Singular perturbations in complex dynamics

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Outline

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Singular perturbations

We take a well understood (dynamically) function $f_0(z)$ and we perturb it adding one pole.

$$f_\lambda(z) = f_0(z) + \frac{\lambda}{(z-a)^d}, \qquad \text{where } \lambda \in \mathbb{C}.$$

If f_0 is a polynomial of degree n, then $f_\lambda(z)$ has degree $n+d$ for $\lambda \neq 0$. They appear new critical points. The dynamics of f_λ is much richer than the dynamics of f_0, although some portions of f_0 persist depending on the location of the pole.

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Singular perturbations

Main Example:

$$f_\lambda(z) = z^n + \frac{\lambda}{z^d}, \text{ with } n, d \geq 2$$

First introduced by C. McMullen in 1988 as an example of a Julia set as a Cantor set of circles. Later on R. Devaney et al. study sistematically this family and other singular perturbations.

Other examples:

1. $f_\lambda(z) = z^n + \frac{\lambda}{(z-a)^d}$, with $a \neq 0$
2. $f_\lambda(z) = z^n + c + \frac{\lambda}{z^d}$
3. $f_\lambda(z) = z^n + c + \frac{\lambda}{\prod_{i=0}^{N-1}(z-c_i)^{d_i}}$

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Singular perturbations

Dynamics of the map $f_0(z) = z^n$ with $n \geq 2$.

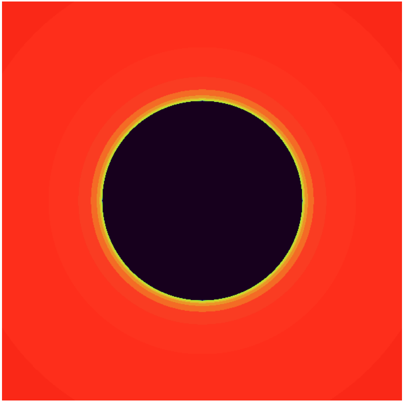

Figura: Dynamical Plane of z^n with $n \geq 2$.

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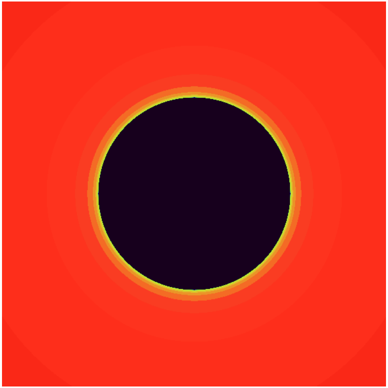

Fatou set of f_0

Set of points with stable behaviour. $F(f_0) = A(0) \cup A(\infty)$.
$A(0) = \{z \in \mathbb{C}; |z| < 1\}$.
$A(\infty) = \{z \in \mathbb{C}; |z| > 1\}$

Julia set of f_0

Set of points with unstable behaviour.
$J(f_0) = \hat{\mathbb{C}} \setminus F(f_0) = \mathbb{S}^1$.

Outline

Basic properties of the McMullen family

$$f_\lambda(z) = z^n + \frac{\lambda}{z^n}, \qquad \text{with } n \geq 2$$

∞ is still a superattracting fixed point

$$A(\infty) = \{z \in \hat{\mathbb{C}}\,;\, f_\lambda^{\circ n}(z) \to \infty\}$$

- B_λ is the immediate basin of attraction of ∞. B_λ Is the he c.c. of $A(\infty)$ containing ∞.
- $f_\lambda^{-1}(\infty) = \{\infty, 0\}$.
- T_λ is the trap door. T_λ is the c.c. of $A(\infty)$ containing 0.

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Critical points of f_λ

- ∞ is a superattracting fixed point.
- 0 is a preimage of ∞.
- The rest of the critical points: $f'_\lambda(z) = 0$

$$c_\lambda = \sqrt[2n]{\lambda}$$

The image of a critical point is a critical value $v_\lambda = f_\lambda(c_\lambda)$.
Observation: There are $2n$ free critical points but only two critical values $v_\lambda = \pm 2\sqrt{\lambda}$.

Basic properties of the McMullen family

$$f_\lambda(z) = z^n + \frac{\lambda}{z^n}, \qquad \text{with } n \geq 2$$

Symmetries

- If n is even, then $f_\lambda(-z) = f_\lambda(z)$.
- If n is odd, then $f_\lambda(-z) = -f_\lambda(z)$.
- If ω verifies $\omega^{2n} = 1$, then $f_\lambda(\omega z) = \omega^n f_\lambda(z)$.
- if $H_\lambda(z) = \sqrt{\lambda}/z$, then $f_\lambda(H_\lambda(z)) = f_\lambda(z)$.

Basic properties of the McMullen family

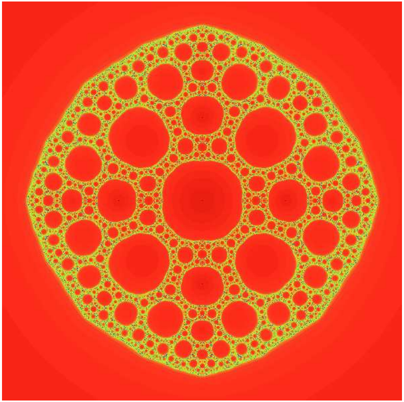

Figura: Dynamical Plane of $z^2 - 0.0625/z^2$.

Basic properties of the McMullen family

$$f_\lambda(z) = z^n + \frac{\lambda}{z^n}, \qquad \text{with } n \geq 2$$

The parameter plane ($\lambda-$plane)

- We have only one critical orbit behaviour. if c_λ is a critical point, $v_\lambda = f(c_\lambda)$ is the critical value

$$\{c_\lambda, v_\lambda, f_\lambda^{\circ 2}(c_\lambda), \cdots, f_\lambda^{\circ n}(c_\lambda), \cdots\}$$

- Possible behaviours of this critical orbit:
  1. captured $c_\lambda \in A(\infty)$, or
  2. disjoint c_λ associated to a Fatou component different from $A(\infty)$, or
  3. associated to the Julia set of f_λ. For example the critical orbit could be preperiodic (Missiurewicz parameter).

Basic properties of the McMullen family

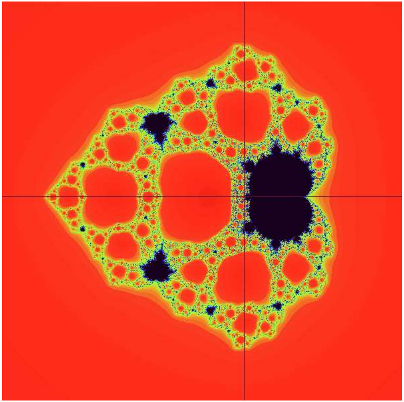

Figura: Parameter Plane of $z^2 + \lambda/z^2$.

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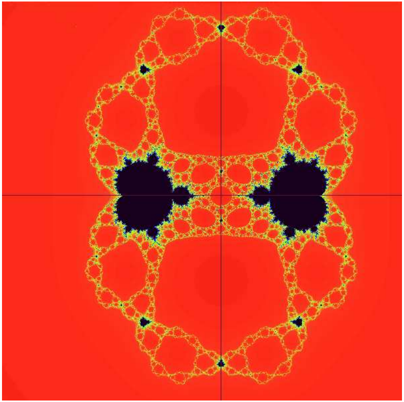

Figura: Parameter Plane of $z^3 + \lambda/z^3$.

Some results about the McMullen family

Which kind of results?

1. Dynamical plane. Topological properties of $J(f_\lambda)$.
   - Is it connected?
   - Is it locally connected?
   - Are there any model for $J(f_\lambda)$? Cantor set, Sierpiński carpet, etc...
2. Parameter plane.
   - Hyperbolic parameters, stable parameters, bifurcation parameters.
   - Limiting processes: $J(f_\lambda) \longrightarrow ?$ as $\lambda \longrightarrow 0$.
3. Measure properties
   - Hausdorff dimension of $J(f_\lambda)$? Hausdorff dimension of the set of bifurcation parameters.

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Some results about the McMullen family

Theorem (Escape Trichotomy)-R. Devaney, D. Look and D. Umisky. (2005))

Suppose that the orbits of the free critical points of f_λ tend to ∞. Then

- *If one of the critical values lies in B_λ, then $J(f_\lambda)$ is a Cantor set.*
- *If one of the critical values lies in T_λ, then $J(f_\lambda)$ is a Cantor set of simple closed curves. ($n > 2$).*
- *If one of the critical values lies in a preimage of T_λ, then $J(f_\lambda)$ is homeomorphic to the Sierpiński carpet.*

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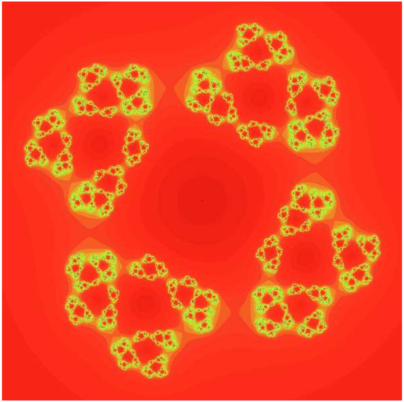

Figura: Dynamical Plane of $z^2 + \frac{0.105+0.22i}{z^2}$.

Basic properties of the McMullen family

If one of the critical values lies in T_λ, then $J(f_\lambda)$ is a Cantor set of simple closed curves.

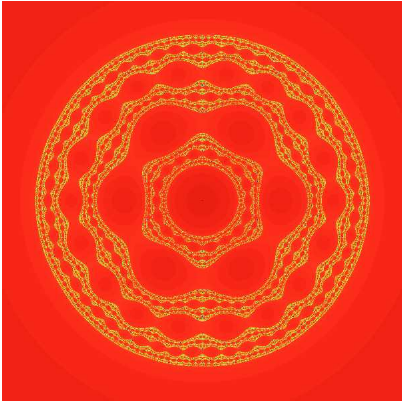

Figura: Dynamical Plane of $z^3 - 0.01/z^3$.

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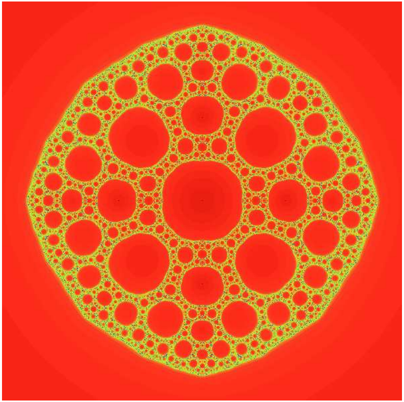

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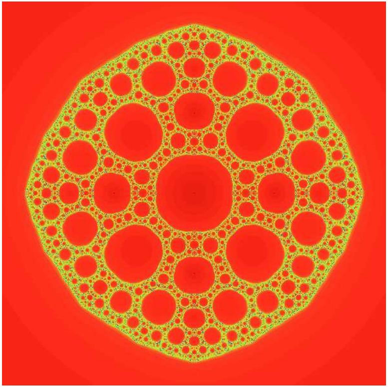

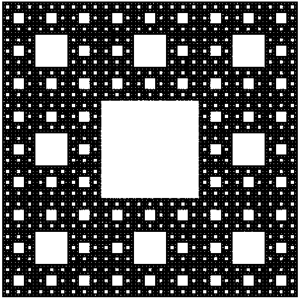

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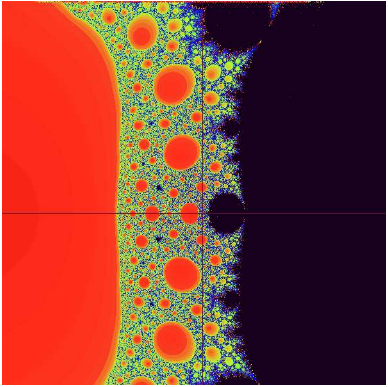
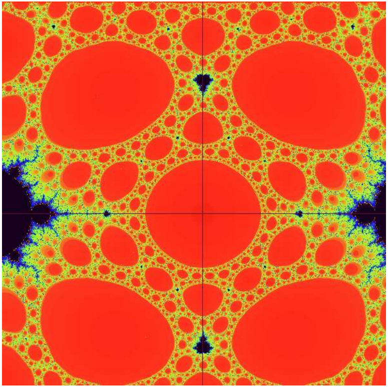
Some results about the McMullen family

Parameter plane
for $n = 2$ near the origin.



Parameter plane
for $n = 3$ near the origin.



Some results about the McMullen family

Theorem (R. Devaney and E. Russell. To appear)

If the critical values of f_λ do not lie in either B_λ or T_λ, then the Julia set of f_λ is always a connected set.

$J(f_\lambda)$ is disconnected iff

- $J(f)$ is a Cantor set $\Leftrightarrow$ $v_\lambda \in B_\lambda$
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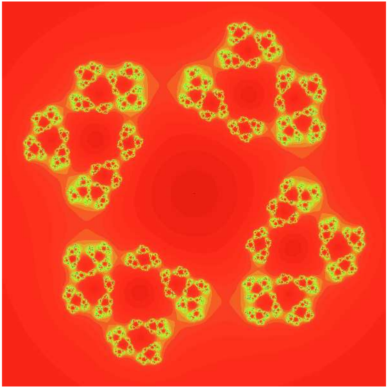
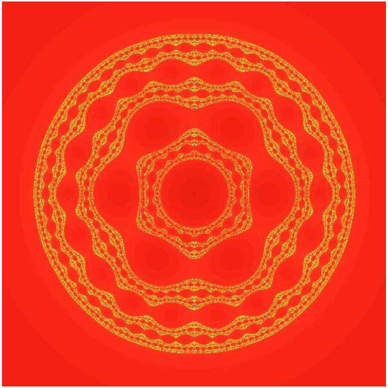
The only two possibilities for a disconnected Julia set.





Some results about the McMullen family

Theorem (W. Qiu, X. Wang and Y. Yin. (2012))

For any $n \geq 3$ and any complex parameter λ, if the Julia set $J(f_\lambda)$ is not a Cantor set, then ∂B_λ is a Jordan curve.

Theorem (F. Yang and X. Wang. (To appear))

Let $n \geq 3$, then for small λ such that $J(f_\lambda)$ is a Cantor set of circles, the Hausdorff dimension of ∂B_λ is

$$\dim_H(\partial B_\lambda) = 1 + \frac{|\lambda|^2}{\log n} + \mathcal{O}(|\lambda|^3)$$

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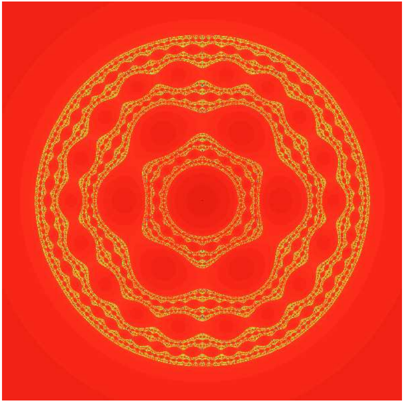

Figura: Dynamical Plane of $z^3 - 0.01/z^3$.

Some results about the McMullen family

Theorem (R. Devaney and T. G. (2008))

When $n = 2$ the Julia set $J(f_\lambda)$ converges to the closed unit disk as $\lambda \longrightarrow 0$ in the Hausdorff topology. On the other hand, when $n > 2$ this is not the case.

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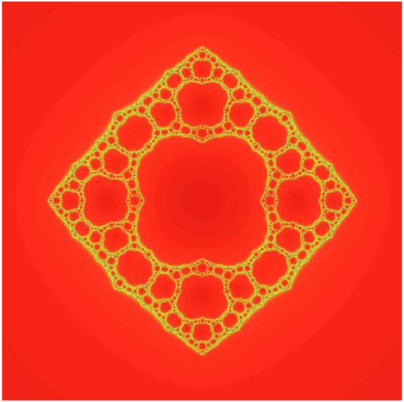

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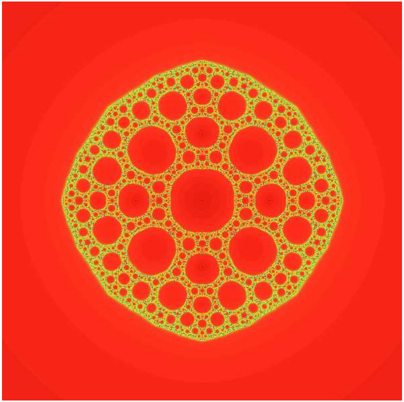

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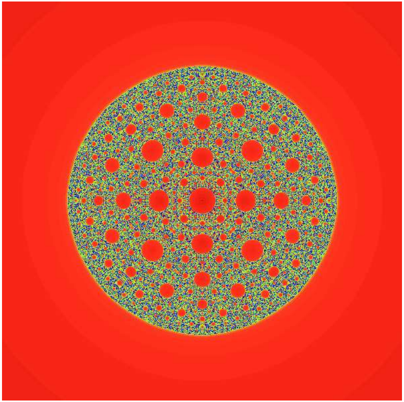

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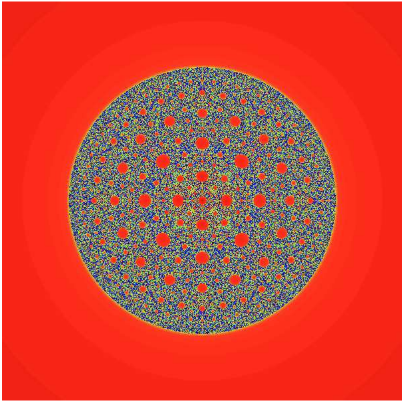

Figura: Dynamical Plane of $z^2 - 0.001/z^2$.

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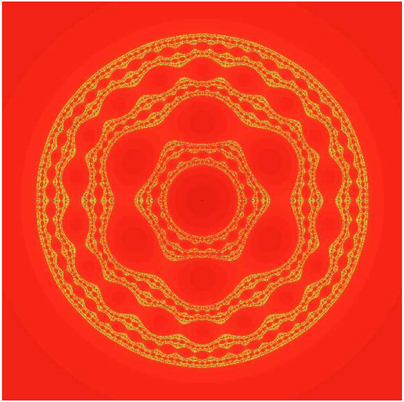
When $n \geq 3$ the Julia set $J(f_\lambda)$ not converge to the closed unit disk as $\lambda \to 0$ in the Hausdorff topology.



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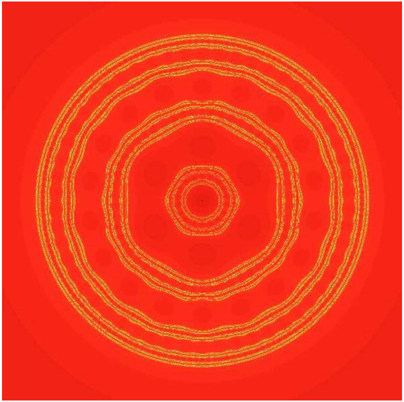

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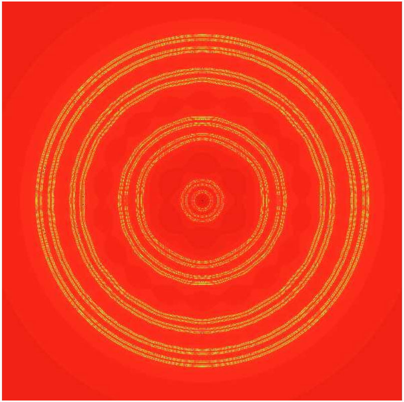

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Singular perturbations with multiple poles

$$f_\lambda(z) = z^n + c + \frac{\lambda}{\prod_{i=0}^{N-1}(z - c_i)^{d_i}}$$

Where:

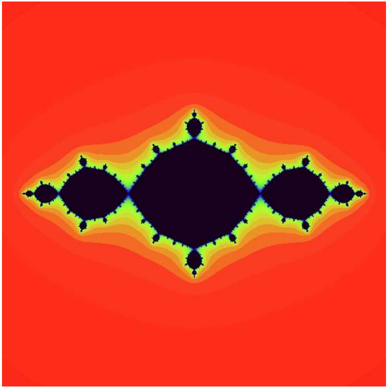
- c is a such that we have a superattracting cycle for f_0, i.e.,

  $$0 = c_0 \to f_0(0) = c_1 \to f_0^{\circ 2}(0) = c_2 \to \cdots f_0^{N-1}(0) = c_{N-1} \to 0.$$

- $d_i \geq 2$ for all $1 \leq i \leq N - 1$.
- λ is a complex parameter.

Singular perturbations with multiple poles

Dynamics of the map $f_0(z) = z^2 - 1$.



Fatou set of f_0

Set of points with stable behaviour. $F(f_0) = A(\{0, -1\}) \cup A(\infty)$. $A(\{0, -1\})$ = basin of attraction of the superattracting cycle $0 \to -1 \to 0$. $A(\infty)$ =basin of attraction of the superattracting fixed point $\infty \to \infty$.

Julia set of f_0

Set of points with unstable behaviour. $J(f_0) = \hat{\mathbb{C}} \setminus F(f_0)$.

Singular perturbations with multiple poles

$$f_\lambda(z) = z^2 - 1 + \frac{\lambda}{z^{d_0}(z+1)^{d_1}}$$

Theorem (T. G., S. Marotta and E. Russell (2012))

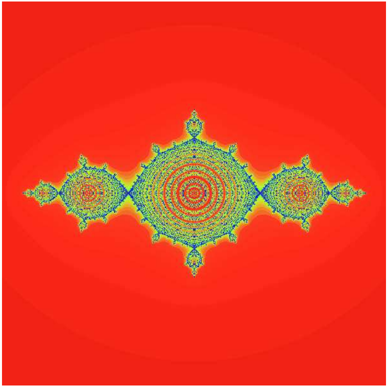
Let $d_0, d_1 \geq 2$ such that $2d_1 > d_0 + 2$ and $d_0 > d_1 + 1$ and λ sufficiently small. Then,

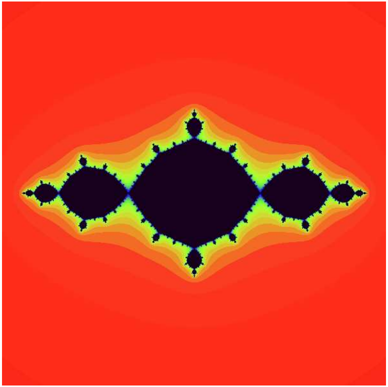
- *∂B_λ is a homeomorphic copy of $\partial B_0 = J(f_0)$.*
- *All the critical points are attracted by ∞.*
- *The Julia set of f_λ is disconnected.*

Singular perturbations with multiple poles

Dynamical plane of f_λ for $d_0 = 7$, $d_1 = 5$ and λ small.

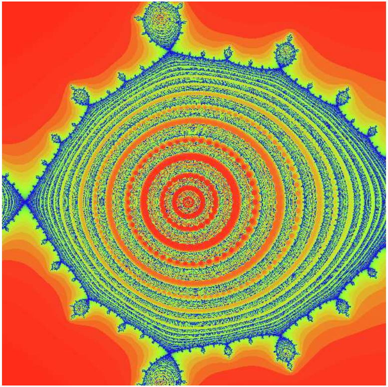
Dynamical plane of f_0

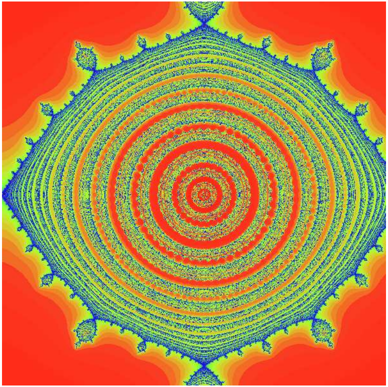




Singular perturbations with multiple poles

In the perturbed basilica we have two trap doors:
one trap door near $z = -1$ and another one trap door near $z = 0$.





Gràcies

Basic properties of the McMullen family

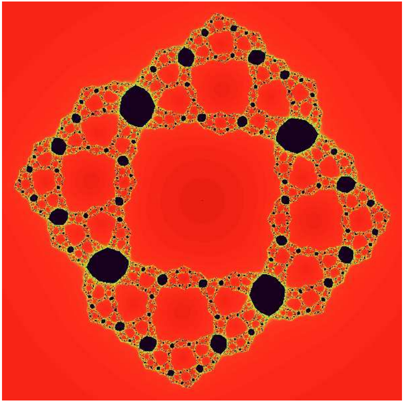


Figura: Dynamical Plane of $z^2 + \frac{-0.16+0.136667i}{z^2}$.