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READING MULTIPLICITY IN UNFOLDINGS FROM ε -NEIGHBORHOODS OF ORBITS

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Limit cycle

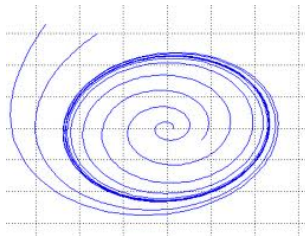


Figure: Limit cycle

Interesting objects in dynamics are *limit cycles*, orbits which are closed curves. Limit cycles appear in bifurcation.

Poincaré map

The appearance of limit cycles is related to the behavior of the Poincaré map near a singular point or limit cycle (or near a polycycle).

Fixed points of the Poincaré map correspond to limit cycles!

Multiplicity of the fixed points is related to the number of limit cycles which can appear after perturbation of the system.

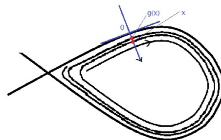
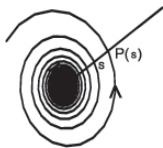


Figure: Poincaré map

HILBERT's 16th PROBLEM

$$\begin{cases} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y), \quad \deg(P, Q) \leq n. \end{cases}$$

Does there exist **an uniform bound** $H(n)$ for the number of closed periodic orbits in a polynomial vector field of degree n , depending only on the degree n .

References

This presentation is mainly based upon the following papers:

- P. Mardešić, M. Resman, V. Županović, *Multiplicity of fixed points and ε -neighborhoods of orbits*, J. Differ. Equations 253 (2012), no. 8, 2493-2514.
- R. Huzak, P. Mardešić, M. Resman, V. Županović, *Reading multiplicity in unfoldings from ε -neighborhoods of orbits*, preprint

Discrete orbit

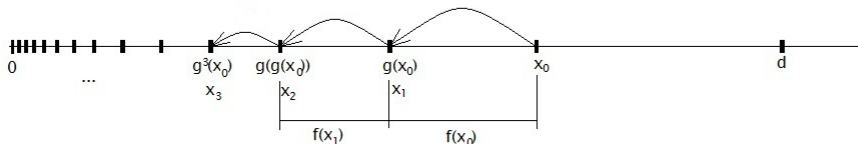


Figure: discrete orbit $\mathcal{O}_g(x_0) \rightarrow 0$ generated by g , $g = id - f$,
 $\mathcal{O}_g(x_0) = \{g^i(x_0) | i \in \mathbb{N}_0\}$.

generator g

multiplicity of
fixed point 0

orbit $\mathcal{O}_g(x_0)$

rate of growth of
 ε -neighborhood

$\leftrightarrow$ *box dimension
(g **diff.** at 0)

*critical Minkowski order
(g **not diff.** at 0)

Definition of box dimension of 1-dimensional set

- *upper s -dimensional Minkowski content of the orbit*,
 $0 \leq s \leq 1$: $\mathcal{M}^{*s}(\mathcal{O}_g(x_0)) = \limsup_{\varepsilon \rightarrow 0} \frac{A_\varepsilon(\mathcal{O}_g(x_0))}{\varepsilon^{1-s}}$

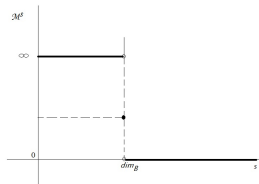


Figure: Minkowski content $\mathcal{M}^{*s}$ as function of $s \in [0, 1]$

- *upper box dimension*:
 $\overline{\dim}_B \mathcal{O}_g(x_0) = \inf\{s \geq 0 \mid \mathcal{M}^{*s}(\mathcal{O}_g(x_0)) = 0\}.$

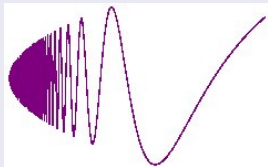
Minkowski content

- lower Minkowski content $\mathcal{M}_*^s$, lower box dimension $\underline{\dim}_B \mathcal{O}_g(x_0)$
- **box dimension** $s = \underline{\dim}_B \mathcal{O}_g(x_0) = \overline{\dim}_B \mathcal{O}_g(x_0)$
- $F(x) \simeq G(x)$, are **comparable** as $x \rightarrow 0$, if there exist $A, B, d > 0$ such that $A \leq F(x)/G(x) \leq B$, $x \in (0, d)$
- $\mathcal{M}^{*s}, \mathcal{M}_*^s \neq 0, \infty \Rightarrow A_\varepsilon(\mathcal{O}_g(x_0)) \simeq \varepsilon^{1-s}$, otherwise not comparable to any power of ε

Examples appearing as orbits of dynamical systems

Chirp $f(x) = x^\alpha \sin x^{-\beta}$, $d = \dim_B \Gamma = 2 - \frac{1+\alpha}{1+\beta}$, $0 < \alpha < \beta \leq 1$

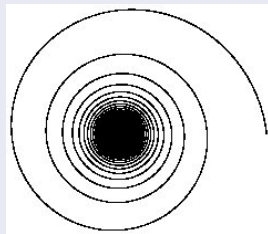
Spiral $r = \varphi^{-\alpha}$, $d = \dim_B \Gamma = \frac{2}{1+\alpha}$, $0 < \alpha \leq 1$ Claude Tricot
formulas from the book *Curves and Fractal Dimension*,
Springer-Verlag New York (1995)



$$\dim_B \Gamma = \frac{7}{4}, \alpha = 0.5, \beta = 1$$

Sequence

$$S \dots \left(\frac{1}{n^\alpha}\right) \quad d = \dim_B S = \frac{1}{1+\alpha}, \alpha > 0$$



$$\dim_B \Gamma = \frac{4}{3}, \alpha = 0.5$$

A motivating example

EXAMPLE 1.

- $g_1(x) = x - x^2$, (diff. generators)
- $g_2(x) = x - x^2(-\log x)$, $g_3(x) = x - x^2 \log(-\log x)$
 (nondiff. generators)
- $A_\varepsilon(\mathcal{O}_{g_1}) \simeq \varepsilon^{1/2}$ – **power-type behavior!**
- $\lim_{\varepsilon \rightarrow 0} \frac{A_\varepsilon(\mathcal{O}_{g_{2,3}})}{\varepsilon^{1/2}} = +\infty$, $\lim_{\varepsilon \rightarrow 0} \frac{A_\varepsilon(\mathcal{O}_{g_{2,3}})}{\varepsilon^{1/(2+\delta)}} = 0$, $\forall \delta > 0$
 – **noncomparable to any power!**
- $\dim_B \mathcal{O}_{g_1} = \frac{1}{2}$, but also $\dim_B \mathcal{O}_{g_{2,3}} = \frac{1}{2}$
- Minkowski content

$$\mathcal{M}^{1/2}(\mathcal{O}_{g_1}) > 0, \text{ but both } \mathcal{M}^{1/2}(\mathcal{O}_{g_{2,3}}) = 0$$

- In nondiff. case find **appropriate gauge functions** (instead of powers) to compare A_ε with $\rightarrow$ generalized Minkowski content (Lapidus); critical Minkowski order

The behavior of the ε -neighborhood of the orbit with respect to nondifferentiable generator

Theorem 1 (Mardešić, Resman, Županović, 2012)

- $f \in C^r(0, d)$, continuous on $[0, d)$, positive on $(0, d)$,
 $f(0) = f'(0) = 0$, $g = id - f$
- f sublinear:

$$m \leq x \cdot (\log f)'(x), \quad x \in (0, d), \quad m > 1.$$

Then

$$A_\varepsilon(\mathcal{O}_g(x_0)) \simeq f^{-1}(\varepsilon) \text{ as } \varepsilon \rightarrow 0.$$

★ e.g. $f(x) = \frac{x}{-\log x}$ not sublinear, $\frac{A_\varepsilon(\mathcal{O}_g(x_0))}{f^{-1}(\varepsilon)} \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Parabolic and hyperbolic case

Theorem 2

Let the conditions of the theorem 1 be satisfied. Let $1 < \mu_0^{fix}(g) < \infty$ that is the parabolic case then

$$A_\varepsilon(\mathcal{O}_g(x_0)) \simeq \varepsilon^{1/\mu_0^{fix}(g)}, \text{ as } \varepsilon \rightarrow 0.$$

Let $\mu_0^{fix}(g) = 1$ that is the hyperbolic case then

$$A_\varepsilon(\mathcal{O}_g(x_0)) \simeq \begin{cases} \varepsilon(-\log \varepsilon), & \text{if } f'(0) < 1 \\ \varepsilon \log(-\log \varepsilon), & \text{if } f'(0) = 1 \end{cases}, \text{ as } \varepsilon \rightarrow 0.$$

Next step-computation of the asymptotic expansion of $A_\varepsilon(\mathcal{O}_g(x_0))$ introducing a bifurcation parameter!

Our admissible class of generating functions

- f with asymptotic development in Chebyshev scale at $x = 0$,

Definition (CHEBYSHEV SCALE;

Mardešić: *Chebyshev systems and the versal unfolding of the cusp of order n*)

$\mathcal{I} = \{u_0, u_1, u_2, \dots\}$, $u_i \in C[0, d] \cap C^r(0, d)$, $r \in \mathbb{N} \cup \{\infty\}$ such that

- i) Division/differentiation algorithm can be performed r times except possibly at $x = 0$ ($\Rightarrow$ extension by continuity to 0):

$$\mathcal{I} = \{u_0, u_1, u_2, \dots\} / : u_0 \Rightarrow D_0(\mathcal{I}) = \left\{ \underbrace{1}_{D_0(u_0)}, \underbrace{\frac{u_1}{u_0}}_{D_0(u_1)}, \underbrace{\frac{u_2}{u_0}}_{D_0(u_2)}, \dots \right\} / ()'$$

$$\{D_0(u_1))', D_0(u_2)', \dots\} / : D_0(u_1)' \Rightarrow D_1(\mathcal{I}) = \left\{ \underbrace{1}_{D_1(u_1)}, \underbrace{\frac{(D_0(u_2))'}{(D_0(u_1))'}}_{D_1(u_2)}, \underbrace{\frac{(D_0(u_3))'}{(D_0(u_0))'}}_{D_1(u_3)}, \dots \right\} / ()'$$

$$\{D_1(u_2))', D_1(u_3)', \dots\} / : D_1(u_2)' \Rightarrow D_2(\mathcal{I}) = \left\{ \underbrace{1}_{D_2(u_2)}, \underbrace{\frac{(D_1(u_3))'}{(D_1(u_2))'}}_{D_2(u_3)}, \underbrace{\frac{(D_1(u_4))'}{(D_1(u_1))'}}_{D_2(u_4)}, \dots \right\} / ()'$$

ii) $D_i(u_{i+1})$ strictly increasing

iii) $D_j u_i(0) = 0$, $j < i$ in the sense of limit

$D_i(f) \dots$ *i -th generalized derivative of f in the scale $\mathcal{I}$*

Our admissible class of generating functions

Examples of Chebyshev scales

- $\mathcal{I} = \{1, x, x^2, x^3, x^4, \dots\}$ -diff. case,
- $\mathcal{I} = \{x^{\alpha_0}, x^{\alpha_1}, x^{\alpha_2}, \dots\}$, $\alpha_i \in \mathbb{R}$, $0 < \alpha_0 < \alpha_1 < \alpha_2 < \dots$
- $\mathcal{I} = \{1, x(-\log x), x, x^2(-\log x), x^2, x^3(-\log x), x^3, \dots\}$
- any set of monomials of the type $x^k(-\log x)^l$, ordered by increasing flatness:
- $x^i(-\log x)^j < x^k(-\log x)^l$ if and only if $(i < k)$ or $(i = k \text{ and } j > l)$.

Multiplicity of fixed point zero in a family

$g \in C^r[0, d)$, 0 fixed point; $f = \text{id} - g$

- $\mu_0(f) = k$, if $f(0) = f'(0) = \dots = f^{(k-1)}(0) = 0$, $f^{(k)}(0) \neq 0$
- $\mu_0^{fix}(g) := \mu_0(f) = k$
- $g, (g_\lambda)$ family
- $\mu_0(g, (g_\lambda)) \geq m$ if for any neighborhood of $x = 0$ there exists some function in (g_λ) , arbitrarily close to g , with at least m fixed points in the given neighborhood (different from 0)
- standard multiplicity in diff. case = multiplicity of f in a family of all diff. functions
- (Mardešić: *Chebyshev systems and the versal unfolding of the cusp of order n*)

Connection between multiplicity and $A_\varepsilon(\mathcal{O}_g(x_0))$

- (f_λ) , asymptotic development in a family of T-scales $\mathcal{I}_\lambda$. The scale consists of u_k .

Theorem 3 (MRŽ, 2012)

$f = f_{\lambda_0}$ satisfies all assumptions of Theorem 2 and the **upper power condition**:

$$x \cdot (\log f(x))' \leq M, \quad x \in (0, d), \quad \text{for some constant } M > 0.$$

Then the following is equivalent:

- 1 $D_i(f)(0) = 0$ for $i = 0, \dots, k-1$ and $D_k(f)(0) > 0$, $k \geq 1$,
 $(f \simeq u_k, \quad k \geq 1)$,
- 2 $A_\varepsilon(\mathcal{O}_g(x_0)) \simeq u_k^{-1}(\varepsilon)$,
- 3 $\mu_0^{fix}(g, (g_\lambda)) = k$.

Differentiable and nondifferentiable Poincaré maps around different limit periodic sets—application of results

- To find the multiplicity of differentiable and nondifferentiable Poincaré maps around different limit periodic sets
weak/strong focus, limit cycle, saddle loop, 2 saddle loop
- **strong/weak focus, limit cycle** $\rightarrow$ Poincaré map differentiable at $x = 0$
- Roussarie's expansion of the Poincaré map around **saddle loop**: logarithmic terms $x^k(-\log x) \rightarrow$ nondiff. at $x = 0$

ε -neighborhood of an orbit generated by f

$$A_\varepsilon(\mathcal{O}_f(x_0)) = N_\varepsilon(\mathcal{O}_f(x_0)) + T_\varepsilon(\mathcal{O}_f(x_0))$$

$$N_\varepsilon(\mathcal{O}_f(x_0)) = f^{n_\varepsilon}(x_0) + 2\varepsilon$$

$$T_\varepsilon(\mathcal{O}_f(x_0)) = n_\varepsilon \cdot 2\varepsilon,$$

where $n_\varepsilon \in \mathbb{N}$ is the discrete critical time separating

- **length of the tail** $T_\varepsilon(\mathcal{O}_f(x_0))$, the union of finitely many disjoint intervals and
- **length of the nucleus** $N_\varepsilon(\mathcal{O}_f(x_0))$, the union of intervals that overlap.

Definition of the continuous ε -neighborhood of an orbit

Assume that f embeds as the time-one map in a flow $\{f^t\}_{t \in (0, +\infty)}$, f^t analytic on $(0, d)$. The **continuous time length of the ε -neighborhood of orbit $\mathcal{O}_f(x_0)$** with respect to the flow $\{f^t\}$ is defined as:

$$A_f^c(x_0, \varepsilon) = (f^{\tau_\varepsilon(x_0)}(x_0) + 2\varepsilon) + \tau_\varepsilon(x_0) \cdot 2\varepsilon. \quad (1)$$

Here, $\tau_\varepsilon(x_0)$ is the *continuous critical time* for the ε -neighborhood of the orbit $\mathcal{O}^f(x_0)$, defined by the equation:

$$f^{\tau_\varepsilon(x_0)}(x_0) - f^{\tau_\varepsilon(x_0)+1}(x_0) = 2\varepsilon \quad (2)$$

Computation of the continuous ε -neighborhood of orbit

Proposition

f analytic on $(0, d)$, $d > 0$, zero a fixed point, $g = \text{id} - f$ increasing, strictly positive. Assume that f embeds as the time-one map in a C^1 -flow $\{f^t\}_{t \in (-\delta, +\infty)}$, $\delta > 0$, f^t analytic on $(0, d)$. Then

$$A_f^c(x_0, \varepsilon) = (g^{-1}(2\varepsilon) + 2\varepsilon) + 2\varepsilon \cdot [\Psi(g^{-1}(2\varepsilon)) - \Psi(x_0)].$$

Example

$\dot{x} = x^2$, $f(x) = \frac{x}{1-x}$ by integrating $\int_x^{f(x)} \frac{dx}{x^2} = \int_0^1 dt$,

from $g = \text{id} - f$ obtain $g(x) = \frac{x^2}{x-1}$.

The Fatou (time) coordinate $\Psi(x) = \int_0^x \frac{dx}{x^2} = -\frac{1}{x}$.

Then we get

$$A_f^c(x_0, \varepsilon) = 2(2\varepsilon + \varepsilon^2)^{1/2} + 2\varepsilon + \frac{2\varepsilon}{x_0} \simeq \varepsilon^{1/2}$$

Saddle-node bifurcation

The saddle-node bifurcation is a generic 1-parameter bifurcation of 1-dimensional vector fields. Let

$$\frac{dx}{dt} = F(x, \nu), \quad (3)$$

with F real, analytic germ in x and in parameter ν , and with a non-hyperbolic singular point $x = 0$ at the bifurcation value $\nu = 0$ (i.e. $F(0, 0) = 0$, $F_x(0, 0) = 0$), satisfying the generic assumptions:

$$F_\nu(0, 0) \neq 0, \quad F_{xx}(0, 0) \neq 0. \quad (4)$$

Under these assumptions, the saddle-node point at $x = 0$ bifurcates at $\nu = 0$ into two hyperbolic points on the real axis: one attracting and one repelling, for $\nu \in (0, \delta)$, or $\nu \in (-\delta, 0)$ depending on the sign of F_ν and F_{xx} in (4).

Saddle-node bifurcation-model

The model system

$$\frac{dx}{dt} = -x^2 + \nu. \quad (5)$$

- 1 $\nu > 0$: one repelling hyperbolic singular point at $-\sqrt{\nu}$ and one attracting hyperbolic point at $\sqrt{\nu}$;
- 2 $\nu < 0$: no singular points on the real axis
- 3 $\nu = 0$: one parabolic point at $x = 0$ stable from the right and unstable from the left.

Nucleus and tail with a parameter

Take x_0 as an initial point. Consider the time-one map f_ν of (3) and its orbit with initial point x_0 :

$$\mathcal{O}_{f_\nu}(x_0) := \{f_\nu^n(x_0) : n \in \mathbb{N}_0\}.$$

By $\mathcal{O}_{f_\nu}(x_0)_\varepsilon$, we denote its ε -neighborhood.

ε -neighborhood of the orbit $\mathcal{O}_{f_\nu}(x_0)$ is a disjoint union of two parts, the tail and the nucleus:

$$\mathcal{O}_{f_\nu}(x_0)_\varepsilon = T_{\varepsilon,\nu} \cup N_{\varepsilon,\nu}. \quad (6)$$

Here, the *tail* $T_{\varepsilon,\nu}$ is the union of finitely many disjoint intervals, and the *nucleus* $N_{\varepsilon,\nu}$ one interval made of a union of infinitely many overlapping ε -neighborhoods of points in the orbit.

Écalle-Roussarie compensator

Definition

Let ν and x be small. The function

$$\omega(x, \nu) := \frac{x^{-\nu} - 1}{\nu}$$

is called the Écalle-Roussarie compensator.

Note that, pointwise, $\omega(x, \nu) \rightarrow -\log x$, as $\nu \rightarrow 0$. The convergence becomes uniform in x , if we multiply by x^δ , $\delta > 0$.

Inverse compensator

Definition (The inverse compensator)

For $x > 0$ and $\nu \in (-\delta, \delta)$, we call the function

$$\alpha(x, \nu) := \frac{1}{\nu} \log \left(1 + \frac{\nu}{x} \right)$$

the inverse compensator.

The name comes from the fact that

$$\alpha(x, \nu) = -\log \circ \omega^{-1} \left(\frac{1}{x}, \nu \right),$$

where ω^{-1} is the inverse of ω with respect to the variable x .

Pointwise, $\alpha(x, \nu) \rightarrow \frac{1}{x}$, as $\nu \rightarrow 0$. For every $\delta > 0$, we get $x^{1+\delta} \alpha(x, \nu) \rightarrow x^\delta$, as $\nu \rightarrow 0$, uniformly in $x > 0$.

Inverse compensator

The asymptotic behavior, as $x \rightarrow 0$, is qualitatively different in the case $\nu = 0$ and $\nu \neq 0$:

$$\alpha(x, \nu) = \begin{cases} \frac{1}{x}, & \nu = 0, \\ \frac{1}{\nu}(-\log x) + \frac{\log \nu}{\nu} + \mathbb{R}_\nu[[x]], & \nu \neq 0. \end{cases} \quad (7)$$

Here $\mathbb{R}[[x]]$ denotes a formal expansion in powers of x .

The square root compensator

Definition (The square root compensator)

For ν small by absolute value and $x > 0$, we define

$$\tilde{\eta}(x, \nu) := \sqrt{x + \nu} - \sqrt{\nu},$$

and call $\tilde{\eta}$ the square root-type compensator.

The asymptotic expansion of $\tilde{\eta}$, as $x \rightarrow 0$, changes qualitatively as ν changes from zero:

$$\tilde{\eta}(x, \nu) = \begin{cases} \sqrt{x}, & \nu = 0, \\ \frac{x}{\sqrt{\nu}} + \sqrt{\nu} \frac{x^2}{\nu^2} \mathbb{R}\{\frac{x}{\nu}\}, & \nu > 0, x \rightarrow 0. \end{cases} \quad (8)$$

$\tilde{\eta}$ is *small* for small x , $\tilde{\eta}(x, \nu) \rightarrow \sqrt{x}$ uniformly in x , as $\nu \rightarrow 0+$.
 Here $\mathbb{R}\{x\}$ is an analytic germ in x .

Theorem—expansion and correspondence

Theorem 4 (HMRŽ, 2023)

Let $\frac{dx}{dt} = F(x, \nu)$ be a generic 1-parameter analytic unfolding of a parabolic singular point, satisfying the generic assumptions (4). Let f_ν be its time-one map and $\mathcal{O}_{f_\nu}(x_0)$ be its attractive orbit starting at x_0 .

There exists a compensator variable $\eta(\varepsilon, \nu)$ and an asymptotic expansion of the length of the tail $A(T_{\nu, \varepsilon})$ in a Chebyshev scale, uniform in $\nu \in [0, d]$, as $\eta \rightarrow 0$.

There is a 1 – 1 correspondence between the expansion of the length $A(T_{\nu, \varepsilon})$ in the η variable and the Taylor expansion of the displacement function $g_\nu := \text{id} - f_\nu$ in the phase x -variable, in the following sense. For every value of the parameter ν , the number of vanishing terms of the expansions of $A(T_{\nu, \varepsilon})$ and g_ν , in the corresponding scales, is the same.

Remark on compensator variable

The inverse compensator appears in the asymptotic expansion.

The variable η is a small variable, in the sense that it tends to 0 as $(\varepsilon, \nu) \rightarrow 0$, and behaves asymptotically as the square root compensator $\tilde{\eta}$.

Thank you!