Quasiconformal surgery in Holomorphic dynamics

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Plan of the talk

1. Brief introduction to holomorphic dynamics
2. Quasiconformal mappings
3. Surgery: constructing dynamical models
4. Applications

1. Holomorphic dynamics (1-D)

We study those dynamical systems generated by the iteration of a **holomorphic** map on a Riemann surface (usually $\mathbb{C}, \mathbb{C}^*$ or $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$).

- Original motivation: Newton's method (1900's).
- It evolved towards a general theory, starting with the simplest cases.
- It persisted: beautiful mathematics + challenging questions (many still open).
- Great success in describing iteration of real and circle maps by complexification.
- Tools from many areas: complex analysis, algebraic geometry, topology, dynamical systems,....

Holomorphic dynamics

Iteration is interesting and nontrivial in the following cases:

- $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$
  - Rational maps $F(z) = \frac{P(z)}{Q(z)}$, P, Q polynomials, $\deg(F) \geq 2$.
  - Particular case: polynomials (rational maps for which ∞ is superattracting with no preimages)
- $F : \mathbb{C} \to \mathbb{C}$ (or $F : \mathbb{C}^* \to \mathbb{C}^*$)
  - Entire transcendental maps (or hol in $\mathbb{C}^*$) – ∞ (and 0) essential singularity – $e^z, sin(z), \cdots$ $(e^{z+1/z}, ...)$
- $F : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$
  - Meromorphic transcendental maps – ∞ is an essential singularity – $e^z/P(z), Tan(z), \cdots$

OBSERVE: Not diffeomorphisms. The degree is always ≥ 2!!! BUT: Local homeos except for a discrete number of points.

As in all dynamical systems, we try to understand:

- Dynamical plane: Asymptotic behaviour of orbits in the phase space for a given map or a class of maps;
- Parameter space: Stability issues and bifurcations for holomorphic families of maps.

We use very powerful tools available because of differentiability:

- Hyperbolic geometry, Coverings theory, complex analysis (Riemann theorem, open mapping principle, maximum principle, analytic continuation, discreteness of zeroes, mapping properties,. . .) . . .

Local study:

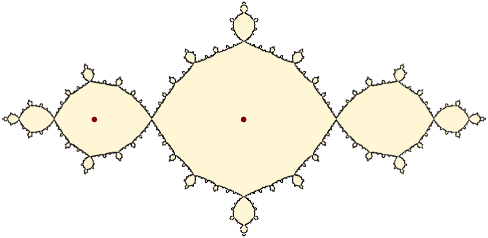
- Attracting cycles, repelling cycles, neutral cycles (parabolic or irrational), ... NO SADDLES.

Global study:

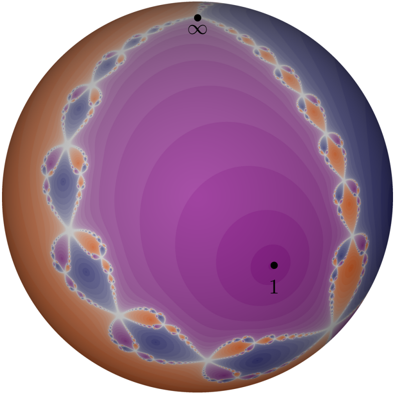
- There is a dynamical partition of the phase space into two **completely invariant** sets:
  - The **Fatou set:** open set formed by stable orbits. Connected components are basins of attraction, (irrational) rotation domains (Siegel disks or Herman rings), preimages of the above, wandering domains, etc. etc. – completely classified.
  - The **Julia set** = closure of the repelling cycles. Orbits are chaotic.
- This partition is preserved under conjugacies of all kinds.

Examples

Attracting basins



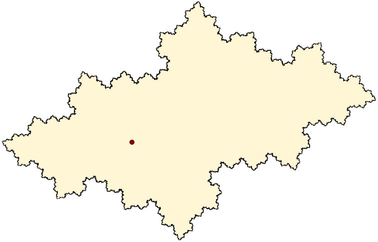
$z \mapsto z^2 - 1$



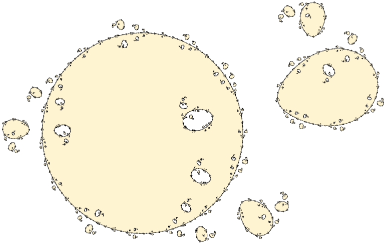
Newton's method for $z(z-1)(z-i)$

Examples

Attracting basins

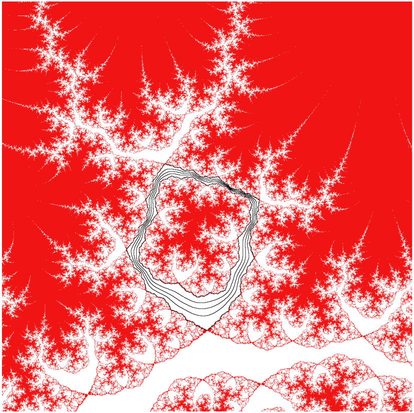


$z \mapsto z^2 + \varepsilon$



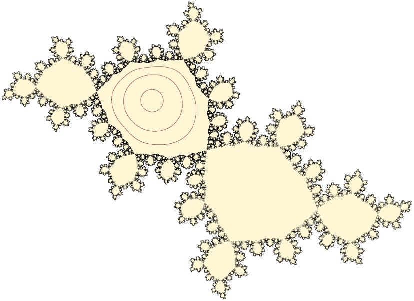
Blaschke product (rational)

$z \mapsto \lambda e^z$

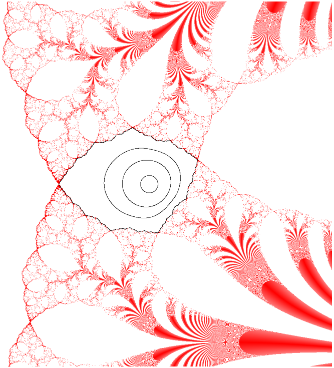


Complex standard map $\lambda z e^{b(z-1/z)}$

Examples
Siegel disks



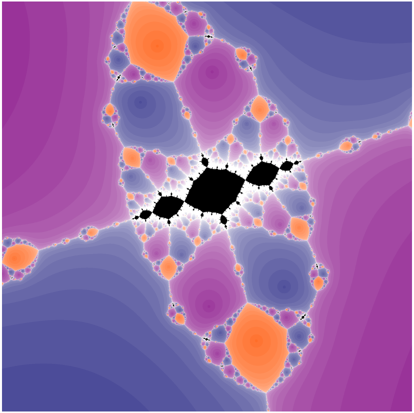
$z \mapsto z^2 + c$



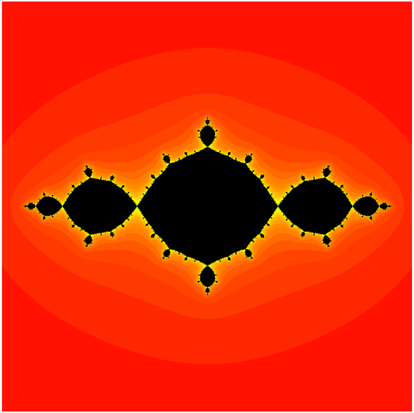
$z \mapsto \lambda z e^z$

Examples

Why do they look so similar????? Keep watching!



Newton of a cubic



$z \mapsto z^2 - 1$

2. Quasiconformal Mappings

- Quasiconformality is a degree of regularity : the right one to study (e.g.) structural stability of holomorphic maps.
  - Topological conjugacies between holomorphic maps can be upgraded to qc conjugacies.

- $QC \Rightarrow \mathcal{C}^0$

- $QC \not\Rightarrow \mathcal{C}^1$ and $\mathcal{C}^1 \not\Rightarrow QC$

- They are very flexible (as opposed to holomorphic maps) — good to construct models.

Quasiconformal maps

- If F is conformal at z_0, it preserves angles between curves crossing at z_0, because

$$DF(z_0) : \mathbb{C} \to \mathbb{C} \text{ is a complex linear map } z \mapsto f'(z_0)\, z.$$

- In general , if F is differentiable at z_0:

$$DF(z_0) : \mathbb{C} \to \mathbb{C} \text{ is a linear map } z \mapsto a\,z + b\,\overline{z}.$$

  with $a = \partial_z F(z_0)$ and $b = \partial_{\overline{z}} F(z_0)$

- The quantities that measure the angle distortion at z_0 are (assume o.p.):
  - $\mu(z_0) = \frac{b}{a} = \frac{\partial_{\overline{z}} F}{\partial_z F}(z_0) \in \mathbb{D}$ and
  - $K(z_0) = \frac{|a|+|b|}{|a|-|b|} \in [1, \infty)$

Definition

A map $f : U \to V$, $U, V \subset \mathbb{C}$ is $K-$quasiconformal if :

- *f is an orientation preserving homeomorphism,*
- *f is absolutely continuous on lines ($\Rightarrow$ differentiable a.e.)*
- *$K(z) < K < \infty$ a.e. where defined.*

Examples:

- f conformal $\Leftrightarrow$ f is $1-$quasiconformal $\Leftrightarrow$ $b = 0$.
- Every o.p. $\mathbb{R}^2-$linear map is qc ($K(z)$ ctant)
- $\mathcal{C}^1$ o.p. homeos are quasiconformal on any compact set.

Quasiconformal/Quasiregular maps

(1) Pasting quasiconformal maps along "reasonable" curves preserves quasiconformality.
(This gives great flexibility!)

(2) If f is K_1-qc and g is K_2-qc $\Rightarrow$ $f \circ g$ is $K_1 \cdot K_2-$qc

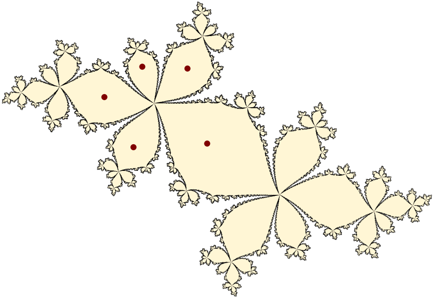
(3) If f is $K-$qc then f^{-1} is also $K-$qc.

Definition

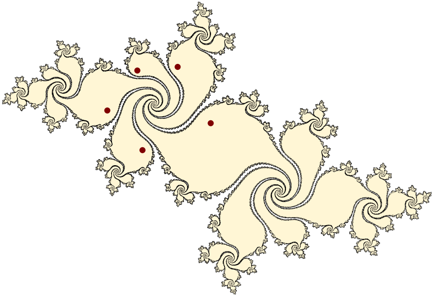
$g : U \to V$ is a $K-$quasiregular map if g is locally $K-$quasiconformal except at a discrete set of points.

conformal	$\longleftrightarrow$	holomorphic
quasiconformal	$\longleftrightarrow$	quasiregular

Quasiconformal conjugate polynomials



$z \mapsto z^2 + c_1$



$z \mapsto z^2 + c_2$

Surgery: constructing dynamical models

Quasiregular maps often are used as dynamical models of holomorphic maps.
We say that a qr map f is a dynamical model of F holomorphic if there exists φ qc such that

$$\varphi \circ f = F \circ \varphi$$

or equivalently, the following diagram conmutes

$$\begin{array}{ccc} U & \xrightarrow{f(qr)} & V \\ \varphi(qc)\downarrow & & \downarrow\varphi(qc) \\ U & \xrightarrow{F(hol)} & V \end{array}$$

We write $f \underset{qc}{\sim} F$

3. Surgery: constructing dynamical models

Question: When is a quasiregular map a dynamical model for some holomorphic map??

The most complete answer is the following.

Theorem (Sullivan's principle)

$f : U \to \mathbb{C}$ is quasiconformally conjugate to some holomorphic $F : U \to \mathbb{C}$

if and only if

$\exists K < \infty$ such that for all $n \geq 1$, the iterates $\{f^n\}$ are (uniformly) $K-$quasiregular.

Moreover, F is unique up to conformal conjugacies.

The proof is based on the celebrated theorem of Ahlfors,Bers,Bojarski and Morrey which proves the existence of solutions (uncer certain conditions) of the Beltrami PDE

Surgery: constructing dynamical models

Examples

(1) Linear maps of $\mathbb{R}^2$ are never qc conjugate to conformal maps unless they are conformal themselves.

(2) (Shishikura's principle) Suppose

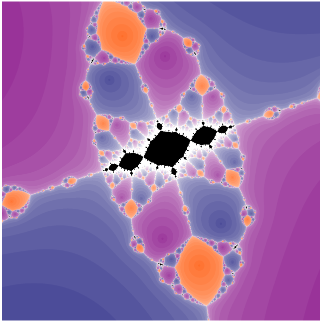
- f is holomorphic in $\mathbb{C} \setminus X$ and $K-$quasiregular in X
- $f^j(X) \cap X = \emptyset$ for all $j \geq N$ (orbits pass through X at most N times).

Then, $\{f^n\}$ are uniformly K^N-quasiregular and hence $f \underset{qc}{\sim} F$ for some F holomorphic.

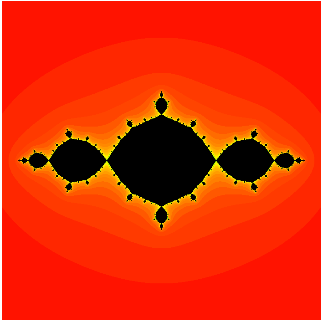
(PICTURE)

This tells us that we can change a holomorphic map in certain regions by a quasiconformal one, as long as the dynamics are controlled.

Question: Why do we see polynomial Julia sets in the dynamical plane of non-polynomial mappings?

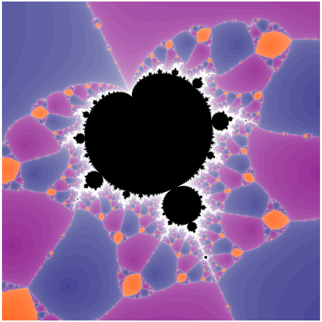


Newton's method of a cubic pol.

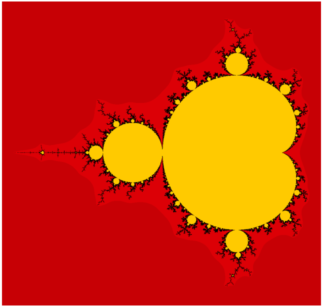


$z \mapsto z^2 - 1$

Even in parameter space, we see that bifurcations occur with the same patterns!



Newton parameter space



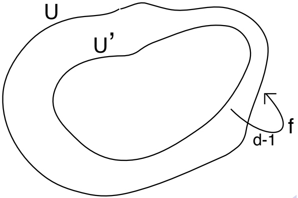
Bifurcations of $z \mapsto z^2 + c$

The answer is given by the Straightening theorem, one of the first surgery applications.

Theorem (Douady and Hubbard, 1985)

Let $U', U \subset \mathbb{C}$ be topological discs such that $\overline{U'} \subset U$. Suppose $f : U' \to U$ is a proper holomorphic map of degree $d \geq 2$. Then, there exists a polynomial $P(z)$ of degree d such that

$$f \underset{qc}{\sim} P \quad \text{on } U'.$$

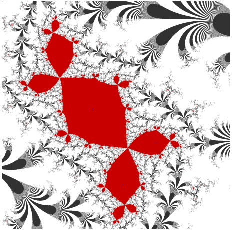


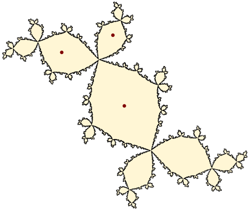
In particular, there is a (qc) homeomo between the small filled Julia set of f

$$\mathcal{K}_f = \{z \in U' \mid f^n(z) \in U' \text{ for all } n \geq 0\}.$$

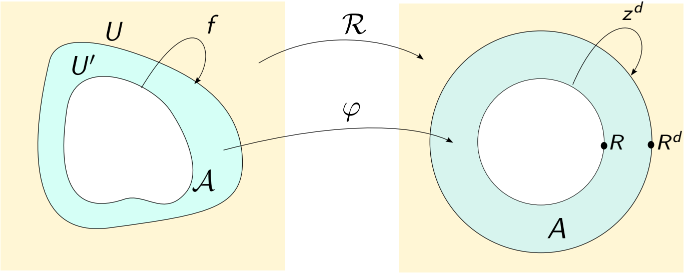
and the filled Julia set of P

$$\mathcal{K}_P = \{z \in \mathbb{C} \mid f^n(z) \nrightarrow \infty\}.$$





(1) Let $\mathcal{R} : \widehat{\mathbb{C}} \setminus \overline{U} \to \widehat{\mathbb{C}} \setminus \mathbb{D}_{R^d}$ be a conformal map. It extends to $\varphi : \partial U \to \mathbb{S}_{R^d}$.



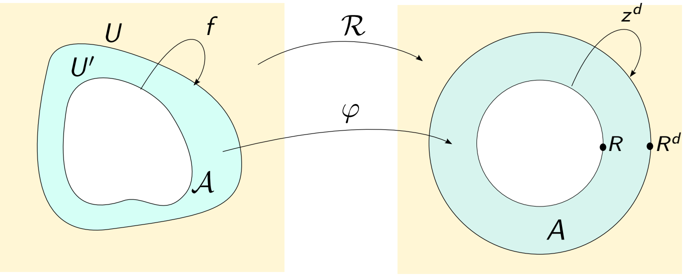
(2) Define $\varphi : \partial U' \to \partial \mathbb{D}_R$ so that

$$\varphi^{-1}\left(\varphi(z)\right)^d = f(z).$$

Extend to $\varphi : \overline{\mathcal{A}} \to \overline{A}$ $K-$quasiconformally (linear interpolation).

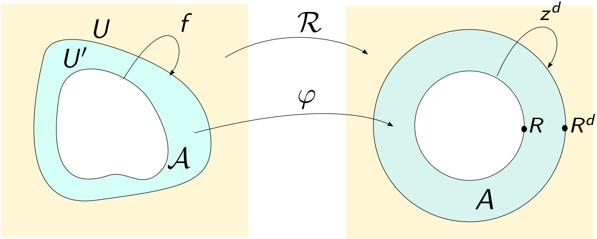
(3) Define a new map (the model)

$$g = \begin{cases} f & \text{on} \quad U \\ \mathcal{R}^{-1}(\varphi(z))^d & \text{on} \quad \overline{\mathcal{A}} \\ \mathcal{R}^{-1}(\mathcal{R}(z))^d & \text{on} \quad \mathbb{C} \setminus U \end{cases}$$



g is a continuous map of $\mathbb{C}$ holomorphic outside $\mathcal{A}$.

(4) All iterates $\{f^m\}$ are uniformly $K-$quasiregular – orbits pass through $\mathcal{A}$ at most once.



(5) (Sullivan's principle) There exists F rational such that $g \underset{qc}{\sim} F$. Normalize so that the conjugacy fixes ∞.

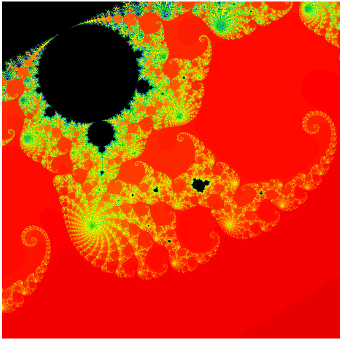
(6) F is entire and ∞ is a superattracting fixed point with no preimages. HENCE F is a polynomial (of degree d).

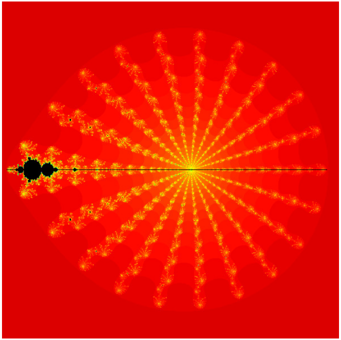
Other applications

Surgery techniques have been used to prove plenty of results of very different nature.

- Nonexistence of wandering domains for rational maps (Sullivan 82)
- Construction of examples of and counterexamples to dozens of conjectures.
  - There exist Siegel discs and Herman rings of any Brjuno rotation number.
  - For rotation numbers of bounded type, Siegel disks have Jordan boundaries with a critical point (rational maps).
- Connectivity of the Julia set under certain hypothesis.
- Bound the number of non-repelling cycles for a given system.
- Parametrization of structurally stable components in parameter spaces.
- Constructing homeomorphisms between parameter spaces of different families.
- and a very large ETC.

Thank you for your attention!!!!





B. Branner, N.Fagella, *Quasiconformal surgery in holomorphic dynamics*, Cambridge Studies in Advanced Mathematics, Cambridge University Press. *To appear*.