

The cyclicity of hyperbolic hemicycles

JORDI VILADELPRAT



Joint work with David Marín

Recent Advances in Dynamical Systems

Santiago de Compostela, July 2024

Acknowledgements

This work has been partially founded by the grants

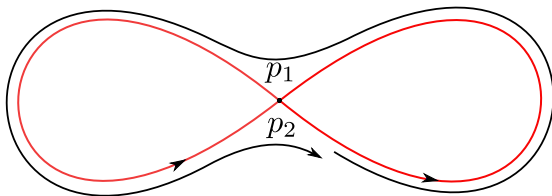
- PID2020-118281GB-C33 of the Ministry of Science, Innovation and Universities (Spain) and
- 2021 SGR 00113 of the Agency for Management of University and Research Grants (Catalonia)



Key definitions

Definition

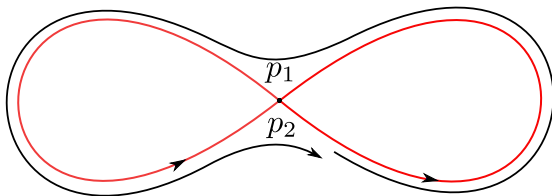
Let X be a vector field on $\mathbb{R}^2$ (or $\mathbb{S}^2$). A **graphic** Γ for X is a compact, non-empty invariant set which is a continuous image of $\mathbb{S}^1$ and consists of a finite number of isolated singularities $\{p_1, \dots, p_m, p_{m+1} = p_1\}$ (not necessarily distinct) and compatibly oriented separatrices $\{s_1, \dots, s_m\}$ connecting them.



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Let X be a vector field on $\mathbb{R}^2$ (or $\mathbb{S}^2$). A **graphic** Γ for X is a compact, non-empty invariant set which is a continuous image of $\mathbb{S}^1$ and consists of a finite number of isolated singularities $\{p_1, \dots, p_m, p_{m+1} = p_1\}$ (not necessarily distinct) and compatibly oriented separatrices $\{s_1, \dots, s_m\}$ connecting them. A graphic is said to be **hyperbolic** if all its singular points are hyperbolic saddles. A **polycycle** is a graphic with a return map defined on one of its sides.



Key definitions

Definition

A non-empty compact subset Γ of a surface S is a **limit periodic set** for a germ of a **family of vector fields** $\{X_\mu\}_{\mu \approx \mu_0}$ on S if there exists a sequence of parameters $\{\mu_n\}_{n \in \mathbb{N}}$ converging to μ_0 such that each X_{μ_n} has a limit cycle γ_n and the sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ converges to Γ as $n \rightarrow \infty$ in the Hausdorff topology of the space $\mathcal{C}(S)$ of compact non-empty subsets of S .

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The **Hausdorff distance** between the compact sets M and N is

$$d_H(M, N) = \inf\{\varepsilon > 0 : M \subset N_\varepsilon \text{ and } N \subset M_\varepsilon\}.$$

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A limit periodic set for an analytic family must be:

- a singular point,
- a periodic orbit, or
- a graphic.

Key definitions

Definition

Let Π be a collection of limit periodic sets for the germ of a family of vector fields $\{X_\mu\}_{\mu \approx \mu_0}$ on $\mathbb{S}^2$. We define the **cyclicity** of Π with respect to $\{X_\mu\}_{\mu \approx \mu_0}$ as

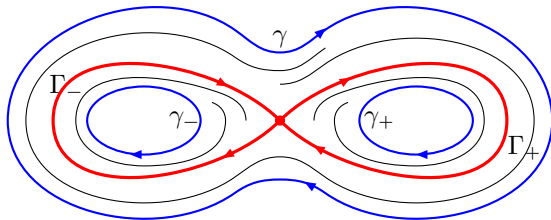
$$\text{Cycl}((\Pi, X_{\mu_0}), X_\mu) := \inf_{\varepsilon, \delta > 0} \sup_{\mu \in B_\delta(\mu_0)} \#\{\gamma \text{ limit cycle of } X_\mu \text{ such that } d_H(\gamma, \Gamma) < \varepsilon \text{ for some } \Gamma \in \Pi\}.$$

Key definitions

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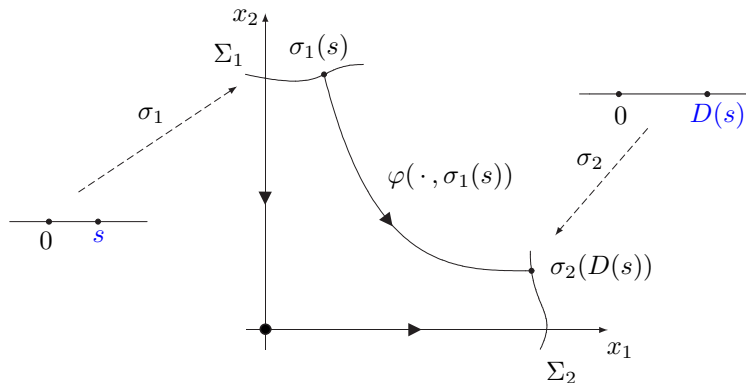
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The Dulac map

The **Dulac map** is a fundamental notion to study the cyclicity of a hyperbolic polycycle.



The Dulac map

We have devoted a series of three papers to study the asymptotic expansion of the Dulac map (and the Dulac time):



Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: local setting,
J. Differential Equations **269** (2020) 8425–8467.



Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: general setting,
J. Differential Equations, **275** (2021) 684–732.



Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: coefficient properties,
J. Differential Equations, **404** (2024) 43–107.

Quadratic hemicycles

For each $(a_0, b_0) \in (-2, 0) \times (0, 2)$ we consider the quadratic differential system

$$\begin{cases} \dot{x} = \frac{b_0-2}{4} + (1-b_0)y + a_0x^2 + b_0y^2, \\ \dot{y} = -2xy. \end{cases} \quad (1)$$

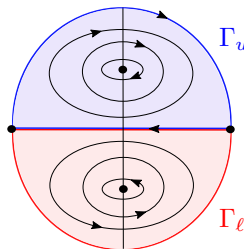
It has two centers, one foliates the whole upper-half plane and the other the whole lower-half plane.

Quadratic hemicycles

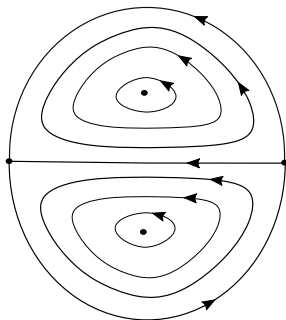
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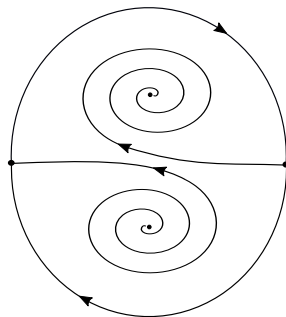
It has two centers, one foliates the whole upper-half plane and the other the whole lower-half plane. Compactified to the [Poincaré disc](#), the outer boundary of the respective period annulus is a polycycle, which in some sources is called *hemicycle*.



The cyclicity of quadratic hemicycles



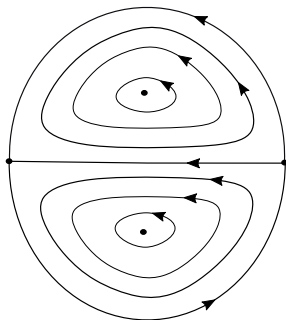
unperturbed



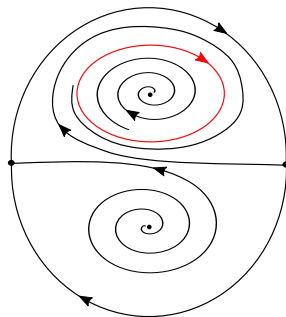
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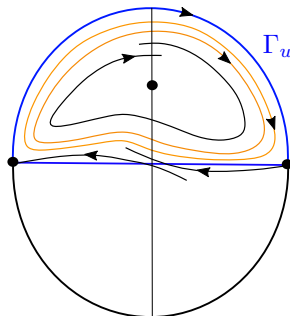
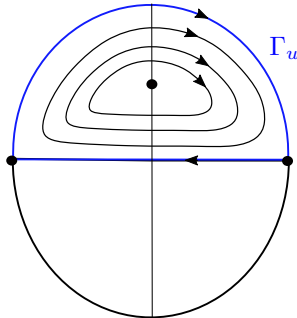
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The cyclicity of quadratic hemicycles

THEOREM A

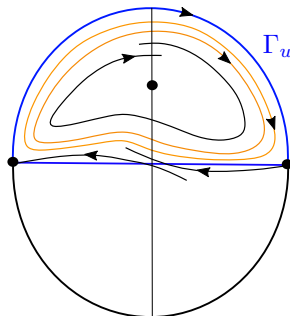
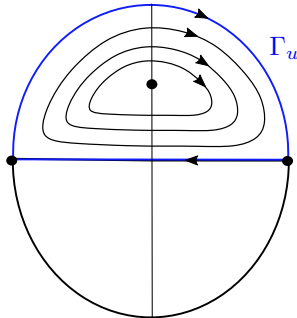
If $(a_0, b_0) \in (-2, 0) \times (0, 2)$ and $a_0 \neq -1$ then the cyclicity of Γ_u when we perturb (1) inside the whole family of quadratic differential systems is exactly 2. Moreover the cyclicity is at least 2 for $(a_0, b_0) \in \{-1\} \times (0, 2)$.



The cyclicity of quadratic hemicycles

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The cyclicity of quadratic hemicycles

It suffices to study the 5-parametric perturbation given by

$$X_\mu \quad \begin{cases} \dot{x} = \frac{b-2}{4} + \varepsilon_1 x + (1-b)y + ax^2 + \varepsilon_2 xy + by^2, \\ \dot{y} = \varepsilon_0 - 2xy, \end{cases} \quad (2)$$

where $\mu := (a, b, \varepsilon_0, \varepsilon_1, \varepsilon_2) \approx (a_0, b_0, 0, 0, 0) =: \mu_0$.

The cyclicity of quadratic hemicycles

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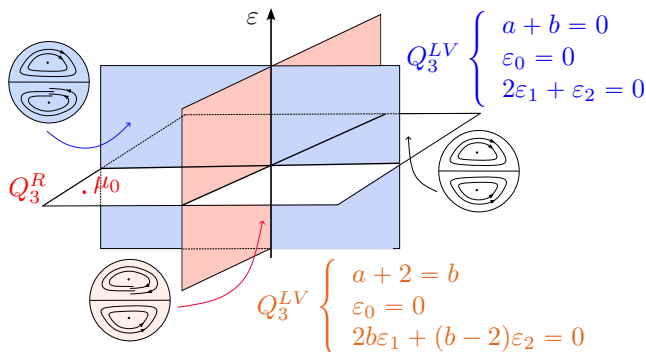
The unperturbed system belongs to the component Q_3^R of reversible centers, which intersects the component Q_3^{LV} of generalized Lotka-Volterra centers. At the intersection things get more complicated.

The cyclicity of quadratic hemicycles

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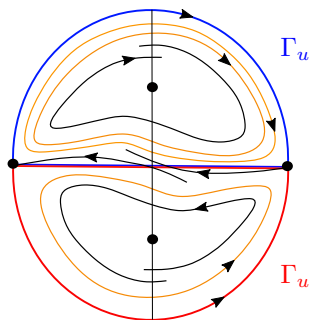
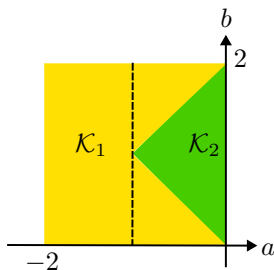
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The cyclicity of quadratic hemicycles

THEOREM B

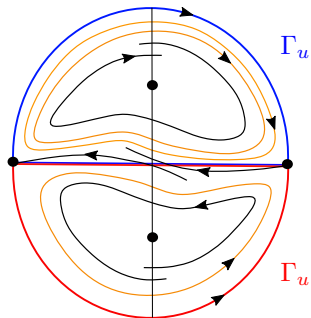
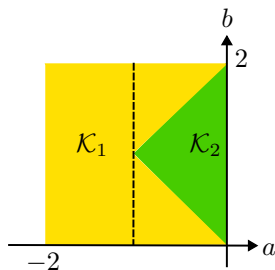
If (a_0, b_0) belongs to $\mathcal{K}_1 \setminus \{a_0 = -1\}$ (respectively, $\mathcal{K}_2$) then the cyclicity of $\Pi = \{\Gamma_u, \Gamma_\ell\}$ when we perturb (1) inside the whole family of quadratic differential systems is exactly 3 (respectively, 2).



The cyclicity of quadratic hemicycles

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If (a_0, b_0) belongs to $\mathcal{K}_1 \setminus \{a_0 = -1\}$ (respectively, $\mathcal{K}_2$) then the cyclicity of $\Pi = \{\Gamma_u, \Gamma_\ell\}$ when we perturb (1) inside the whole family of quadratic differential systems is exactly 3 (respectively, 2). Moreover it is at least 3 for $(a_0, b_0) \in \{-1\} \times (0, 2)$.



The cyclicity of quadratic hemicycles

COMPACTIFICATION MATTERS

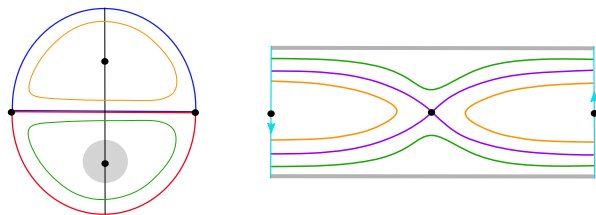
In the [projective plane](#), the antipodal points at infinity are identified and, consequently, the polycycles Γ_u and Γ_ℓ are the same set. This identification is not compatible with the direction of the vector field but we can think in terms of a [foliation](#). In this context, the “cyclicity” of $\Gamma_u = \Gamma_\ell$ in the projective plane would be three.

The cyclicity of quadratic hemicycles

COMPACTIFICATION MATTERS

In the **projective plane**, the antipodal points at infinity are identified and, consequently, the polycycles Γ_u and Γ_ℓ are the same set. This identification is not compatible with the direction of the vector field but we can think in terms of a **foliation**. In this context, the “cyclicity” of $\Gamma_u = \Gamma_\ell$ in the projective plane would be three.

By removing a disk, the projective plane becomes a Möbius strip.



The cyclicity of quadratic hemicycles

THEOREM C

If $(a_0, b_0) \in \{(-1, 1), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{3}{2})\}$ then an alien limit cycle bifurcation occurs at $\Gamma_u \cup \Gamma_\ell$ when we perturb (1) inside the whole family of quadratic differential systems.

The cyclicity of quadratic hemicycles

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If $(a_0, b_0) \in \{(-1, 1), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{3}{2})\}$ then an **alien limit cycle bifurcation** occurs at $\Gamma_u \cup \Gamma_\ell$ when we perturb (1) inside the whole family of quadratic differential systems.

The original notion of alien limit cycle was introduced in



F. Dumortier and R. Roussarie, *Abelian integrals and limit cycles*, J. Differential Equations, **224** (2006) 296–313.

where the authors bring to light that there are limit cycles bifurcating from a polycycle which cannot be detected as a zero of the first Melnikov function.

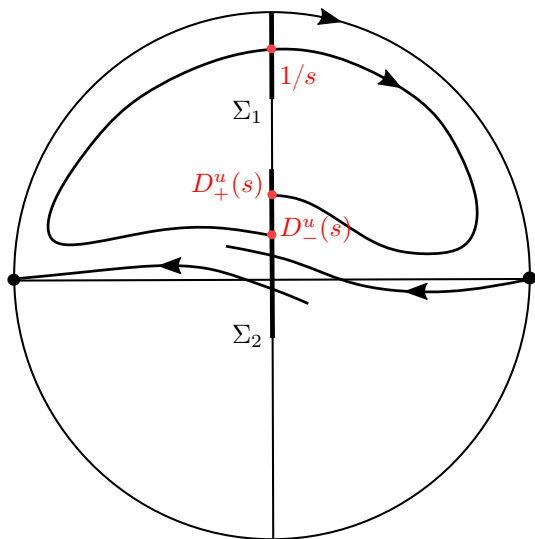
The cyclicity of quadratic hemicycles

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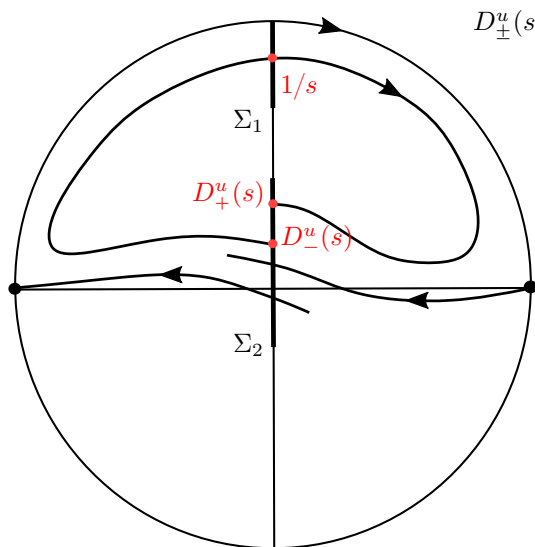
If $(a_0, b_0) \in \{(-1, 1), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{3}{2})\}$ then an alien limit cycle bifurcation occurs at $\Gamma_u \cup \Gamma_\ell$ when we perturb (1) inside the whole family of quadratic differential systems.

Our definition differs from the one used by Dumortier and Roussarie because we account only for limit cycles which cannot be detected as zeroes of any Melnikov function of any order. In addition our definition is more intrinsic and geometric.

The cyclicity of quadratic hemicycles

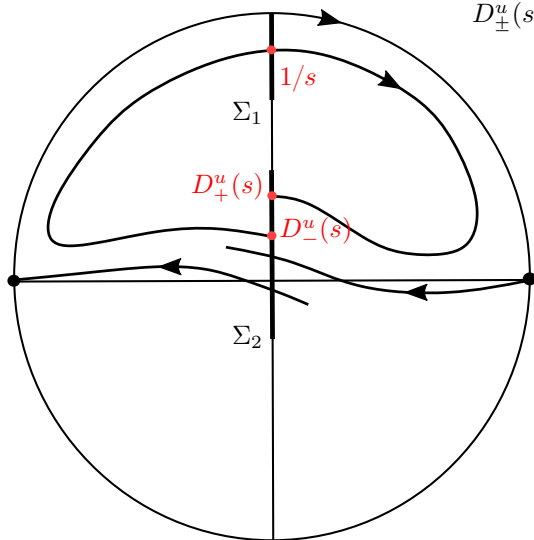


The cyclicity of quadratic hemicycles



$$D_{\pm}^u(s; \mu) = \delta_{\pm} + \Delta_0^{\pm} s^{\lambda} + \mathcal{F}_{\ell}^{\infty}(\mu_0)$$

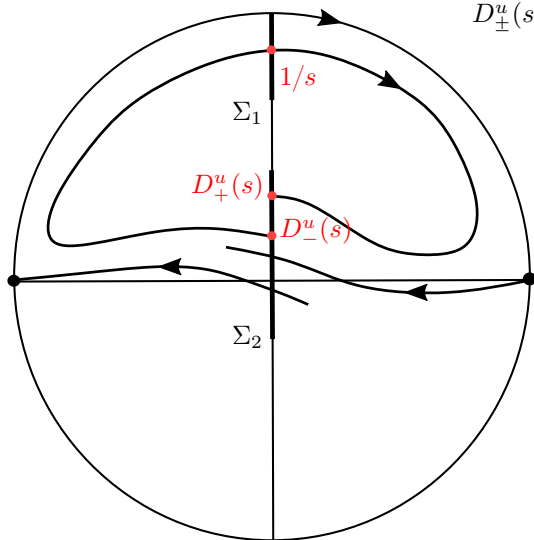
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$$D_{\pm}^u(s; \mu) = \delta_{\pm} + \Delta_0^{\pm} s^{\lambda} + \mathcal{F}_{\ell}^{\infty}(\mu_0)$$

$\delta_{\pm}$ and $\Delta_0^{\pm}$ are smooth functions at $\mu \approx \mu_0$

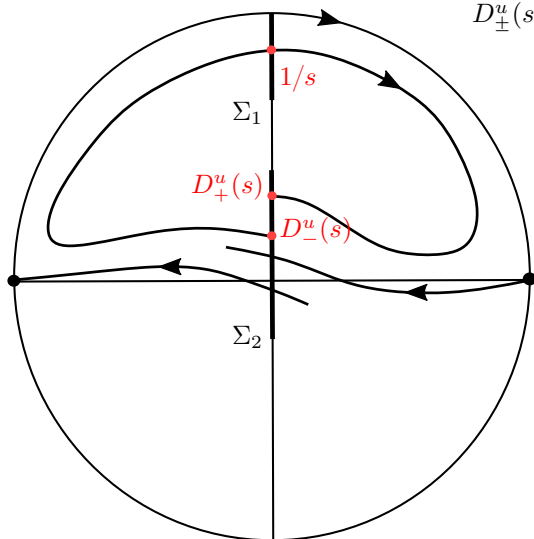
The cyclicity of quadratic hemicycles



$$D_{\pm}^u(s; \mu) = \delta_{\pm} + \Delta_0^{\pm} s^{\lambda} + \mathcal{F}_{\ell}^{\infty}(\mu_0)$$

$$\delta_{\pm}|_{\varepsilon_0=0} \equiv 0$$

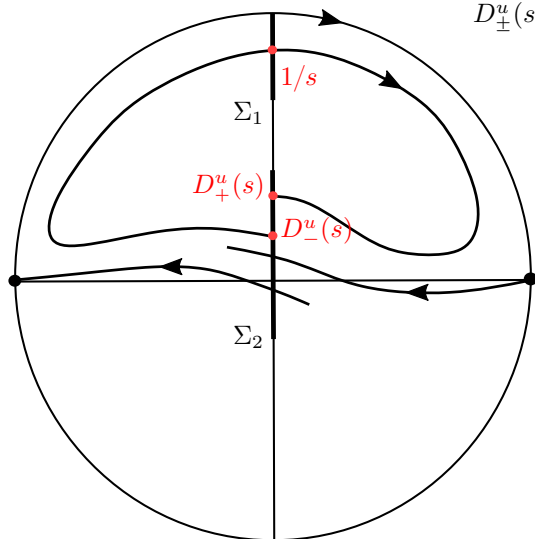
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$$D_{\pm}^u(s; \mu) = \delta_{\pm} + \Delta_0^{\pm} s^{\lambda} + \mathcal{F}_{\ell}^{\infty}(\mu_0)$$

$\lambda = \lambda(\mu)$ is the
hyperbolicity ratio of the
saddle at infinity

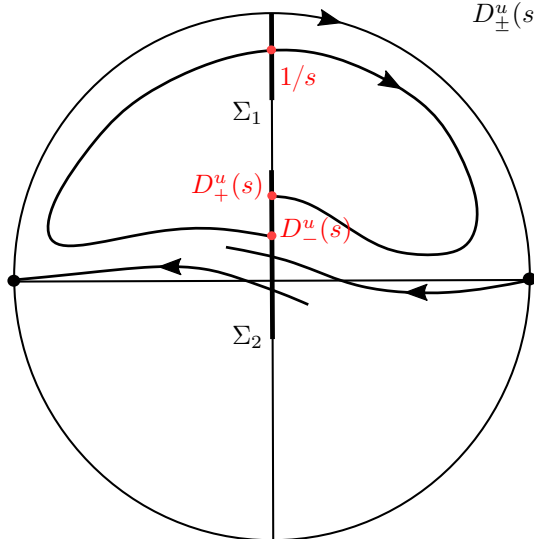
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$$D_{\pm}^u(s; \mu) = \delta_{\pm} + \Delta_0^{\pm} s^{\lambda} + \mathcal{F}_{\ell}^{\infty}(\mu_0)$$

$\mathcal{F}_{\ell}^{\infty}(\mu_0)$ is the class of ℓ -flat functions

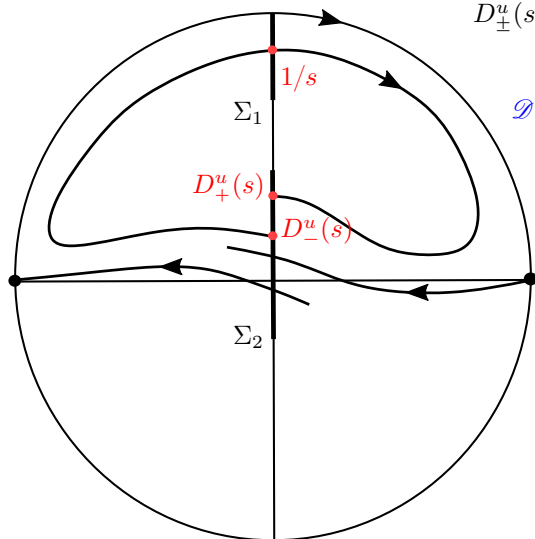
The cyclicity of quadratic hemicycles



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$\mathcal{F}_{\ell}^{\infty}(\mu_0)$ is the class of ℓ -flat functions and in this case $\ell \in [\lambda_0, \min(2\lambda_0, \lambda_0 + 1))$

The cyclicity of quadratic hemicycles



$$D_{\pm}^u(s; \mu) = \delta_{\pm} + \Delta_0^{\pm} s^{\lambda} + \mathcal{F}_{\ell}^{\infty}(\mu_0)$$

The limit cycles are zeros of

$$\mathcal{D}(s; \mu) := D_+^u(s; \mu) - D_-^u(s; \mu)$$

The cyclicity of quadratic hemicycles

Definition

Given $L \in \mathbb{R}$ and $\mu_\star \in \mathbb{R}^N$, we say that $\psi(s; \mu)$ is *L-flat with respect to s at $\mu_\star$* , and we write $\psi \in \mathcal{F}_L^\infty(\mu_\star)$, if for each $\ell = (\ell_0, \ell_1, \dots, \ell_N) \in \mathbb{Z}_{\geq 0}^{N+1}$ there exist a neighbourhood V of $\mu_\star$ and $C, s_0 > 0$ such that

$$\left| \frac{\partial^{|\ell|} \psi(s; \mu)}{\partial s^{\ell_0} \partial \mu_1^{\ell_1} \dots \partial \mu_N^{\ell_N}} \right| \leq C s^{L-\ell_0} \text{ for all } s \in (0, s_0) \text{ and } \mu \in V.$$



D. Marín and J. Villadelprat, *Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: local setting*, J. Differential Equations **269** (2020) 8425–8467.

The cyclicity of quadratic hemicycles

Lemma

Fix any $\mu_0 = (a_0, b_0, \varepsilon_0, \varepsilon_1, \varepsilon_2)$ with $(a_0, b_0) \in (-2, 0) \times (0, 2)$ and $\varepsilon_i \approx 0$ for $i = 0, 1, 2$. Then, setting $\lambda_0 := \lambda(\mu_0) = -\frac{a_0+2}{a_0}$,

$$D_{\pm}^u(s; \mu) = \delta_{\pm} + \Delta_0^{\pm} s^{\lambda} + \begin{cases} \Delta_1^{\pm} s^{\lambda+1} + \mathcal{F}_{\ell_1}^{\infty}(\mu_0) & \text{if } \lambda_0 > 1, \\ \Delta_2^{\pm} s^{2\lambda} + \mathcal{F}_{\ell_2}^{\infty}(\mu_0) & \text{if } \lambda_0 < 1, \end{cases}$$

for any $\ell_1 \in [\lambda_0 + 1, \min(2\lambda_0, \lambda_0 + 2))$ and $\ell_2 \in [2\lambda_0, \min(3\lambda_0, \lambda_0 + 1))$.



D. Marín and J. Villadelprat, *Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: general setting*, J. Differential Equations, **275** (2021) 684–732.

The cyclicity of quadratic hemicycles

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$\lambda_0 = 1 \Leftrightarrow a_0 = -1$ and we need to introduce a compensator to study this case (work in progress).

The cyclicity of quadratic hemicycles

Proposition

Fix any $\mu_0 = (a_0, b_0, 0, 0, 0)$ with $(a_0, b_0) \in (-2, 0) \times (0, 2)$. Then

$$\mathcal{D}_u(s; \mu) = \delta_u + \Delta_0^u s^\lambda + \begin{cases} \Delta_1^u s^{\lambda+1} + \mathcal{F}_{\ell_1}^\infty(\mu_0) & \text{if } a_0 > -1, \\ \Delta_2^u s^{2\lambda} + \mathcal{F}_{\ell_2}^\infty(\mu_0) & \text{if } a_0 < -1. \end{cases}$$

Moreover $\mathcal{D}_u(s; \mu_0) \equiv 0$, $\partial_{\varepsilon_0} \delta_u(\mu_0) \neq 0$, $\partial_{\varepsilon_1} \delta_u(\mu_0) = \partial_{\varepsilon_2} \delta_u(\mu_0) = 0$ and

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$$\Delta_0^u - \eta_1 \left(2 \frac{\sqrt{b(a+2)}}{\sqrt{a(b-2)}} \varepsilon_1 + \varepsilon_2 \right) \in \langle \delta_u \rangle$$

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Moreover $\mathcal{D}_u(s; \mu_0) \equiv 0$, $\partial_{\varepsilon_0} \delta_u(\mu_0) \neq 0$, $\partial_{\varepsilon_1} \delta_u(\mu_0) = \partial_{\varepsilon_2} \delta_u(\mu_0) = 0$ and

$$\Delta_0^u - \eta_1 \left(2 \frac{\sqrt{b(a+2)}}{\sqrt{a(b-2)}} \varepsilon_1 + \varepsilon_2 \right) \in \langle \delta_u \rangle$$

$$\Delta_1^u - \eta_2 \left(\varepsilon_1 + \frac{a(b-1)}{2(a+1)b} \varepsilon_2 + o(\|(\varepsilon_1, \varepsilon_2)\|) \right) \in \langle \delta_u, \Delta_0^u \rangle$$

The cyclicity of quadratic hemicycles

Proposition

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$$\Delta_2^u - \eta_3 \left(\frac{2(a+2)(b-1)}{(a+1)(b-2)} \varepsilon_1 + \varepsilon_2 + o(\|(\varepsilon_1, \varepsilon_2)\|) \right) \in \langle \delta_u, \Delta_0^u \rangle$$

The cyclicity of quadratic hemicycles

SKETCH OF THE PROOF OF THEOREM A

(1/2)

Suppose for instance that $a_0 \in (-2, -1)$. In this case

$$\mathcal{D}(s; \mu) = \delta + \Delta_0 s^\lambda + \Delta_1 s^{\lambda+1} + \mathcal{F}_L^\infty(\mu_0) \text{ with } L \in [\lambda_0 + 1, \min(2\lambda_0, \lambda_0 + 2))$$

and we define $\Psi(\mu) := (a - a_0, b - b_0, \delta(\mu), \Delta_0(\mu), \Delta_1(\mu))$. Hence

$$\det(D\Psi)(\mu_0) \sim \left| \begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline & & \star & 0 & 0 \\ \hline & & & 2\frac{\sqrt{b_0(a_0+2)}}{\sqrt{a_0(b_0-2)}} & 1 \\ & & & 1 & \frac{a_0(b_0-1)}{2b_0(a_0+1)} \end{array} \right| \sim a_0 + b_0$$

If $a_0 + b_0 \neq 0$ then $\nu = \Psi(\mu)$ is a change of coordinates at $\mu = \mu_0$.

The cyclicity of quadratic hemicycles

SKETCH OF THE PROOF OF THEOREM A

(2/2)

Then we can consider

$$\mathcal{D}(s; \Psi^{-1}(\nu)) = \nu_3 + \nu_4 s^\lambda + \nu_5 s^{\lambda+1} + f(s; \nu),$$

where $f \in \mathcal{F}_L^\infty(0_5)$ with $f(s; \nu)|_{\nu_3=\nu_4=\nu_5} \equiv 0$. Thus the remainder can be divided as $f(s; \nu) = \nu_3 f_1(s; \nu) + \nu_4 f_2(s; \nu) + \nu_5 f_3(s; \nu)$ with $f_i \in \mathcal{F}_L^\infty(0_5)$ so that,

$$\mathcal{D}(s; \Psi^{-1}(\nu)) = \nu_3(1 + \mathcal{F}_L^\infty) + \nu_4(s^\lambda + \mathcal{F}_L^\infty) + \nu_5(s^{\lambda+1} + \mathcal{F}_L^\infty).$$

From here the proof follows using Bautin's trick and the derivation-division algorithm.

The case $a_0 + b_0 = 0$ is more delicate because, using that

$$\mathcal{D}(s; \mu)|_{a_0+b_0=\varepsilon_0=2\varepsilon_1+\varepsilon_2=0} \equiv 0,$$

a different division has to be performed.

Alien limit cycles: A beyond all order phenomenon

$$\text{Cycl}((\Gamma, X_{\mu_0}), X_\mu) = \inf_{\varepsilon, \delta > 0} \sup_{\mu \in B_\delta(\mu_0)} Z_{(0, \varepsilon)}(\mathcal{D}(\cdot; \mu)) =: \mathcal{Z}$$

Here, given an interval I and a function f , $Z_I(f)$ stands for the number of zeros of f in I .

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In contrast, for each analytic arc $\mu = \xi(\epsilon)$ with $\xi(0) = \mu_0$ such that $\mathcal{D}(s; \xi(\epsilon)) \not\equiv 0$, we can take the Taylor's expansion at $\epsilon = 0$ and write

$$\mathcal{D}(s; \xi(\epsilon)) = \epsilon^{k_\xi} (M_\xi(s) + O(\epsilon)).$$

$M_\xi(s)$ is the first non-identically zero [Melnikov function](#).

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A [sufficient condition](#) for an alien limit cycle bifurcation is that $\mathcal{M} < \mathcal{Z}$.

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The definition of Dumortier and Roussarie is this one but taking the supremum over the [radial arcs \$\xi\$ for which \$k_\xi = 1\$](#) .

Thanks very much!!

Moitas grazas!!