Connectivity of the Julia set: From polynomials to meromorphic transcendental maps

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Introduction: A motivational result...

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- A weakly repelling fixed point of f (wrfp) is either:
  - a repelling (fixed point), or
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Theorem (Barański, Bergweiler, Fagella, Karpińska, Taixés, J) Let f a meromorphic transcendental map.

If $\mathcal{J}(f)$ is disconnected then f has at least one wrfp.

Julia set for polynomials

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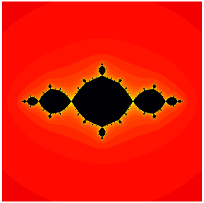
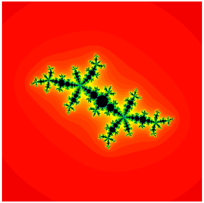
Proposition: $\mathcal{J}(f) = \partial K(P) = \partial A^\star(\infty)$.

Connected and non connected polynomial Julia sets



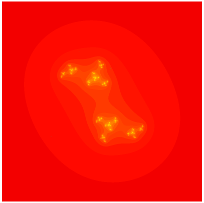
(a) $Q_{-1}(z) = z^2 - 1$

(b) $Q_i(z) = z^2 + i$



(c) $Q_{c_0}(z) = z^2 + c_0$

(d) $Q_{-2.05}(z) = z^2 - 2.05$



(e) $Q_{c_1}(z) = z^2 + c_1$

Connectivity I: The dicotomy

Theorem Let P be a polynomial of degree $d \geq 2$. Let $\mathcal{C} = \{f(c_1), \ldots f(c_d)\}$ be the set of critical values (images of the critical points, c_i, $i = 1, \ldots d$).

$$\mathcal{J}(P) \text{ is connected } \iff \mathcal{C} \cap A^\star(\infty) = \emptyset$$

Moreover if $\mathcal{C} \subset A^\star(\infty)$ then $\mathcal{J}(f)$ is totally disconnected.

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Corollary If $\deg(P) = 2$ (P writes as $P_\lambda(z) = z^2 + \lambda$ and so $\mathcal{C} = \{\lambda\}$) then

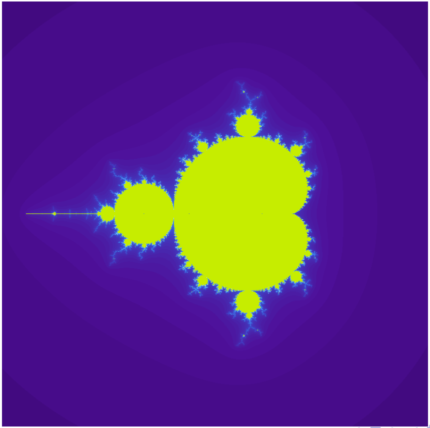
$$\text{If } \lambda \notin A^\star(\infty) \text{ then } \mathcal{J}(P) \text{ is connected}$$

$$\text{If } \lambda \in A^\star(\infty) \text{ then } \mathcal{J}(P) \text{ is totally disconnected}$$

Connectivity I: The dicotomy

Definition Let $P_\lambda(z) = z^2 + \lambda,\ \lambda \in \mathbb{C}$ be the quadratic family. The Mandelbrot set is defined as

$$\mathcal{M} = \{\lambda \in \mathbb{C} \mid \mathcal{J}(P_\lambda) \text{ is connected}\}$$



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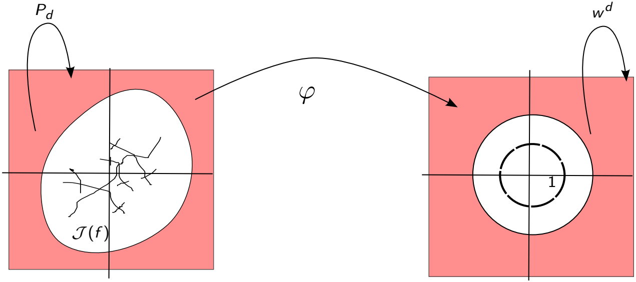


Main tool: Böttcher coordinate which conformally conjugates $P(z)$ in some neighborhood of infinity with the map $g(w) = w^n$ in $\mathbb{D}(0, r)$, $r > 1$.

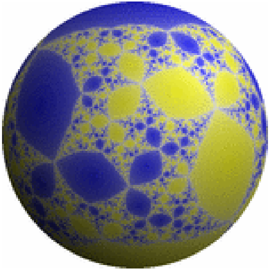
Then we use a pull-back argument to extend the local coordinate.

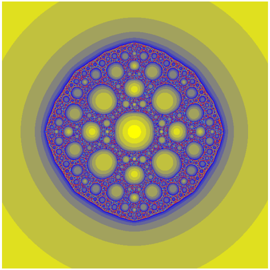
The rational scenario (part I)

Let

$$R_d(z) = \frac{P(z)}{Q(z)}, \quad P, Q \quad \text{polynomials}$$

be a rational map of degree d (hol. map in the Riemann sphere).





Two examples of rationals maps (the colors mean different stable dynamics).

The rational scenario (part I)

Let $\mathcal{F}(R_d) = \mathbb{C} \setminus \mathcal{J}(R_d)$ be the Fatou set.

Definition: Each connected component of $\mathcal{F}(R_d)$ is called a Fatou component or a Fatou domain.

Remark: Fatou domains maps to Fatou domains.

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Remark: Fatou domains maps to Fatou domains.

Theorem (Fatou, Sullivan) Let U a Fatou component. Then U is either pre-periodic, or periodic. In the later case it is either:

- The attracting basin of an attracting periodic point, or
- The parabolic basin of a parabolic periodic point, or
- a $p-$cycle of Siegel discs ($U \simeq \mathbb{D}$ and $f^p\,|_U$ is conjugate to an irrational rotation), or
- a $p-$cycle of Herman rings ($U \simeq A$ annulus, and $f^p\,|_U$ is conjugate to an irrational rotation).

The rational scenario (part I)

Theorem (Milnor 1993) Let R_d be a rational map of degree $d \geq 2$. Let $\mathcal{C} = \{R_d(c_1), \ldots R_d(c_d)\}$ be the set of critical values. Assume $\mathcal{C} \subset U$ for some U Fatou component (so, attracting or parabolic). Then

$$\mathcal{J}(R_d) \text{ is totally disconnected.}$$

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Theorem (Makienko, Przytycki, Milnor, Youngchen)

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Idea of the proof ($d = 2$):

If $f = R_2$ there are no Herman rings (Shishikura)

$\mathcal{J}(f)$ **is connected** $\iff$ **all Fatou components are simply connected**

The rational scenario (part II): wrfp

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Tool: Rational fixed point theorem

Theorem (Shishikura 1990-2009):
If $\mathcal{J}(R)$ is disconnected $\Rightarrow$ R has at least 2 wrfp.

Tools:

Quasi-conformal surgery

$\mathcal{J}(f)$ is connected $\iff$ all Fatou components are simply connected

Motivational corollary: Newton's method

Let P be a polynomial map. Then

$$N_P(z) = z - \frac{P(z)}{P'(z)}$$

is the Newton's method for the polynomial P.

- $$P(z_0) = 0 \iff N_P(z_0) = z_0$$
- $|(N_P)'(z_0)| < 1$, and
- $|(N_P)'(\infty)| > 1$ (only one wrfp!)

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Corollary (of Shishikura's Theorem): $\mathcal{J}(N_P(z))$ is connected.

Remark: *Previous* results in this direction are due to Przytyck, Meier and Tan Lei.

The transcendental case

Theorem Let f be a transcendental map. Let U a Fatou component. Then U is either:

- wandering domain ($f^k(U) \cap f^m(U) = \emptyset$ for all $k \neq j$), or
- preperiodic to a cycle of Fatou components, or
- periodic. In this case U is either:
  - The attracting basin of an attracting periodic point, or
  - The parabolic basin of a parabolic periodic point, or
  - a $p-$cycle of Siegel discs ($U \simeq \mathbb{D}$ and $f^p\,|_U$ is conjugate to an irrational rotation), or
  - a $p-$cycle of Herman rings ($U \simeq A$ annulus, and $f^p\,|_U$ is conjugate to an irrational rotation), or
  - a $p-$cycle of Baker domains ($f^{np}\,|_U \rightarrow \zeta \in \partial U$ and $f^p(\zeta)$ is not defined).

The entire transcendental case

Theorem: Assume $f : \mathbb{C} \to \mathbb{C}$ is a transcendental entire map.

- (Baker) Then every periodic or preperiodic component of the Fatou set is simply connected (no Herman rings!).
- (Bergweiler-Terglane) If f has a multiply connected wandering domain then f has infinitely many wrfp's.

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Corollary: Assume $f : \mathbb{C} \to \mathbb{C}$ is a entire transcendental map.

If $\mathcal{J}(f)$ is disconnected then f has at least one wrfp.

Motivational corollary: Newton's method II

Let g be an entire transcendental map. Then

$$N_g(z) = z - \frac{g(z)}{g'(z)}$$

is the Newton's method for g. Of course, in general, N_g is a meromorphic transcendental map.

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Theorem: If $\mathcal{J}(f)$ is disconnected then f has at least one wrfp.

Answer: Yes.

About the proof:

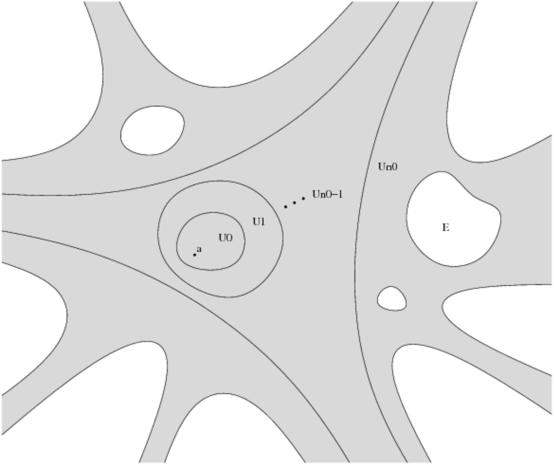
Theorem: If $\mathcal{J}(f)$ is disconnected then f has at least one wrfp.

$\mathcal{J}(f)$ is connected $\iff$ all Fatou components are simply connected

- Step 1: (F, J, Jordi Taixés) The case where U is either an attracting or parabolic basin, or a pre-periodic component.

Tools: Quasiconformal sugery *a la Shishikura* (but leading with infinity!!) and the pull–back of absorbing domains.

About the proof:



About the proof:

- Step 2 (Bergweiler): The case where U is a wandering domain.
- Step 3 (Barański, Fagella, Karpińska, J): The case where U is a Herman ring.
- Step 4 (Barański, Fagella, Karpińska, J): The case of Baker domains.

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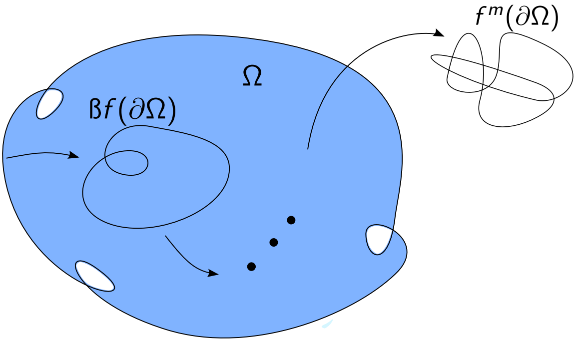
Tools: Quasiconformal surgery + Proving the existence of absorbing domains for non simply connected Baker domains (hyperbolic geometry) + results of the following *shape*

About the proof:

Theorem: Let $f : \mathbb{C} \to \mathbb{C}$. Let $\Omega \subset \mathbb{C}$ be bounded, simply connected domain. Assume

- $f^j(\partial\Omega) \subset \overline{\Omega}$ for $j = 1, \ldots, m-1$,
- $f^m(\partial\Omega) \cap \overline{\Omega} = \emptyset$.

Then, f has a weakly repelling fixed point in Ω.



Thank you !!