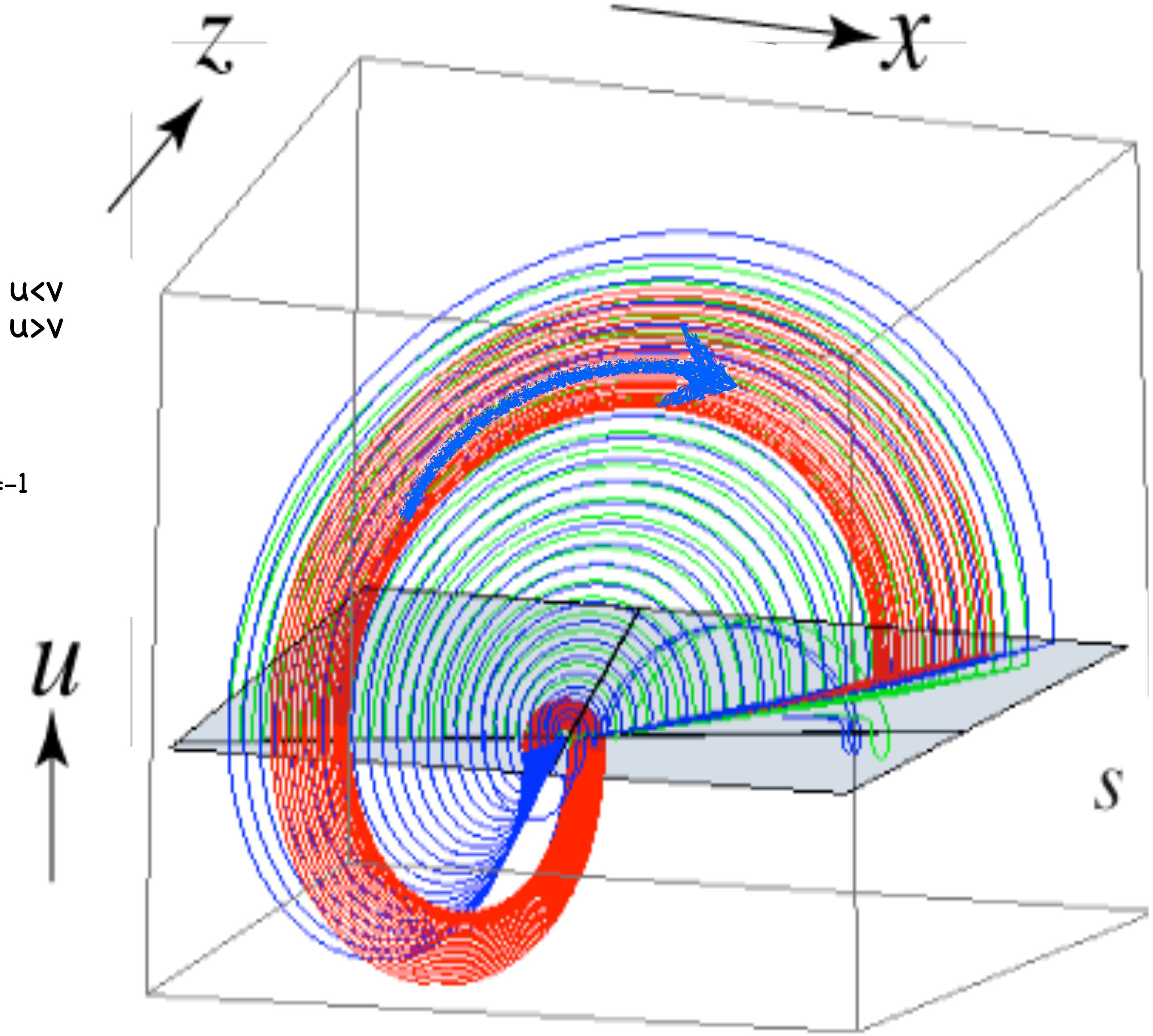
Non-deterministic chaos
in a negatively damped oscillator
with a switching force

$$\frac{dx}{dt} = u$$

$$\frac{du}{dt} = (u-v)b - x + \begin{cases} r\,z & \text{if } u<v \\ r'z & \text{if } u>v \end{cases}$$

$$\frac{dz}{dt} = \begin{cases} 1 & \text{if } u<v \\ a+(u-v)c & \text{if } u>v \end{cases}$$

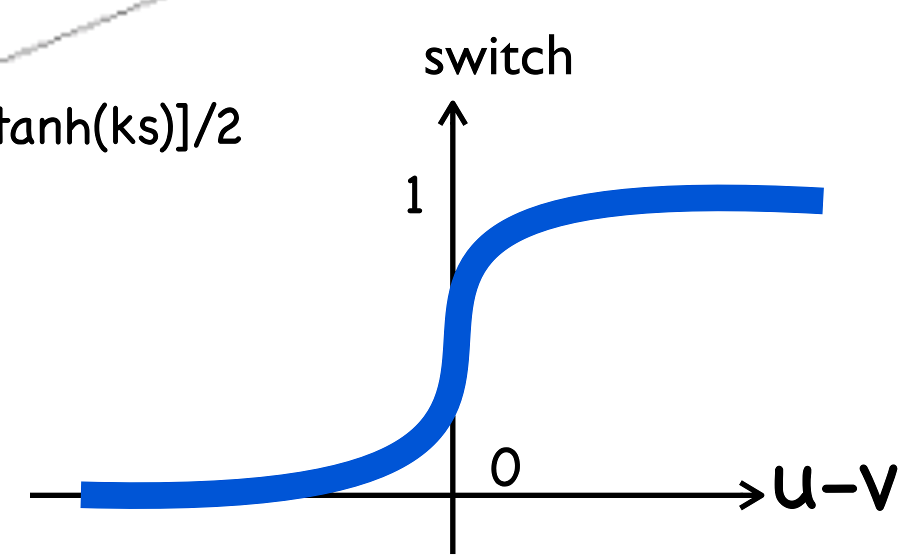
$a=-1.3,\ b=0.1,\ c=0.2,\ v=-1,\ r=12,\ r'=-1$



Smoothing:

$\lambda(s) = [1+\tanh(ks)]/2$

$$\frac{d(x,u,z)}{dt} = \begin{cases} f_1(x,u,z) & \text{if } u<v \\ f_2(x,u,z) & \text{if } u>v \end{cases} \approx \lambda f_1 + (1-\lambda) f_2$$



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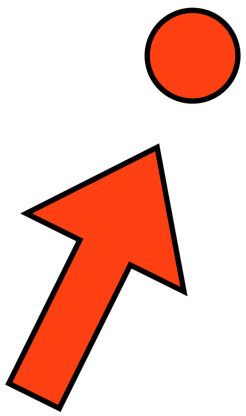
"Non-determinism" and "non-deterministic chaos"



USTED
ESTA AQUI

First comes the intuition:

- what behaviour does the phenomenon represent?

The perception of randomness in a non-random system

(chaos => repeating yet unpredictable!)

Then comes the mathematical rigour:

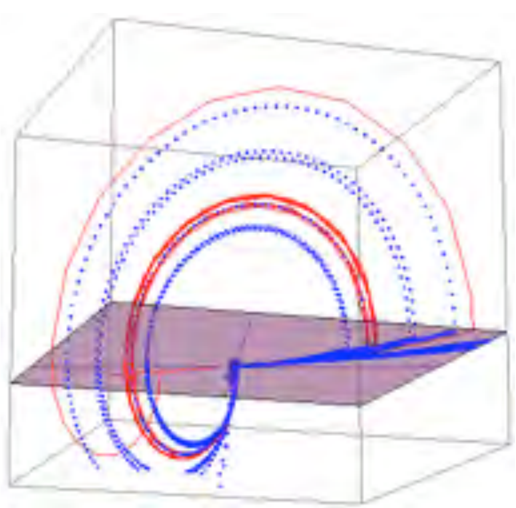
- how do we define it?

Set-valued flow
(chaos => transitive & sensitive,

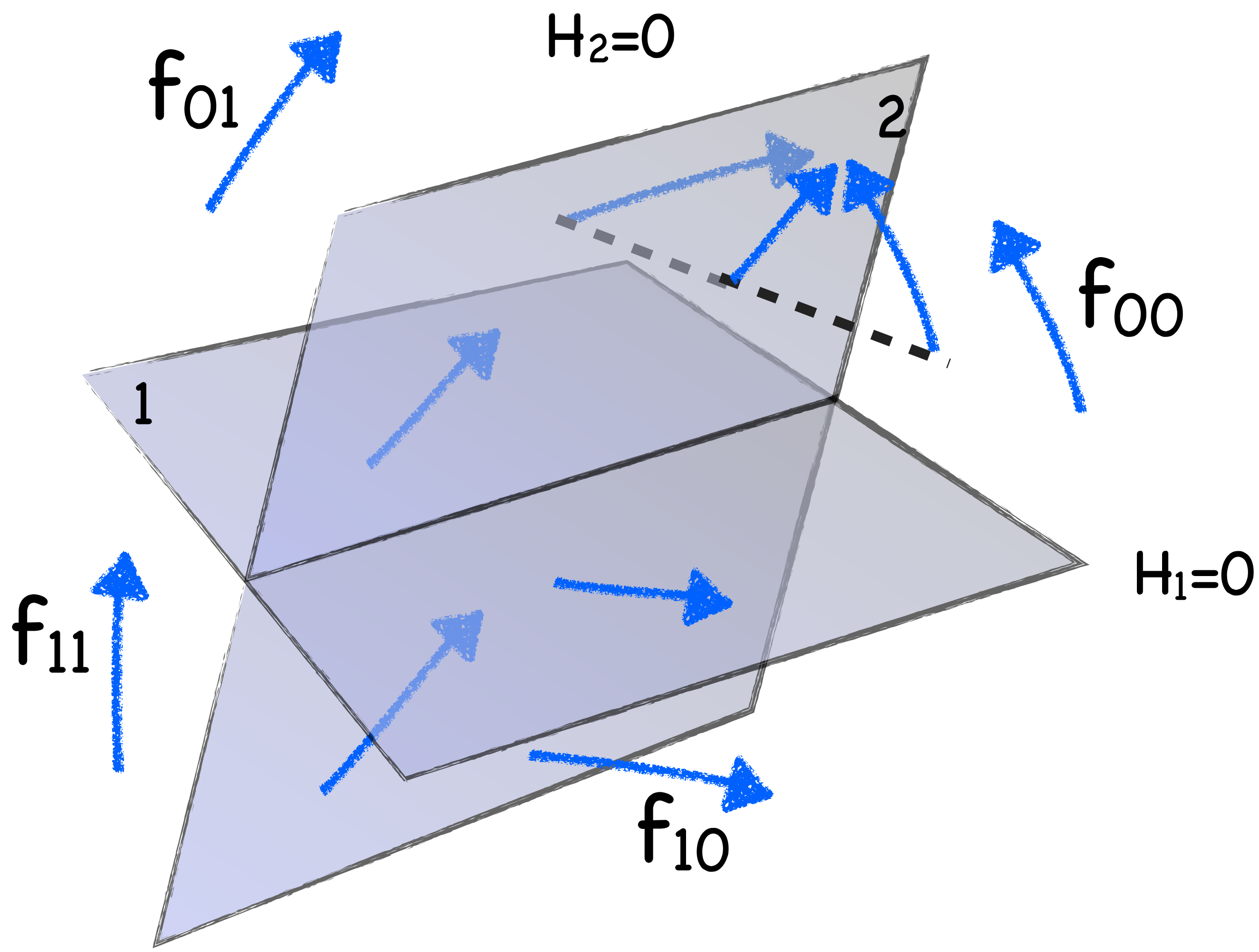
the full set consists of a single dense periodic orbit

infinite sensitivity to initial conditions)

- is it robust?



Beyond the plane



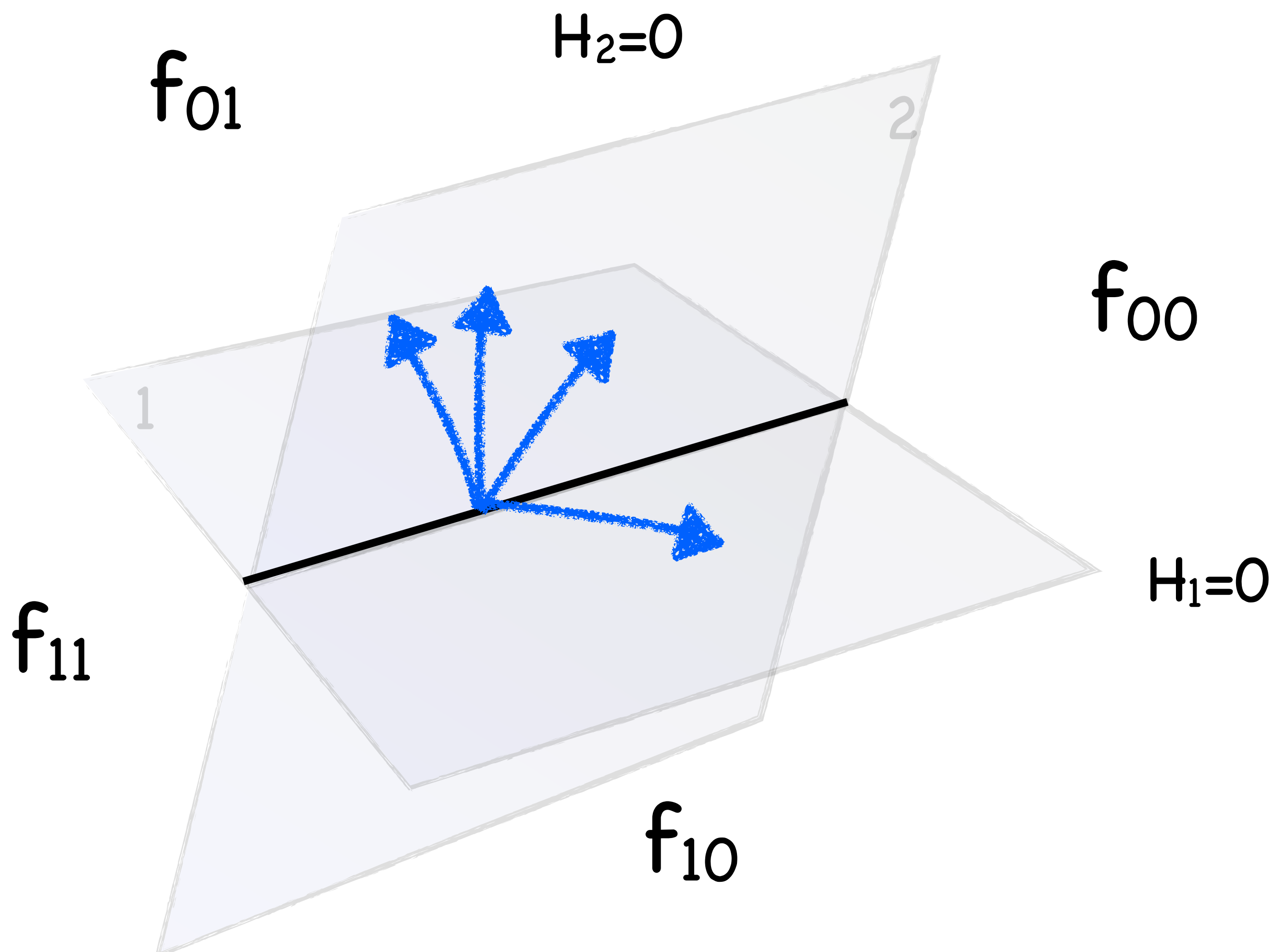
$$\frac{d}{dt}\underline{x} = \lambda_1 f_0 + (1-\lambda_1) f_1$$

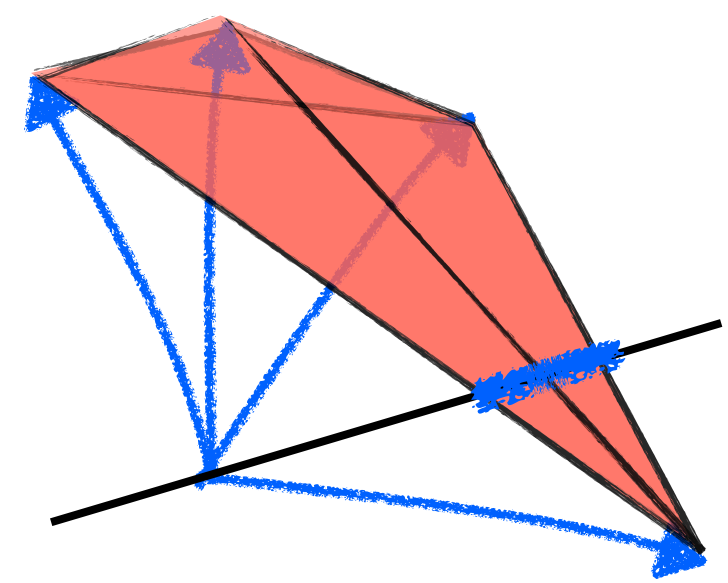
for sticking on $H_r=0$,
solve tangency condition

$$\frac{dx}{dt}\cdot\nabla H_r(x)=0$$

$r=1,2$

Beyond the plane : Convex combination





$$\frac{d}{dt}\underline{x} = \sum_k \lambda_k f_k \quad \text{with} \quad \sum \lambda_k = 1$$

for sticking on $H_r=0$,
solve tangency condition

$$\frac{dx}{dt} \cdot \nabla H_r(x) = 0$$
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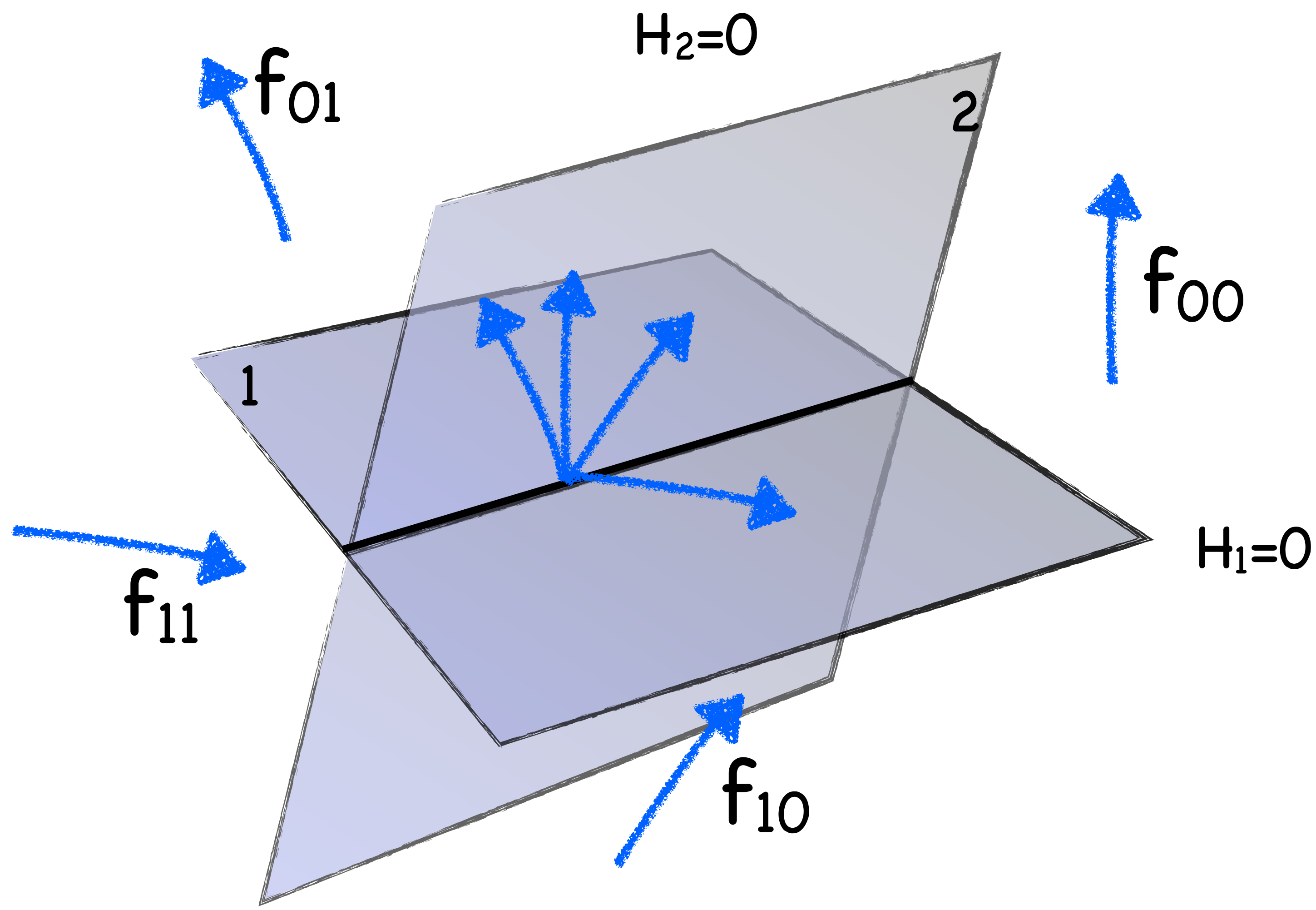

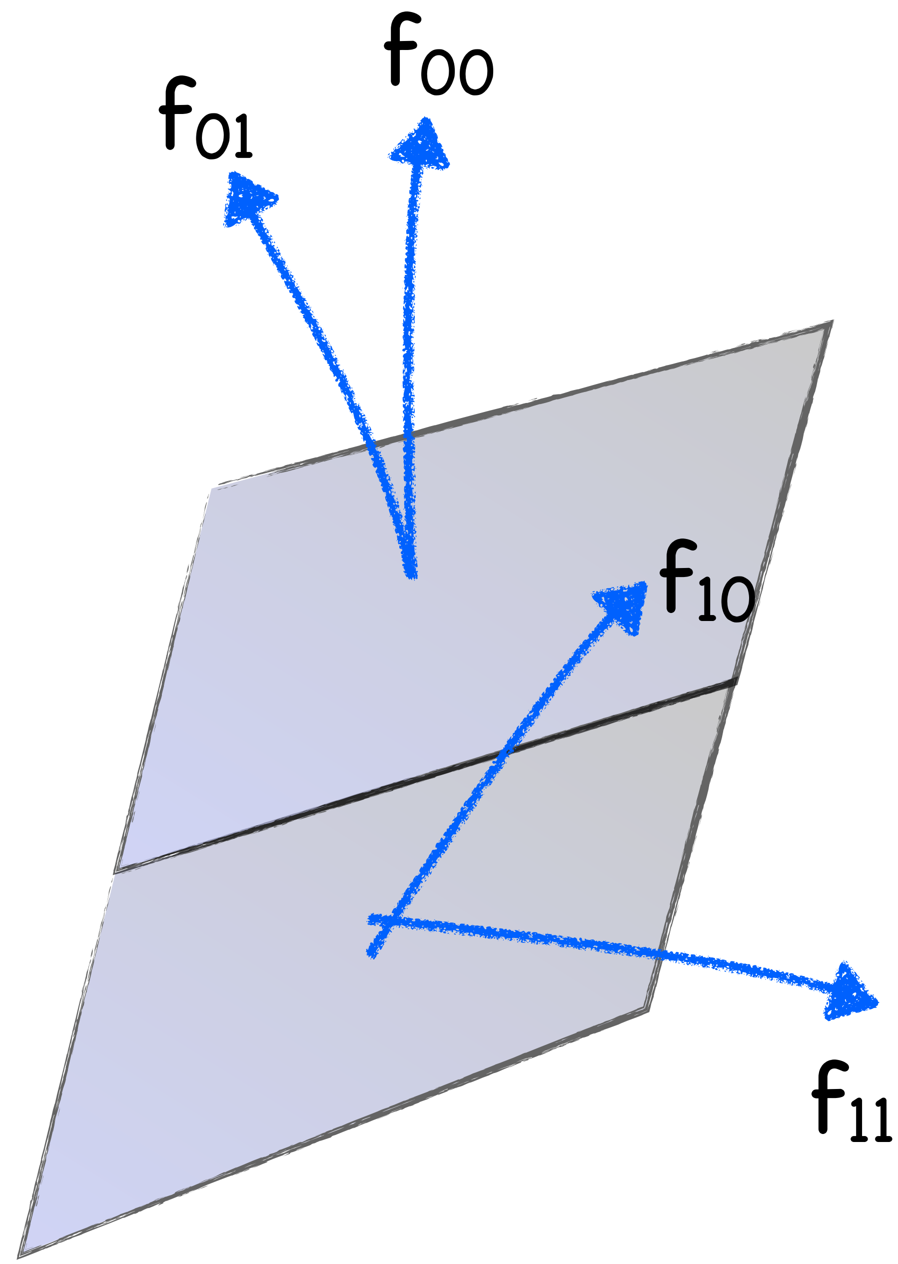

$$\frac{d}{dt}\underline{x} = \sum_k \lambda_k f_k \qquad \text{with} \quad \sum \lambda_k = 1$$

for sticking on $H_r=0$,
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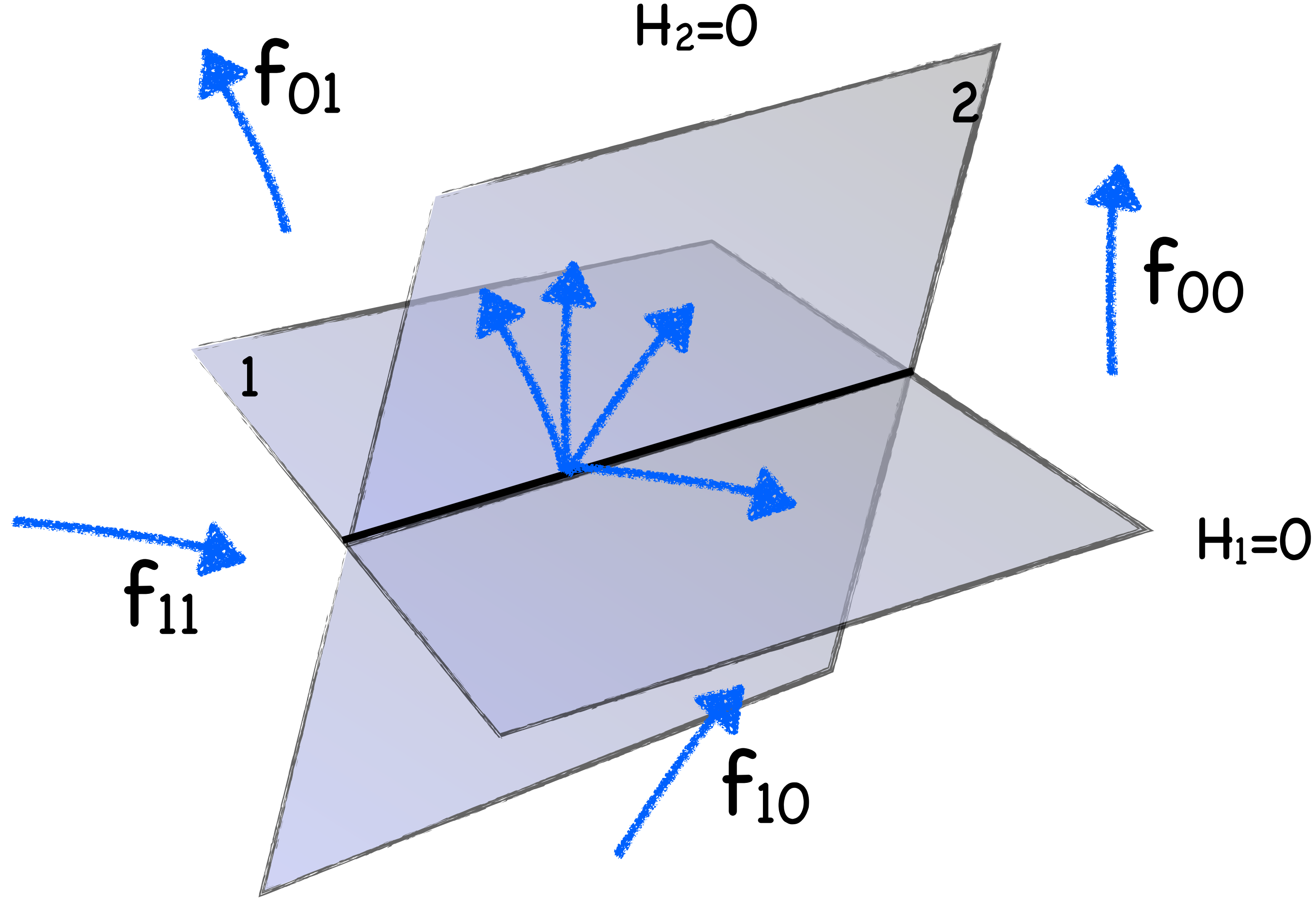
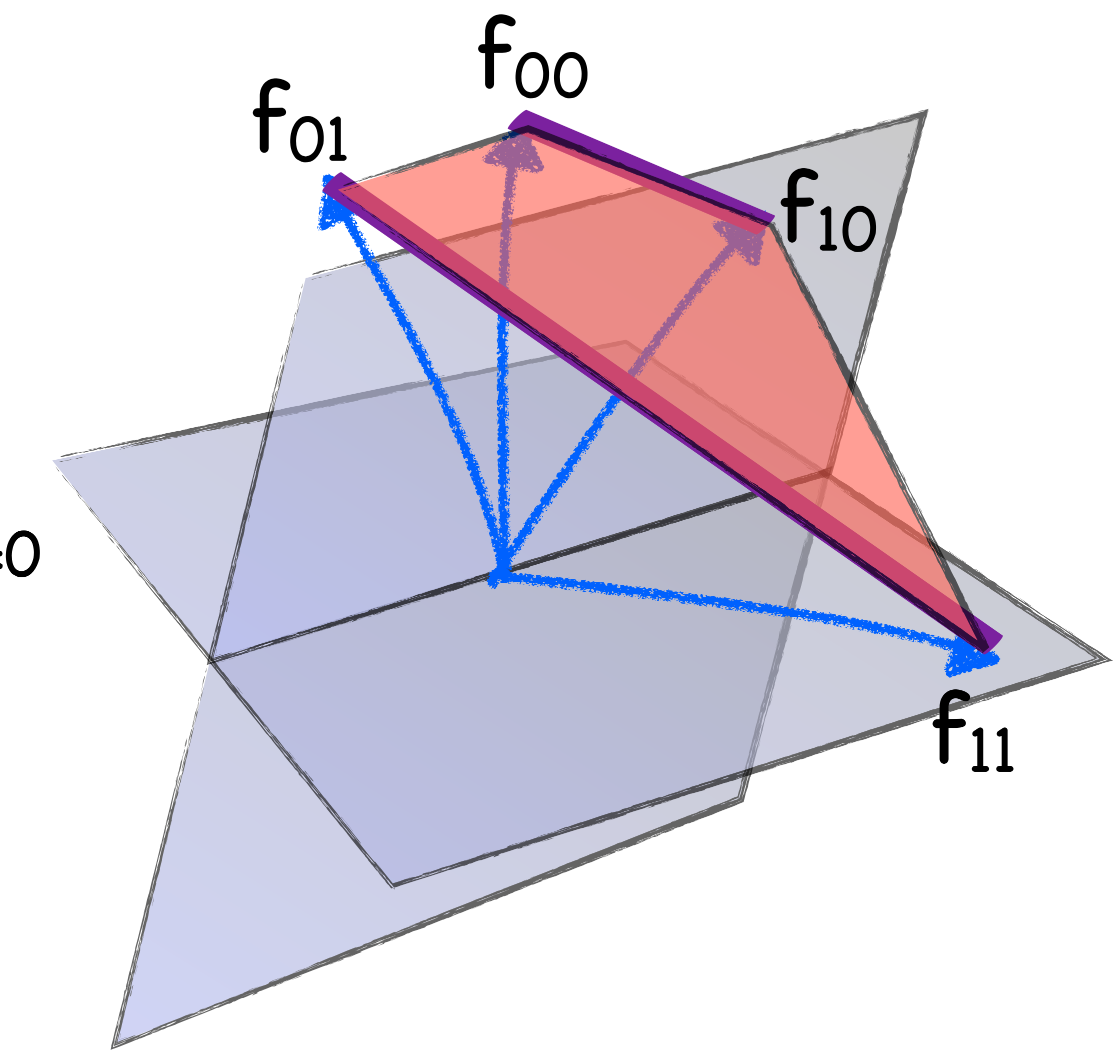
Beyond the plane : Convex ~~combination~~ canopy





$$\frac{d}{dt}\underline{x} = \sum_k \lambda_k f_k \qquad \text{with} \quad \sum \lambda_k = 1$$

$$= \lambda_1[\, \lambda_2 f_{00} + (1-\lambda_2) f_{01} \,] + (1-\lambda_1)[\, \lambda_2 f_{10} + (1-\lambda_2) f_{11} \,]$$

for sticking on $H_r=0$,
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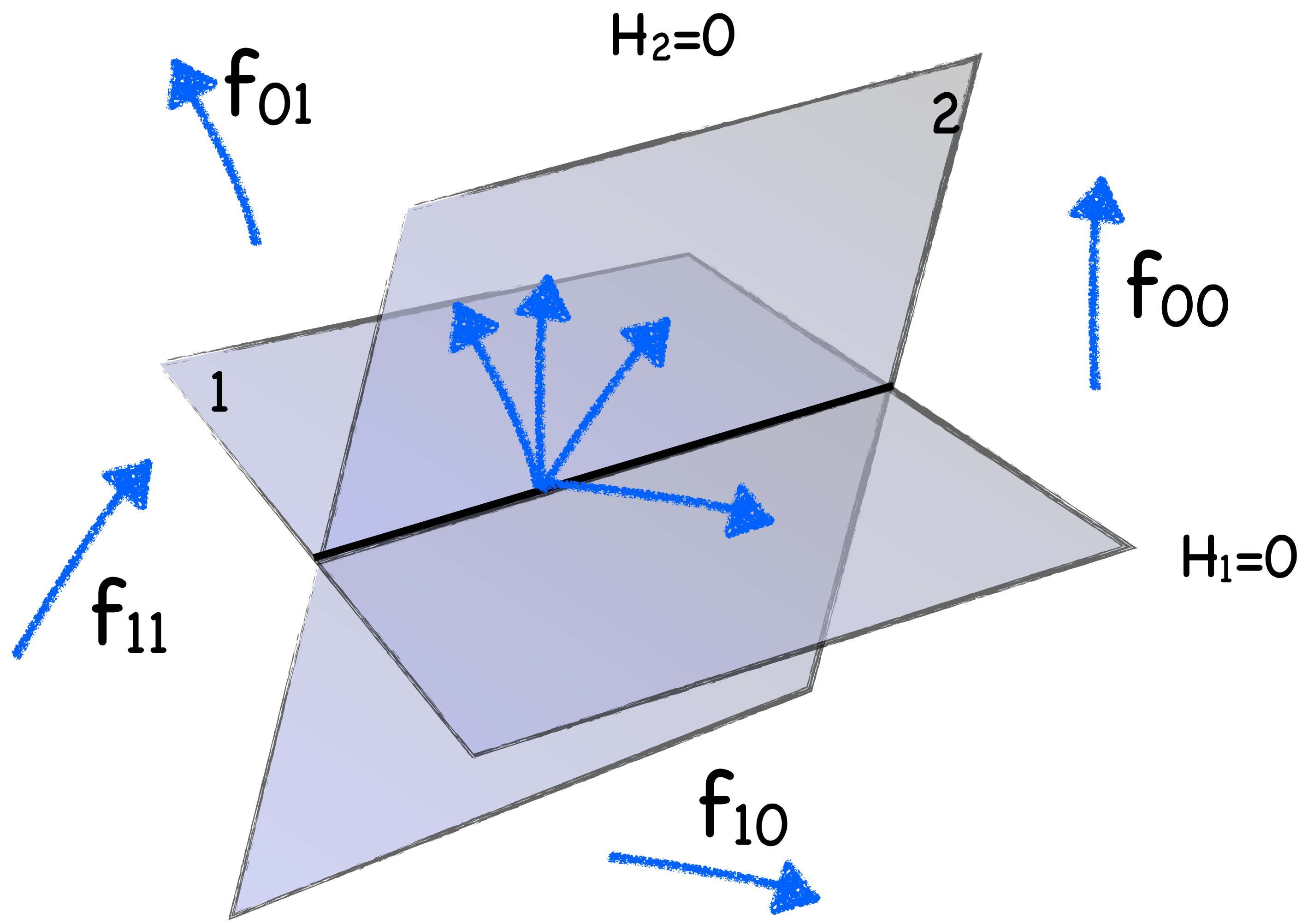

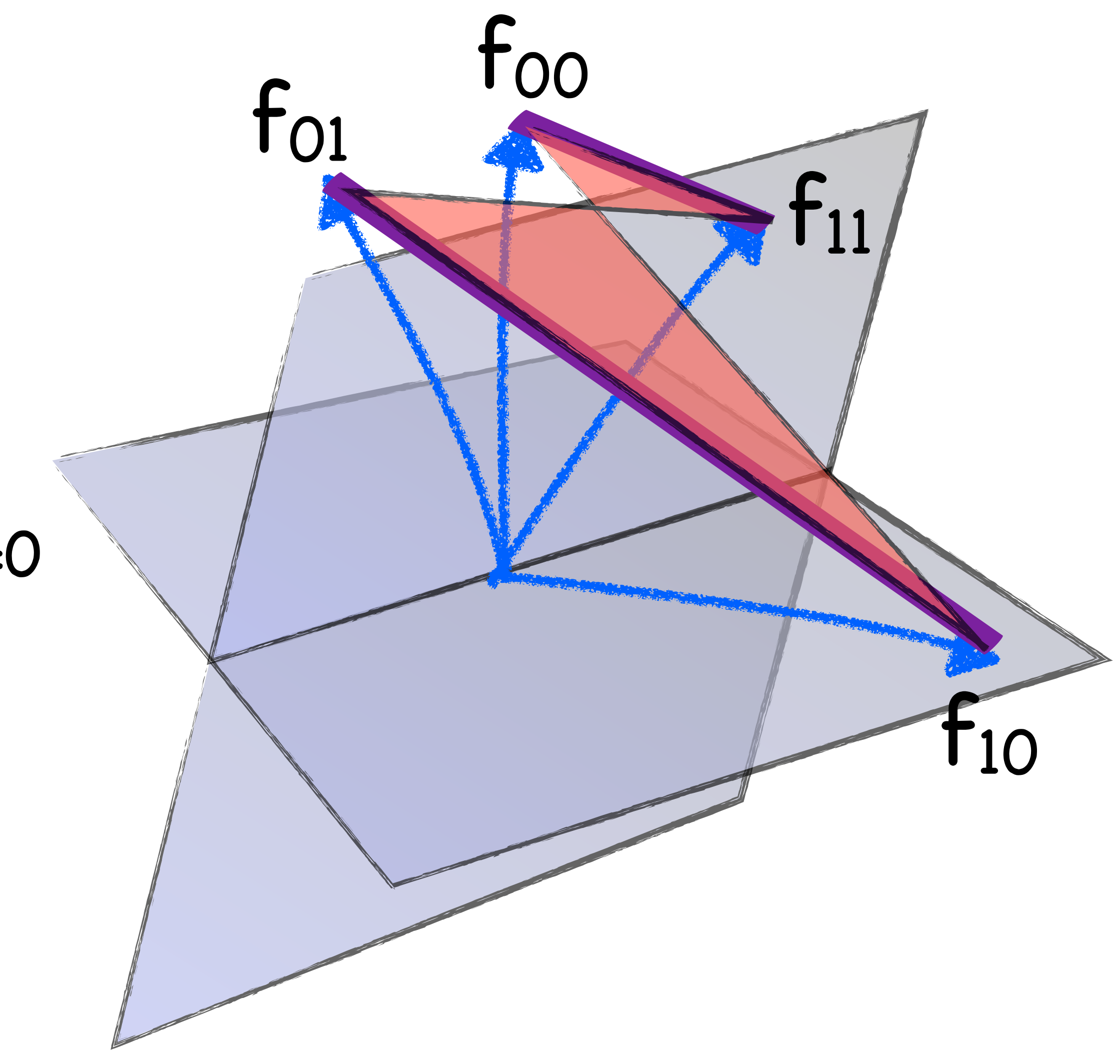

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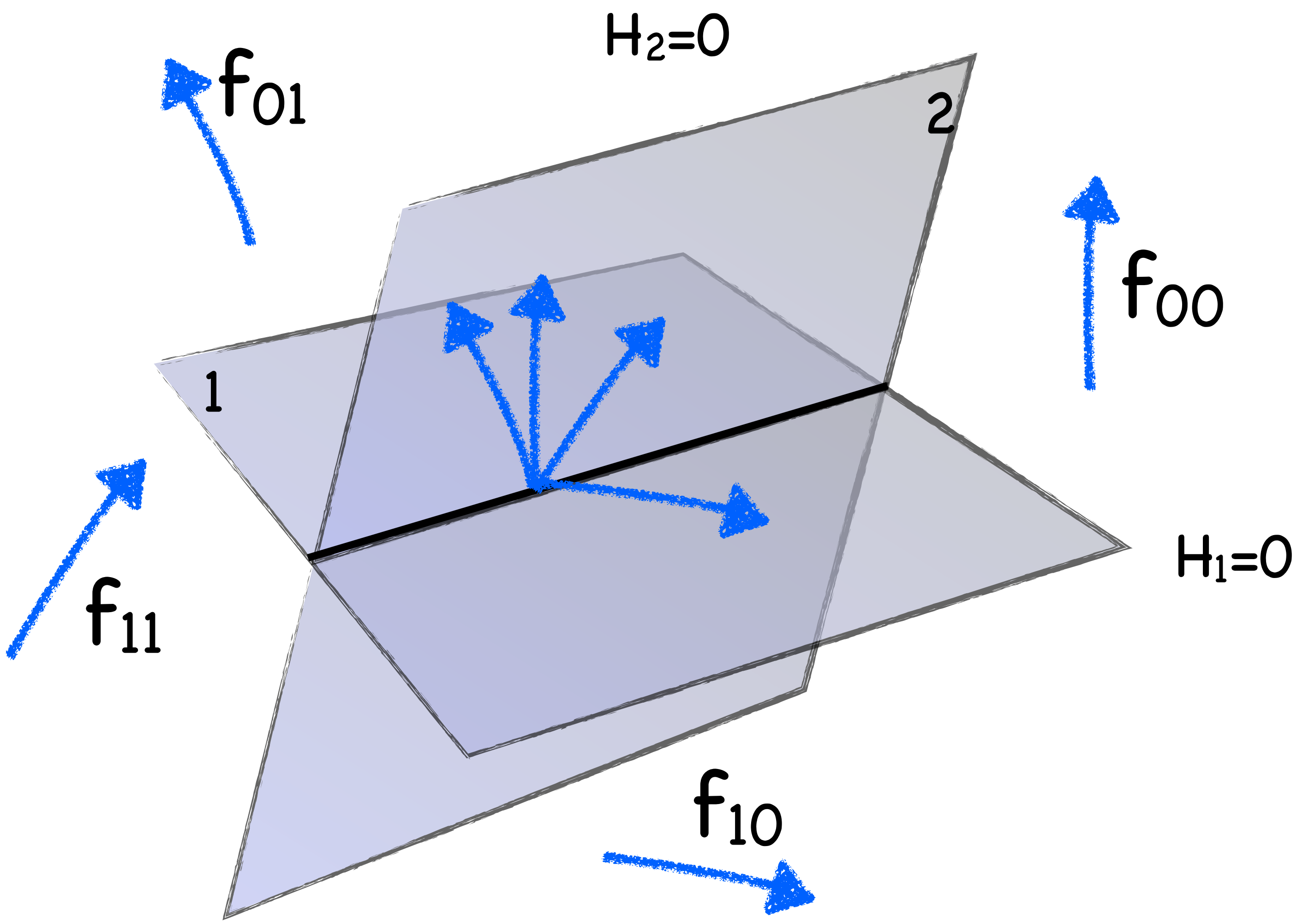
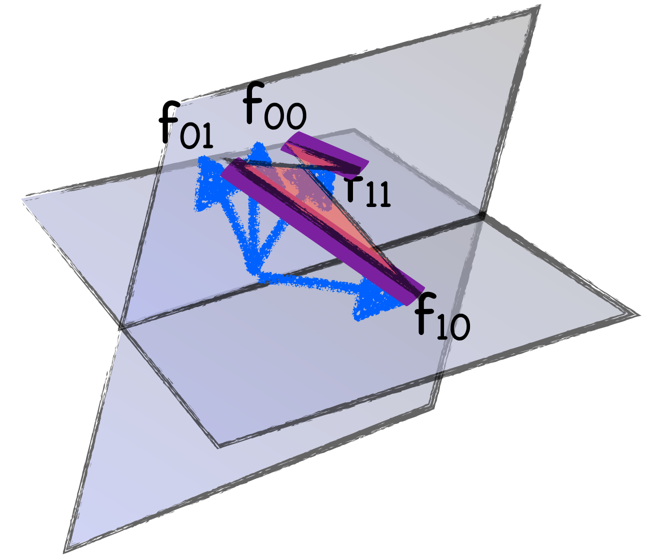
$$= \lambda_1[\, \lambda_2 f_{00} + (1-\lambda_2) f_{01} \,] + (1-\lambda_1)[\, \lambda_2 f_{10} + (1-\lambda_2) f_{11} \,]$$

for sticking on $H_r=0$,
solve tangency condition

$$\frac{dx}{dt} \cdot \nabla H_r(x) = 0 \qquad r=1,2$$

Beyond the plane : Convex combination canopy





for sticking on $H_r=0$,
solve tangency condition

$$\frac{dx}{dt} \cdot \nabla H_r(x) = 0$$

$r=1,2,\ldots,k$

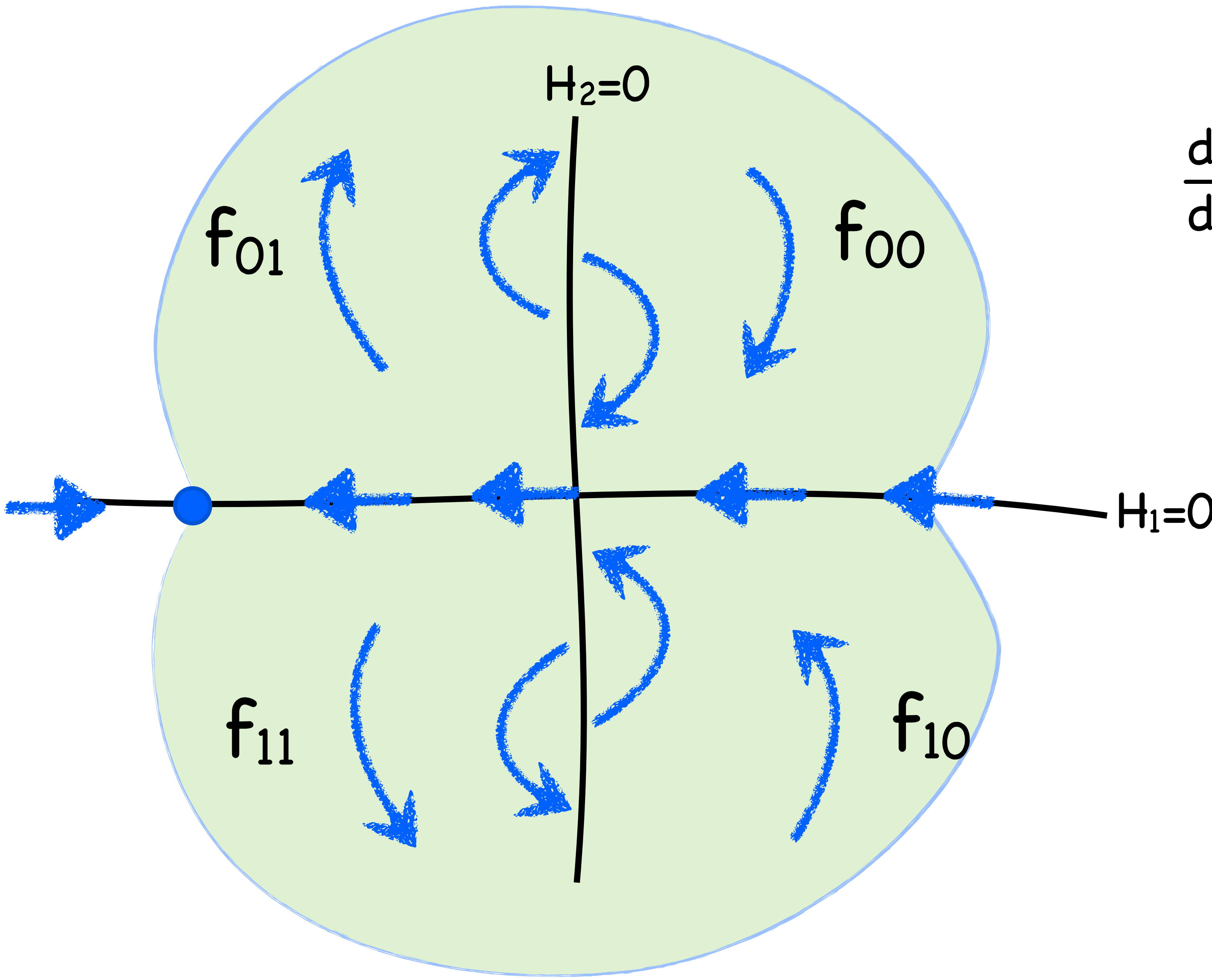
$$\frac{d}{dt}\underline{x} = \sum_k \lambda_k f_k \qquad \text{with} \quad \sum \lambda_k = 1$$

$$= \lambda_1[\,\lambda_2 f_{00} + (1-\lambda_2) f_{01}\,] + (1-\lambda_1)[\,\lambda_2 f_{10} + (1-\lambda_2) f_{11}\,]$$

$$= \lambda_1 \ldots \lambda_2 \ldots \lambda_3 f_{..0} + \ldots \quad = \quad . \; . \; .$$

$$= \sum_{p_i=0,1} \lambda_1^{p_1} \lambda_2^{p_2} \ldots \lambda_k^{p_k} f_{p_1 p_2 \cdots p_k} \qquad \text{where} \qquad \lambda_i^0 = \lambda_i \qquad \lambda_i^0 = 1-\lambda_i$$

Beyond determinism : The non-deterministic chaotic cross



$$\frac{d}{dt}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}c_1-1+|x_2|+a\,\mathrm{sign}(x_1)\\c_2-b\,\mathrm{sign}(x_1x_2)\end{pmatrix}$$

$H_1(x)=x_2 \qquad H_2(x)=x_1$

for sticking on $H_r=0$, solve tangency condition

$$\frac{dx}{dt}\cdot\nabla H_r(x)=0$$

$r=1,2$

$$\frac{d}{dt}\underline{x} = \lambda_1[\,\lambda_2 f_{00}+(1-\lambda_2)f_{01}\,]+(1-\lambda_1)[\,\lambda_2 f_{10}+(1-\lambda_2)f_{11}\,]$$

$$= \sum_{p_i=0,1}\lambda_1^{p_1}\lambda_2^{p_2}\ldots\lambda_k^{p_k}\,f_{p_1p_2\cdots p_k}$$

where $\lambda_i^0=\lambda_i \qquad \lambda_i^0=1-\lambda_i$