

Calculating the fractal dimension for slow-fast Liénard equation

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1.1 Introduction: Hilbert 16

Consider a polynomial vector fields in the real plane:

$$\begin{cases} \frac{dx}{dt} = P(x, y) \\ \frac{dy}{dt} = Q(x, y) \end{cases}$$

where both P and Q are real polynomials of degree n .

Hilbert 16th: Decide an upper bound for the number of limit cycles in polynomial vector fields of degree n . (Difficult)

1.2 Introduction: Fractal Approach

Solve it locally in the (x, y) -phase space for Liénard System.

Our goal in slow-fast setting:

Calculate the fractal dimension $\rightarrow$ The number of limit cycles

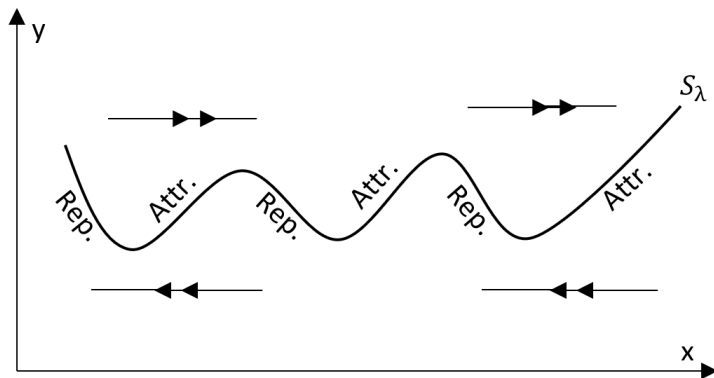
1.3 Slow-Fast System: Liénard System

Parameters $0 \leq \varepsilon \ll 1$ and $\lambda \in \mathbb{R}^p$ on vector fields $\mathbb{R}^2$ with functions f_λ (fast dynamics) and g_λ (slow dynamics) smooth

$$X_{\varepsilon,\lambda} = \begin{cases} \frac{dx}{dt} = y - f_\lambda(x) \\ \frac{dy}{dt} = \varepsilon g_\lambda(x) \end{cases}$$

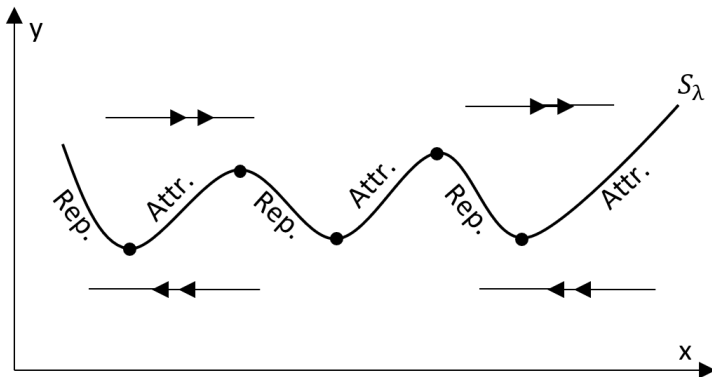
1.3 Critical Curve

$$S_\lambda = \{(x, y) \in \mathbb{R}^2 \mid f_\lambda(x) = y\}$$



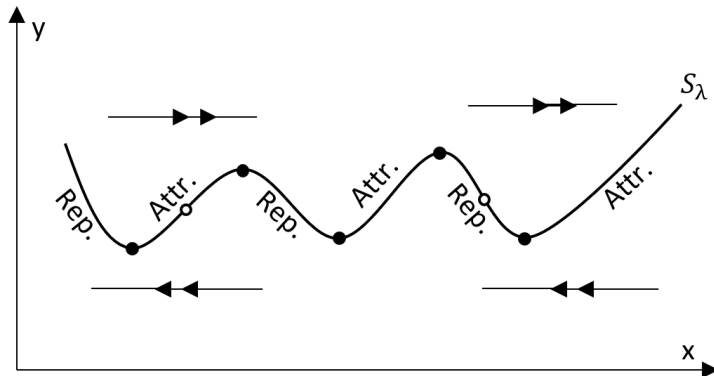
1.3 Contact Points

$$C_\lambda = \{(x, y) \in S_\lambda \mid f'_\lambda(x) = 0\}$$

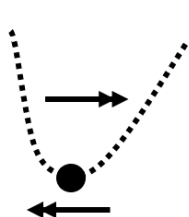


1.3 Slow Singularity

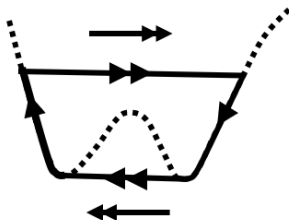
$$\Sigma_\lambda = \{(x, y) \in S_\lambda \setminus C_\lambda \mid g_\lambda(x) = 0\}$$



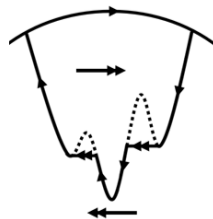
1.4 Calculate box-dimension $\rightarrow$ The number of limit cycles.



Contact Point



Canard Cycle



Canard Cycle

[De Maesschalck, Dumortier, Roussarie, 2021],
[Crnkovic, Huzak, Vlah, 2021]

1.4 The Minkowski/Box dimension (Falconer, Lapidus, ...)

- Let $0 < \delta \ll 1$.
- $U(\delta) \stackrel{\text{def}}{=} \text{the } \delta\text{-neighborhood of a bounded } U \subseteq \mathbb{R}$.
- $|U(\delta)| \stackrel{\text{def}}{=} \text{the Lebesgue measure of } U(\delta)$.

$\Rightarrow$ Lower Box dimension:

$$\underline{\dim}_B U = \liminf_{\delta \rightarrow 0} \left(1 - \frac{\ln |U(\delta)|}{\ln(\delta)} \right) \quad (1)$$

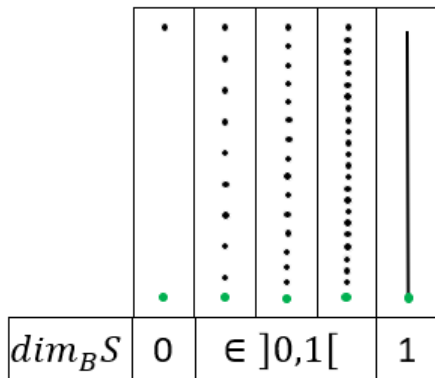
$\Rightarrow$ Upper Box dimension:

$$\overline{\dim}_B U = \limsup_{\delta \rightarrow 0} \left(1 - \frac{\ln |U(\delta)|}{\ln(\delta)} \right) \quad (2)$$

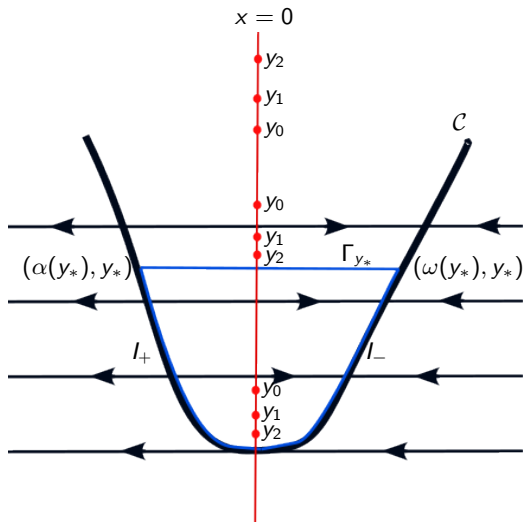
$$\underline{\dim}_B U = \dim_B U = \overline{\dim}_B U. \quad (3)$$

1.4 Box-dimension

Box-dimension measures the density of U : the bigger the box dimension, the higher the density.



1.5 Entry-Exit



1.6 Divergence Integral

The slow-fast Liénard system:

$$X_{\varepsilon,\lambda} \begin{cases} \frac{dx}{dt} = y - f(x\lambda) \\ \frac{dy}{dt} = \varepsilon g(x, \lambda) \end{cases}$$

Let σ be a curve with on the x-axis $]x_1, x_2[\subset S_\lambda \setminus (C_\lambda \cup \Sigma_\lambda)$:

$$I(x_1, x_2) = - \int_{x_1}^{x_2} \frac{\left(\frac{\partial f}{\partial x}(x, \lambda)\right)^2}{g(x, \lambda)} dx. \quad (4)$$

1.7 Comparability & Codimension

Let $(x_k)_{k \geq 0}$ and $(y_k)_{k \geq 0}$ be two sequences of positive real numbers. We say that $(x_k)_{k \geq 0}$ and $(y_k)_{k \geq 0}$ are comparable and write $x_k \simeq y_k$ as $k \rightarrow \infty$ if $1/\rho \leq x_k/y_k \leq \rho$ for some $\rho > 0$ and for all $k \geq 0$.

The number of limit cycles depends on the codimension of the slow-fast Hopf point (the higher the codimension, the more limit cycles can be born). A singular Hopf point has codimension $j + 1$ if

$$g(x, \lambda_0) + g(-x, \lambda_0) = \alpha x^{2j} + O(x^{2j+2}), \quad \alpha \neq 0.$$

2.1 Normal form Hopf Point: Assumptions

$$\begin{cases} \dot{x} = y - f(x, \lambda) \\ \dot{y} = \epsilon (g(x, \epsilon, \lambda) + (y - f(x, \lambda)) h(x, y, \epsilon, \lambda)) \end{cases} \quad (5)$$

where f, g, h are smooth, $f(0, \lambda_0) = \frac{\partial f}{\partial x}(0, \lambda_0) = 0$

$$f(x, \lambda_0) = x^n \tilde{f}(x) \text{ with } n \geq 2$$

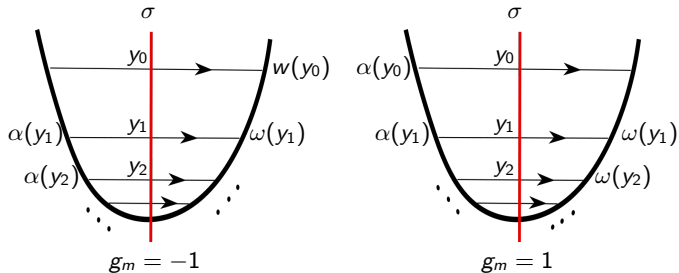
$$g(x, 0) = g_m x^m + x^{m+1} \tilde{g}(x)$$

where $g_m = \pm 1$ and $\tilde{f}, \tilde{g}$ are smooth functions.

Assumption A:

n is even and m is odd $\Rightarrow$ gives a “parabola-like” critical curve.

2.1 Normal form Hopf Point: Entry-Exit



2.1 Normal form Hopf Point: Assumptions

Smooth diffeomorphism:

$$(x, y) \rightarrow (x\tilde{f}(x)^{\frac{1}{n}}, y) \left(\text{resp. } (x, y) \rightarrow (-x(-\tilde{f}(x))^{\frac{1}{n}}, -y) \right)$$
$$\begin{cases} \dot{x} = y - x^n \\ \dot{y} = \epsilon (g(x, \epsilon) + (y - x^n) h(x, y, \epsilon)) \end{cases}, \quad (6)$$

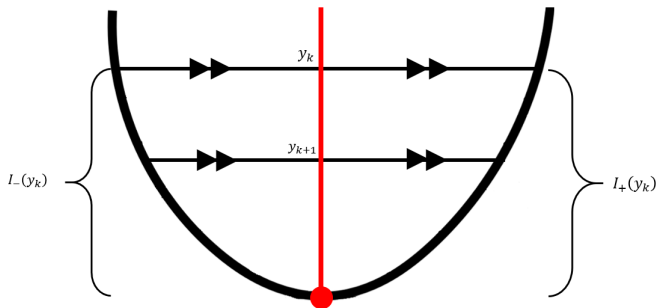
Assumption B:

Finite slow divergence: $m \leq 2(n-1)$.

Assumption C:

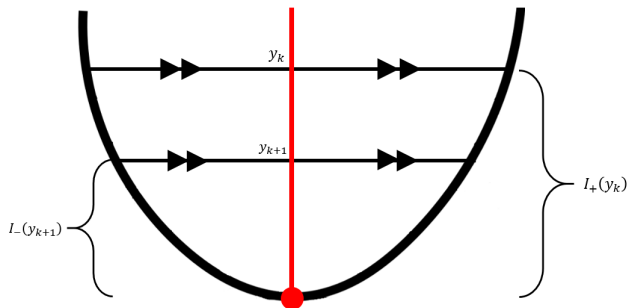
We assume that $y = 0$ is an isolated zero of $\hat{I}(y)$, meaning that there is a small $y^* > 0$ such that $\hat{I}$ is nonzero on $]0, y^*[$.

2.2 Idea of Proof



$$I_+(y_k) - I_-(y_k) = \tilde{I} \text{ with } \tilde{I} \rightarrow 0 \text{ as } k \Rightarrow \infty$$

2.2 Idea of Proof



$$I_-(y_{k+1}) - I_+(y_k) = 0 \Rightarrow I_+(y_k) - I_-(y_k) = \tilde{I} + I_-(y_{k+1}) - I_+(y_k)$$

2.2 Idea of Proof

After calculations and Taylor:

$$y_k - y_{k+1} \simeq y_k^{\frac{n+2j+1}{n}}, \quad k \rightarrow \infty., \quad k \rightarrow \infty,$$

As $\eta := \frac{n+2j+1}{n} > 1$ use theorem 1 of [Elezović, N. and Županović, V. and Žubrinić, D.]

$$\Rightarrow \dim_B(y_k)_{k \geq 0} = 1 - \frac{1}{\eta} = \frac{2j+1}{n+2j+1},$$

and these results do not depend on the choice of y_0 .

3.1 Slow-fast Liénard equations to Infinity

$$\begin{cases} \dot{x} = y - \sum_{k=0}^{n+1} B_k x^k & B_{n+1} \neq 0 \\ \dot{y} = -\epsilon \sum_{k=0}^m A_k x^k & A_m \neq 0 \end{cases} \quad (7)$$

$$\frac{F'(x)^2}{G(x)} = -\frac{(n+1)^2}{A} x^{2n-m} (1 + o(1)), \quad x \rightarrow \pm\infty. \quad (8)$$

$$m < 2n + 1$$

$$m = 2n + 1$$

$$m > 2n + 1$$

3.1 Slow-fast Liénard equations to Infinity

$$\begin{cases} \dot{x} = y - \sum_{k=0}^{n+1} B_k x^k & B_{n+1} \neq 0 \\ \dot{y} = -\epsilon \sum_{k=0}^m A_k x^k & A_m \neq 0 \end{cases} \quad (9)$$

$$m < 2n + 1$$

$$m = 2n + 1$$

$$m > 2n + 1$$

$m \leq 2n + 1$: Poincaré-Lyapunov disc of degree $(1, n + 1)$

$$x = \frac{\bar{x}}{r}, y = \frac{1}{r^{n+1}}$$

$m \geq 2n + 1$: Poincaré-Lyapunov disc of degree $(2, m + 1)$ (resp. of degree $(1, \frac{m+1}{2})$ for m even (resp. m odd))

$$x = \frac{\pm 1}{r}, y = \frac{\bar{y}}{r^{\frac{m+1}{2}}}$$

3.1 Slow-fast Liénard equations to Infinity

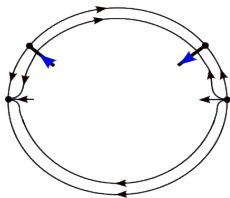
$$\begin{cases} \dot{x} = y - \sum_{k=0}^{n+1} B_k x^k & B_{n+1} \neq 0 \\ \dot{y} = -\epsilon \sum_{k=0}^m A_k x^k & A_m \neq 0 \end{cases} \quad (10)$$

After rescaling in x, y, t

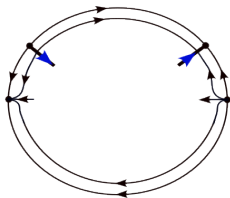
$$\begin{cases} \dot{x} = y - (x^{n+1} \sum_{k=0}^n b_k x^k) \\ \dot{y} = -\epsilon (Ax^m + \sum_{k=0}^{m-1} A_k x^k) \end{cases} \quad (11)$$

- $A = 1$: if m is even and $m \neq 2n + 1$
- $A = \text{sign}(A_m)$: if m is odd and $m \neq 2n + 1$
- $A \neq 0$: if $m = 2n + 1$

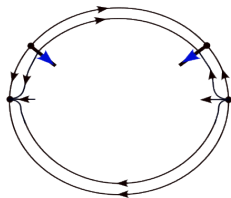
3.2 $m < 2n + 1$



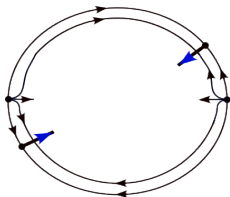
$A = 1, m \text{ odd}, n \text{ odd}$



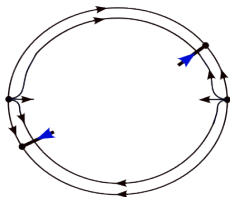
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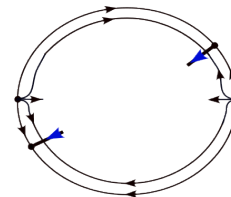
$A = 1, m \text{ even}, n \text{ odd}$



$A = 1, m \text{ odd}, n \text{ even}$

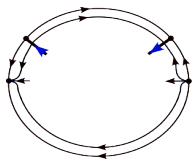


$A = -1, m \text{ odd}, n \text{ even}$

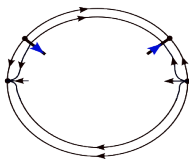


$A = 1, m \text{ even}, n \text{ even}$

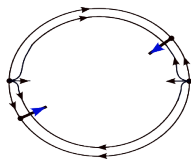
3.2 $m = 2n + 1$



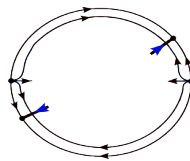
$A > 0, n \text{ odd}$



$A < 0, n \text{ odd}$

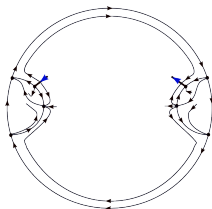


$A > 0, n \text{ even}$

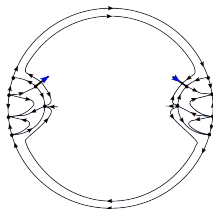


$A < 0, n \text{ even}$

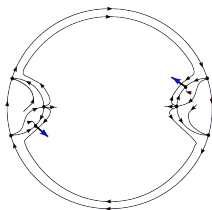
3.2 $m > 2n + 1$



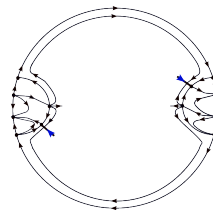
$A = 1, n$ odd



$A = -1, n$ odd

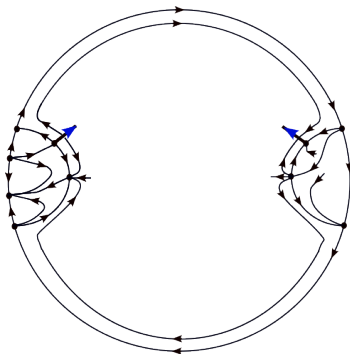


$A = 1, n$ even

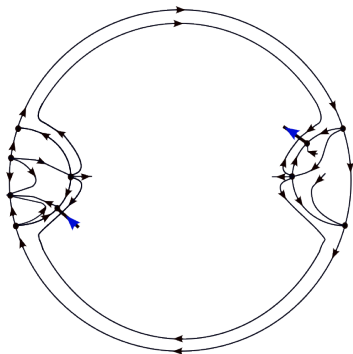


$A = -1, n$ even

3.2 $m > 2n + 1$



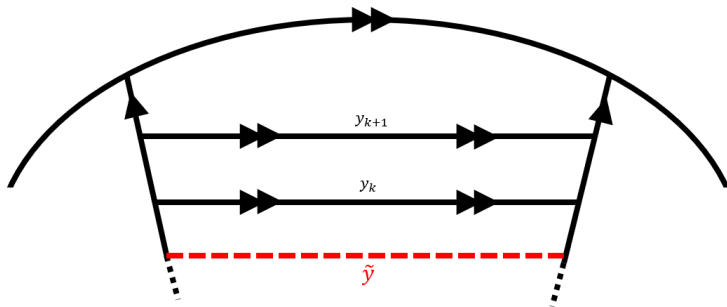
$A = 1, n$ odd



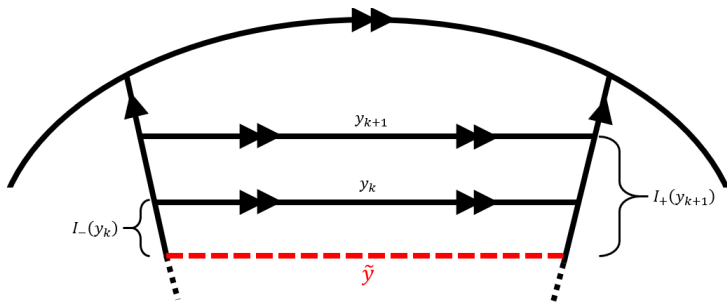
$A = 1, n$ even

3.3 Idea of the proof

Take $\tilde{y}$ fixed



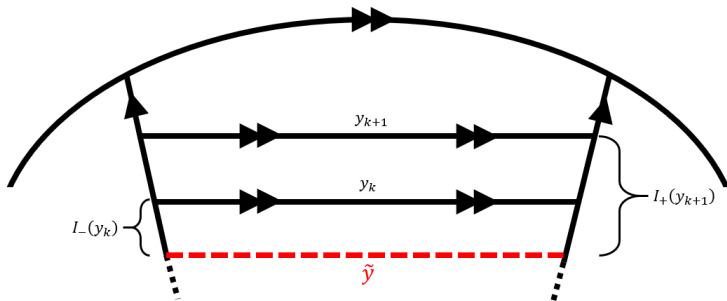
3.3 Idea of the proof



Let $\tilde{I}$ be a real constant. Assume sequence $(y_k)_{k \in \mathbb{N}}$ generated by

$$I_-(y_k) - I_+(y_{k+1}) = \tilde{I}, \quad k \in \mathbb{N},$$

3.3 Idea of the proof



Let $\tilde{I}$ be a real constant.

$$I_-(y_k) - I_+(y_k) = \tilde{I} + I_+(y_{k+1}) - I_+(y_k), \quad k \in \mathbb{N}.$$

$\Rightarrow$ Blow up

3.3 Dimensions: $m < 2n + 1$

Case A: $(n - 2j_b < m - 2j_a \text{ or } a_e = 0)$ and $m - n - 2j_b - 1 < 0$.

$$r_l - r_{l+1} \simeq r_l^{n+1-2j_b}, \quad l \rightarrow \infty \Rightarrow \dim_B(r_l)_{l \in \mathbb{N}} = \frac{n - 2j_b}{n + 1 - 2j_b} \quad (12)$$

Case B: $(n - 2j_b < m - 2j_a \text{ or } a_e = 0)$ and $m - n - 2j_b - 1 > 0$.

$$r_l - r_{l+1} \simeq r_l^{2n+2-m}, \quad l \rightarrow \infty \Rightarrow \dim_B(r_l)_{l \in \mathbb{N}} = \frac{2n + 1 - m}{2n + 2 - m} \quad (13)$$

Case C: $(n - 2j_b > m - 2j_a \text{ or } b_o = 0)$ and $2m - 2n - 2j_a - 1 < 0$.

$$r_l - r_{l+1} \simeq r_l^{m+1-2j_a}, \quad l \rightarrow \infty \Rightarrow \dim_B(r_l)_{l \in \mathbb{N}} = \frac{m - 2j_a}{m + 1 - 2j_a} \quad (14)$$

Other Cases: Trivial $\Rightarrow \dim_B(r_l)_{l \in \mathbb{N}} = 0$.

3.3 Dimensions: $m = 2n + 1$

Case A: $(n - 2j_b < 2n + 1 - 2j_a$ **or** $a_e = 0)$ **and** $n - 2j_b < 0$.

$$r_l - r_{l+1} \simeq r_l^{n+1-2j_b}, \quad l \rightarrow \infty \Rightarrow \dim_B(r_l)_{l \in \mathbb{N}} = \frac{n - 2j_b}{n + 1 - 2j_b} \quad (15)$$

Case B: $(n - 2j_b > 2n - 1 - 2j_a$ **or** $b_o = 0)$ **and** $2n - 2j_a + 1 < 0$.

$$r_l - r_{l+1} \simeq r_l^{2n+2-2j_a}, \quad l \rightarrow \infty \Rightarrow \dim_B(r_l)_{l \in \mathbb{N}} = \frac{2n + 1 - 2j_a}{2n + 2 - 2j_a} \quad (16)$$

Other Cases: Trivial $\Rightarrow \dim_B(r_l)_{l \in \mathbb{N}} = 0$.

3.3 Dimensions: $m > 2n + 1$

Case A: $n - 2j_b < m - 2j_a$ or $a_e = 0$.

$$\hat{r}_l - \hat{r}_{l+1} \simeq \hat{r}_l^{\frac{(n-2j_b)(m+1)}{2(n+1)} + 1} \Rightarrow \dim_B(\hat{r}_l)_{l \in \mathbb{N}} = \frac{(n - 2j_b)(m + 1)}{(n - 2j_b)(m + 1) + 2(n + 1)}$$

Case B: $n - 2j_b > m - 2j_a$ or $b_o = 0$.

$$\hat{r}_l - \hat{r}_{l+1} \simeq \hat{r}_l^{\frac{(n-2j_b)(m+1)}{2(n+1)} + 1} \Rightarrow \dim_B(\hat{r}_l)_{l \in \mathbb{N}} = \frac{(n - 2j_b)(m + 1)}{(n - 2j_b)(m + 1) + 2(n + 1)}$$

Other Cases: Trivial $\Rightarrow \dim_B(\hat{r}_l)_{l \in \mathbb{N}} = 0$.

3.4 Extra Case: $b_{2j_b+1}(m - n + 2j_b + 1) - a_{2j_a}(n + 1) = 0$

For $m < 2n + 1$:

$$r_l - r_{l+1} \simeq r_l^{n+3} \Rightarrow \dim_B(r_l)_{l \in \mathbb{N}} = \frac{n+2}{n+3} \quad (17)$$

For $m = 2n + 1$:

$$r_l - r_{l+1} \simeq r_l^{n+3} \Rightarrow \dim_B(r_l)_{l \in \mathbb{N}} = \frac{n+2}{n+3} \quad (18)$$

For $m > 2n + 1$:

$$\begin{aligned} \hat{r}_l - \hat{r}_{l+1} &\simeq \hat{r}_l^{\frac{(3n-m+3)(m+1)(n+2)+2(n+1)}{2(n+1)}} \\ \Rightarrow \dim_B(\hat{r}_l)_{l \in \mathbb{N}} &= \frac{(3n-m+3)(m+1)(n+2)}{(3n-m+3)(m+1)(n+2)+2(n+1)} \end{aligned} \quad (19)$$

35. Calculating the fractal dimensions

Let $\mathcal{S} = (y_k)_{k \geq 0} \rightarrow 0$ with an initial point y_0 such that $y_k - y_{k+1} \simeq y_k^\eta$ as $k \rightarrow \infty$, for $\eta > 1$.

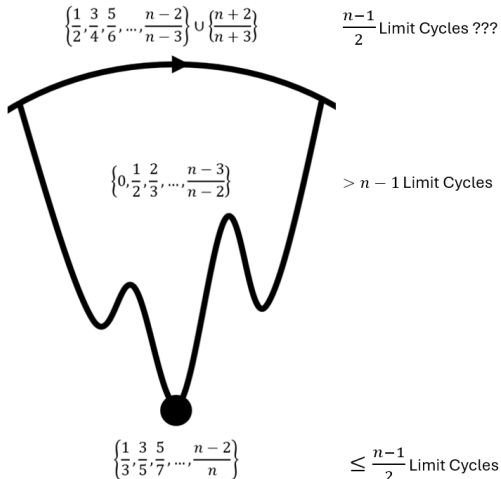
$$\dim_B \mathcal{S} = \lim_{k \rightarrow \infty} \frac{\ln k}{-\ln(y_k - y_{k+1})}, \quad (20)$$

$$\dim_B \mathcal{S} = \lim_{k \rightarrow \infty} \frac{1}{1 - \frac{\ln y_k}{\ln k}}, \quad (21)$$

and

$$\dim_B \mathcal{S} = \lim_{k \rightarrow \infty} \left(1 - \frac{\ln(k(y_k - y_{k+1}) + y_k)}{\ln\left(\frac{y_k - y_{k+1}}{2}\right)} \right). \quad (22)$$

3.6 Classical Liequations ($1 = m < 2n + 1$)



Thank You for Listening & Questions?



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