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J. Morlet
J-O. Strömberg

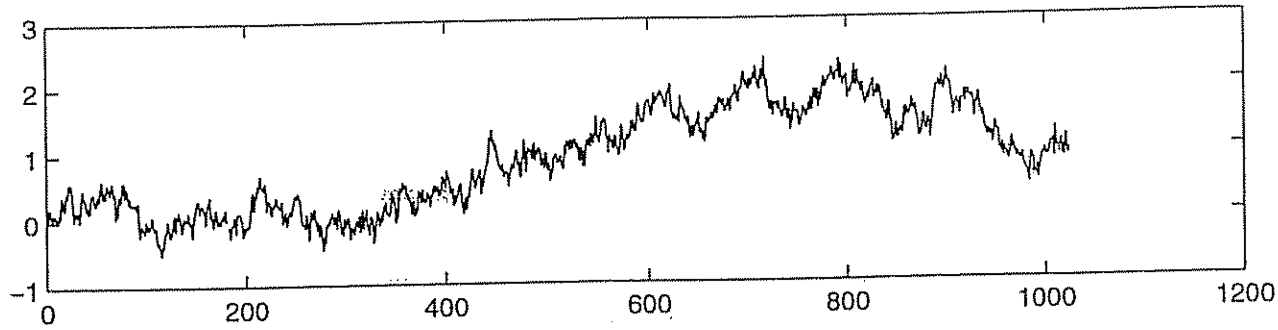
Uniform Hölder regularity

Definition: Let $f : \mathbb{R}^d \to \mathbb{R}$; let $0 < s < 1$;
$f \in C^s(\mathbb{R}^d)$ if $f \in L^\infty(\mathbb{R}^d)$ and if
$\exists C > 0, \forall x, y \quad |f(x) - f(y)| \leq C|x-y|^s$.

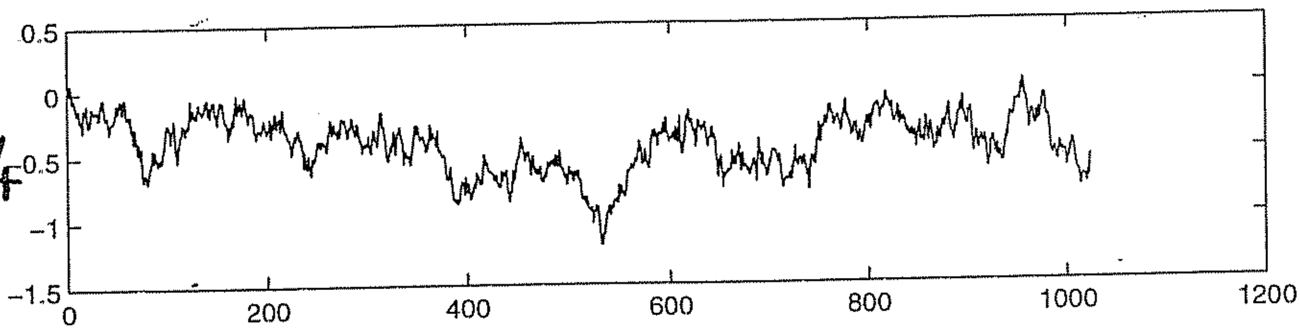
If $s \geq 1$, $f \in C^s(\mathbb{R}^d)$ if $\forall \alpha$ such that $|\alpha| \leq s$, $\partial^\alpha f$ is continuous and bounded, and if $|\alpha| = [s]$, then
$\exists C > 0, \forall x, y \quad |\partial^\alpha f(x) - \partial^\alpha f(y)| \leq C|x-y|^{s-[s]}$.

Fractional Brownian Motions

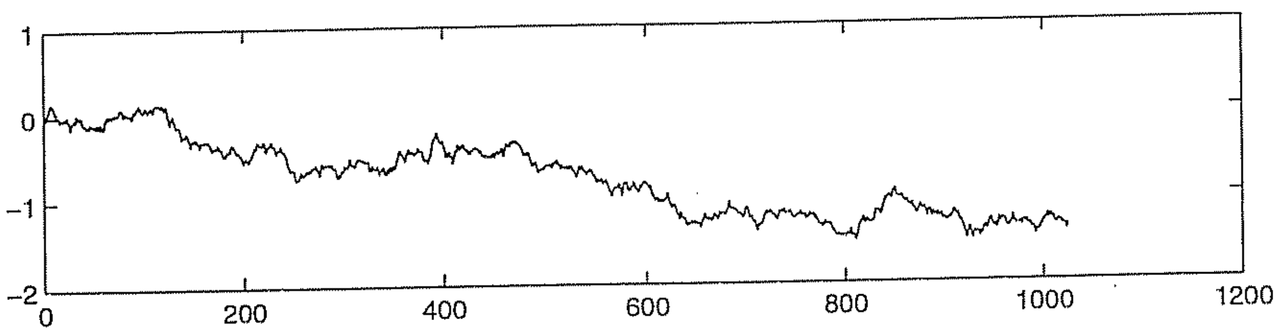
S=0,3



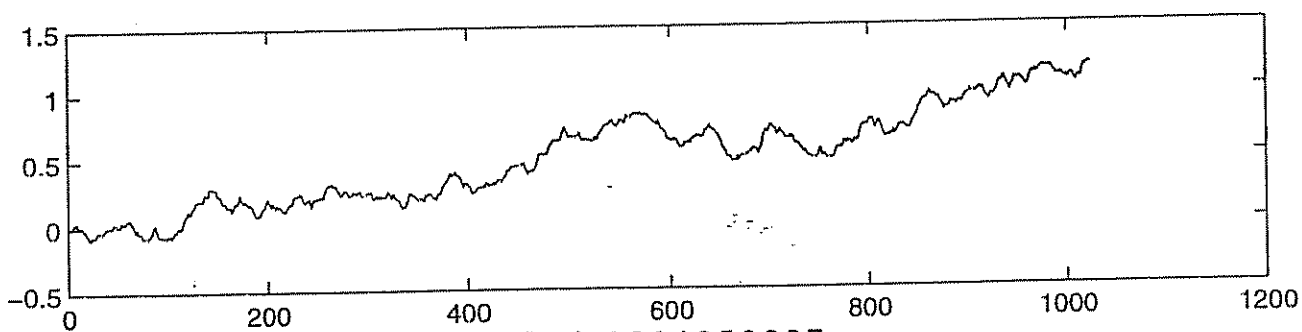
S=0,4



0,5

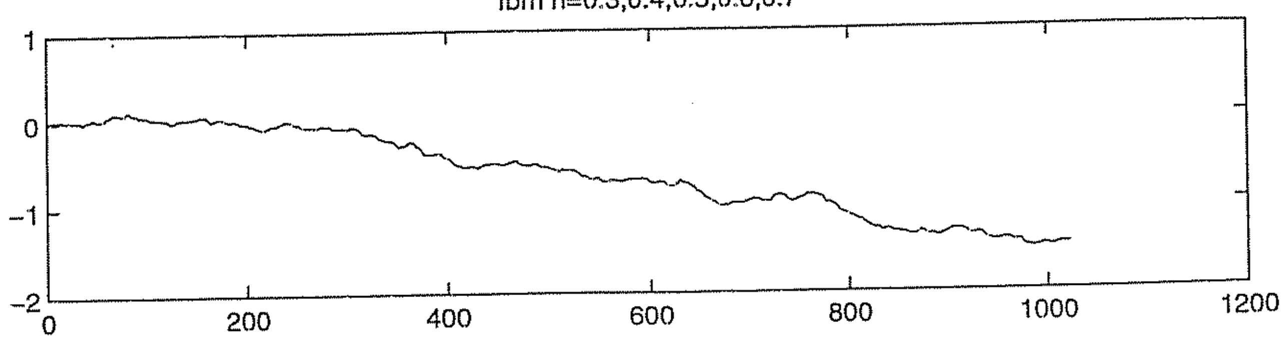


0,6

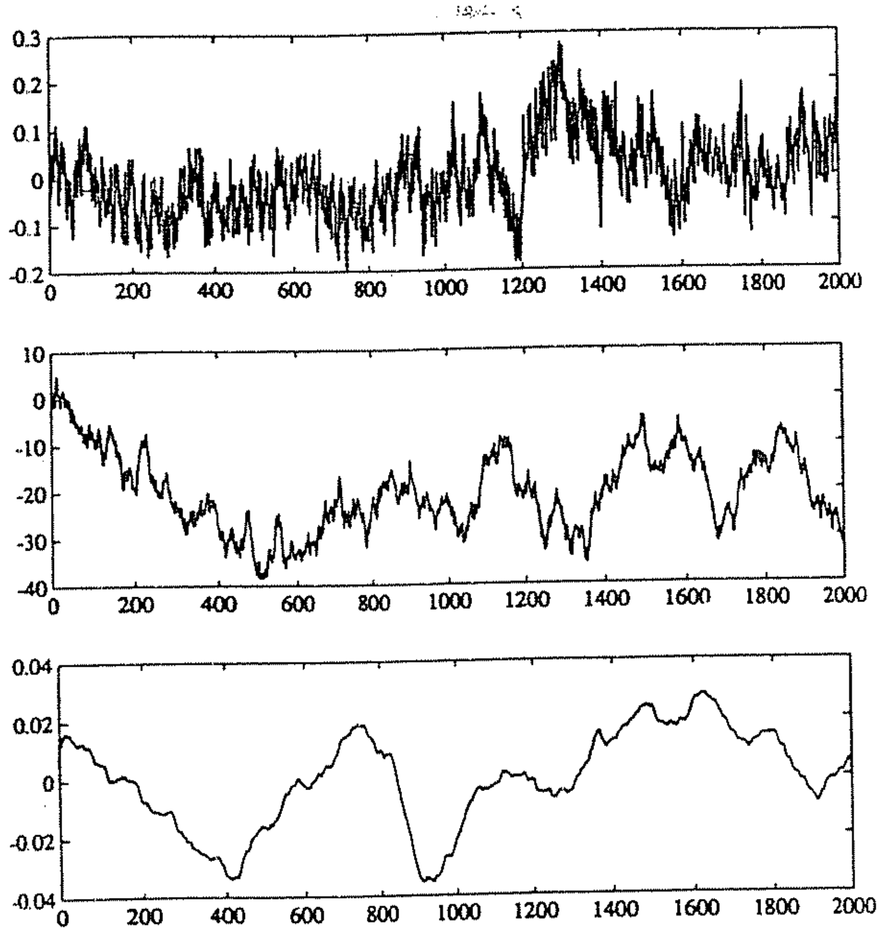


fbm h=0.3,0.4,0.5,0.6,0.7

0,7



Fractional Brownian motions



S = 0,05

S = 0,5

S = 0,95

Figure 1. Examples of fBm realizations
Top: $H = 0.05$, $D = 1.95$; middle: $H = 0.5$, $D = 1.5$; bottom: $H = 0.95$, $D = 1.05$. As the index H is increased, the fractal dimension D is decreased, as well as the "roughness" of the fBm realization, considered as a curve

First attempts for analysing or synthesising fBm *via* wavelets have been proposed in [6] [9] and, more recently, in [10] [11] and [12]. Although it would be equally possible to make use of continuous wavelet representations [6] [11], we will here focus on *discrete* and *orthonormal* wavelet decompositions.

Weierstrass functions

Let $A < 1$, $B > 1$ and $AB > 1$; then

$$W_{A,B}(x) = \sum_{n=1}^{\infty} A^n \cos(B^n x)$$

Proposition: Let $\alpha = -\dfrac{\operatorname{Log} A}{\operatorname{Log} B}$; then $W_{A,B} \in C^{\alpha}(\mathbb{R})$.

Proof: $$|W_{A,B}(x) - W_{A,B}(y)| \leq \sum_{n=1}^{\infty} A^n |\cos(B^N x) - \cos(B^N y)|$$

$$\leq \sum_{n=1}^{N} A^n B^n |x-y| + 2 \sum_{n=N+1}^{\infty} A^n$$

(N is the largest integer such that $B^N |x-y| \leq 1$)

$$\leq C(A^N B^N |x-y| + A^N) \leq C' A^N \leq C'' |x-y|^{-\frac{\operatorname{Log} A}{\operatorname{Log} B}}$$

$$\left(A^N = e^{N \operatorname{Log} A} = e^{\operatorname{Log} B \, N \frac{\operatorname{Log} A}{\operatorname{Log} B}} = B^{N \frac{\operatorname{Log} A}{\operatorname{Log} B}} \right)$$

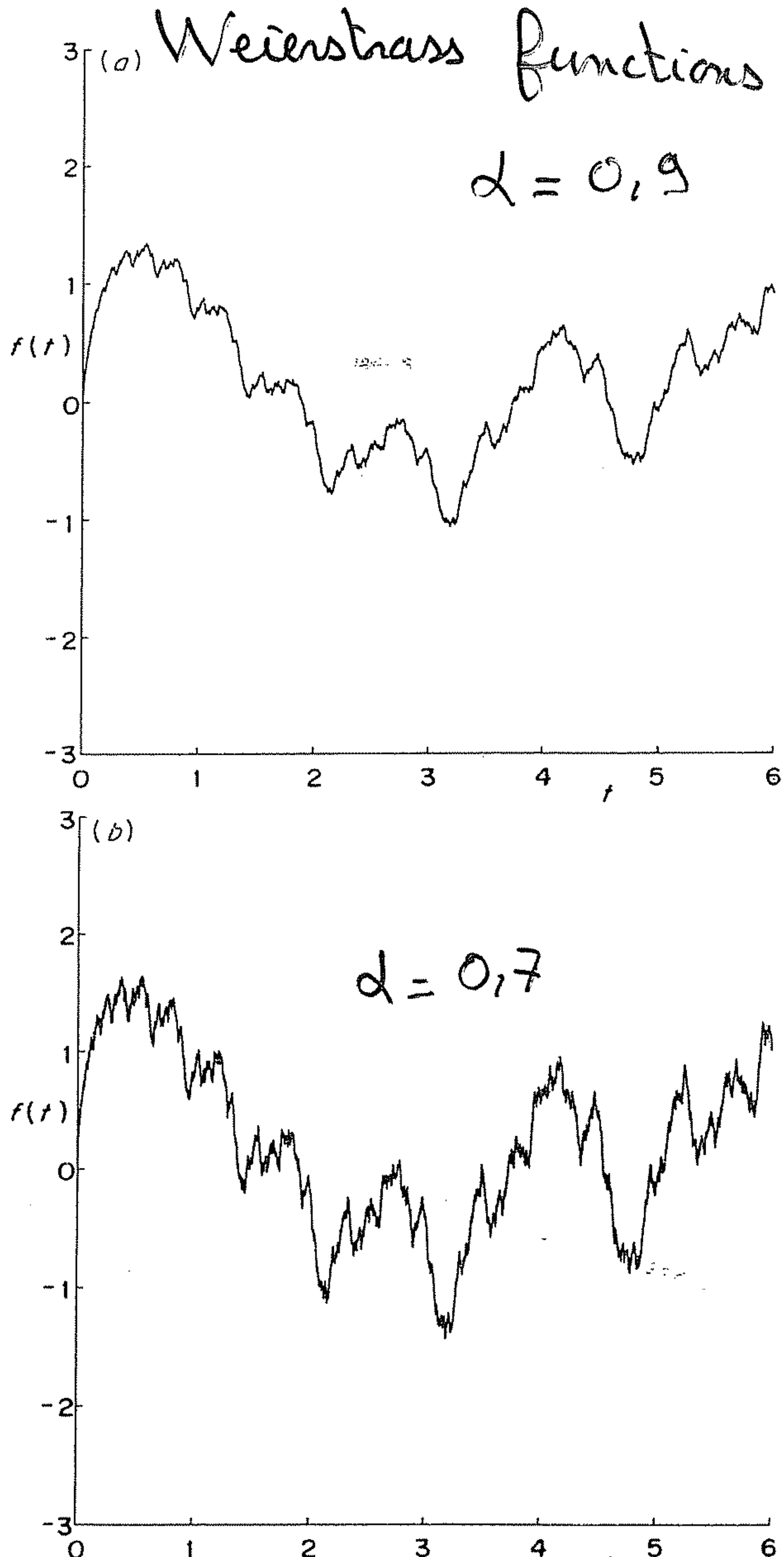


Figure 11.2 The Weierstrass function $f(t) = \sum_{k=0}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k t)$ with $\lambda = 1.5$ and $(a) s = 1.1$, (b) $s = 1.3$, (c) $s = 1.5$, (d) $s = 1.7$

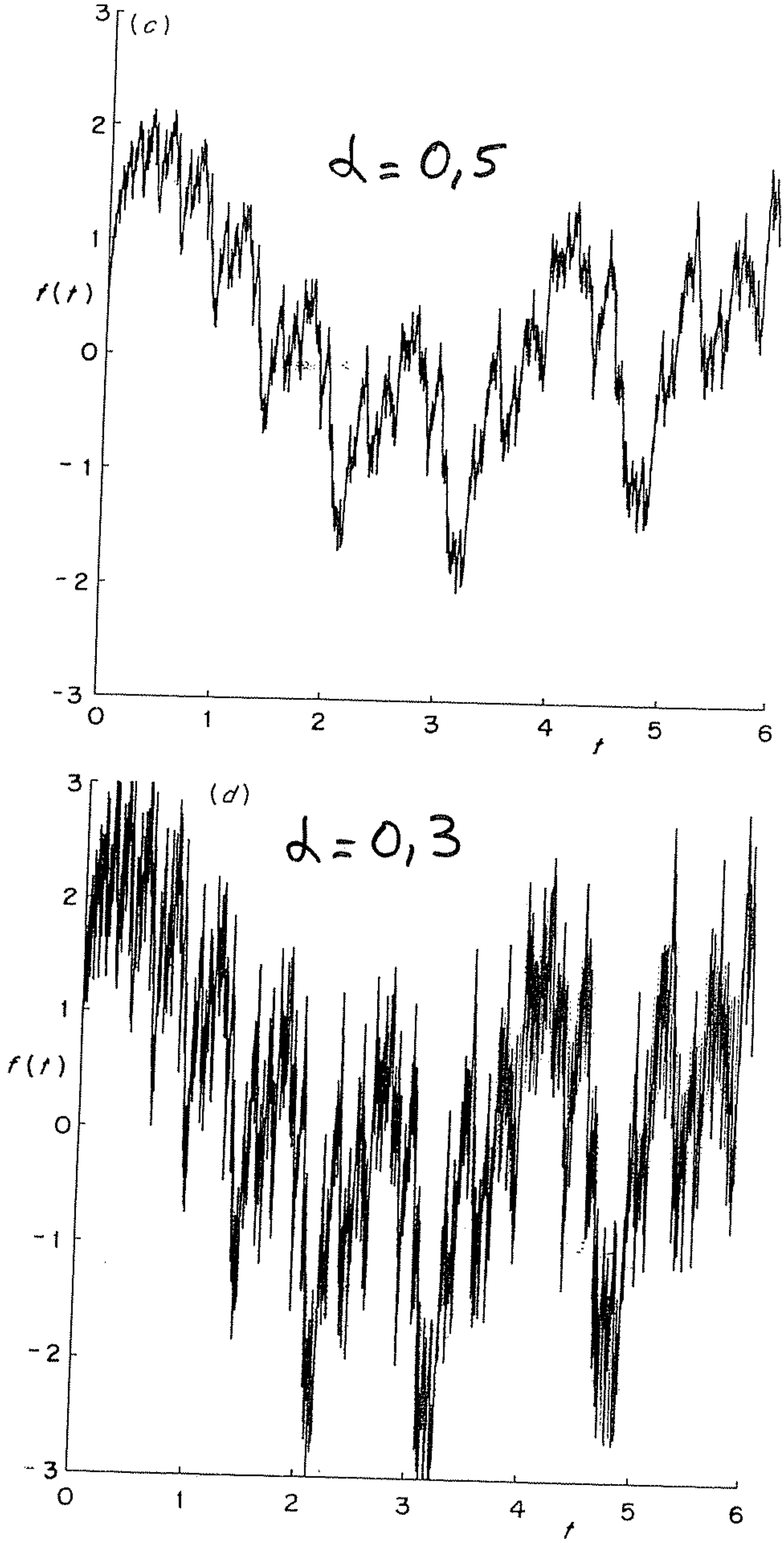


Figure 11.2 (*Continued*)

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Brownian motion

[illegible]: The Brownian motion $(B_t)_{t \geq 0}$ is a random function satisfying:

- If $t > s$, $B_t - B_s$ is independent of B_s
- $B_t - B_s \overset{\mathcal{L}}{\approx} B_{t-s}$
- $t \to B_t$ is almost surely continuous

Proposition: For any $\varepsilon > 0$, B_t is $C^{1/2 - \varepsilon}(\mathbb{R}^+)$; and

$$|B_t - B_s| \leq C \sqrt{\mathrm{Log}\left(\frac{1}{t-s}\right)} \sqrt{|t-s|} \quad \text{if } |t-s| \leq \frac{1}{2}$$

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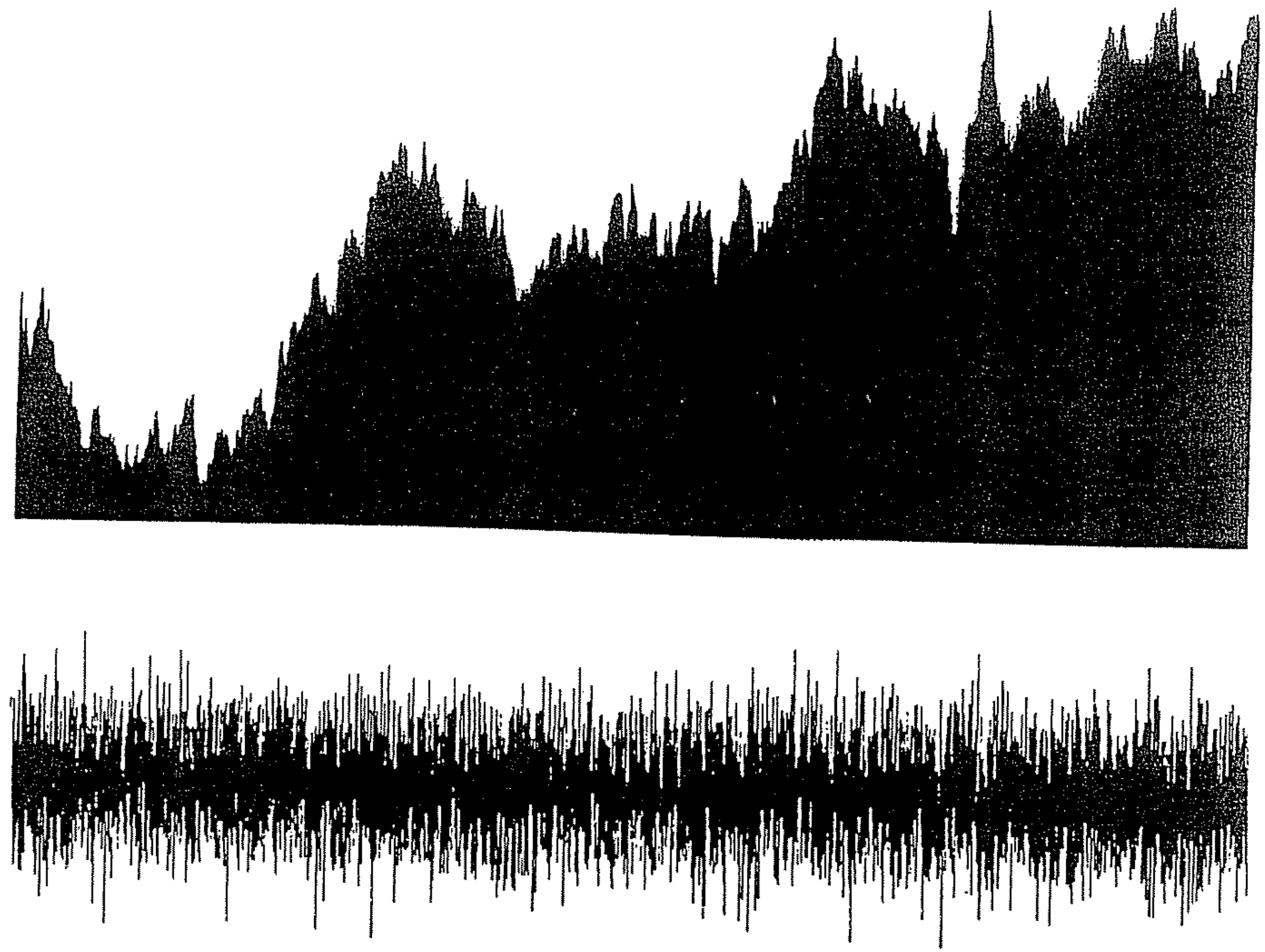
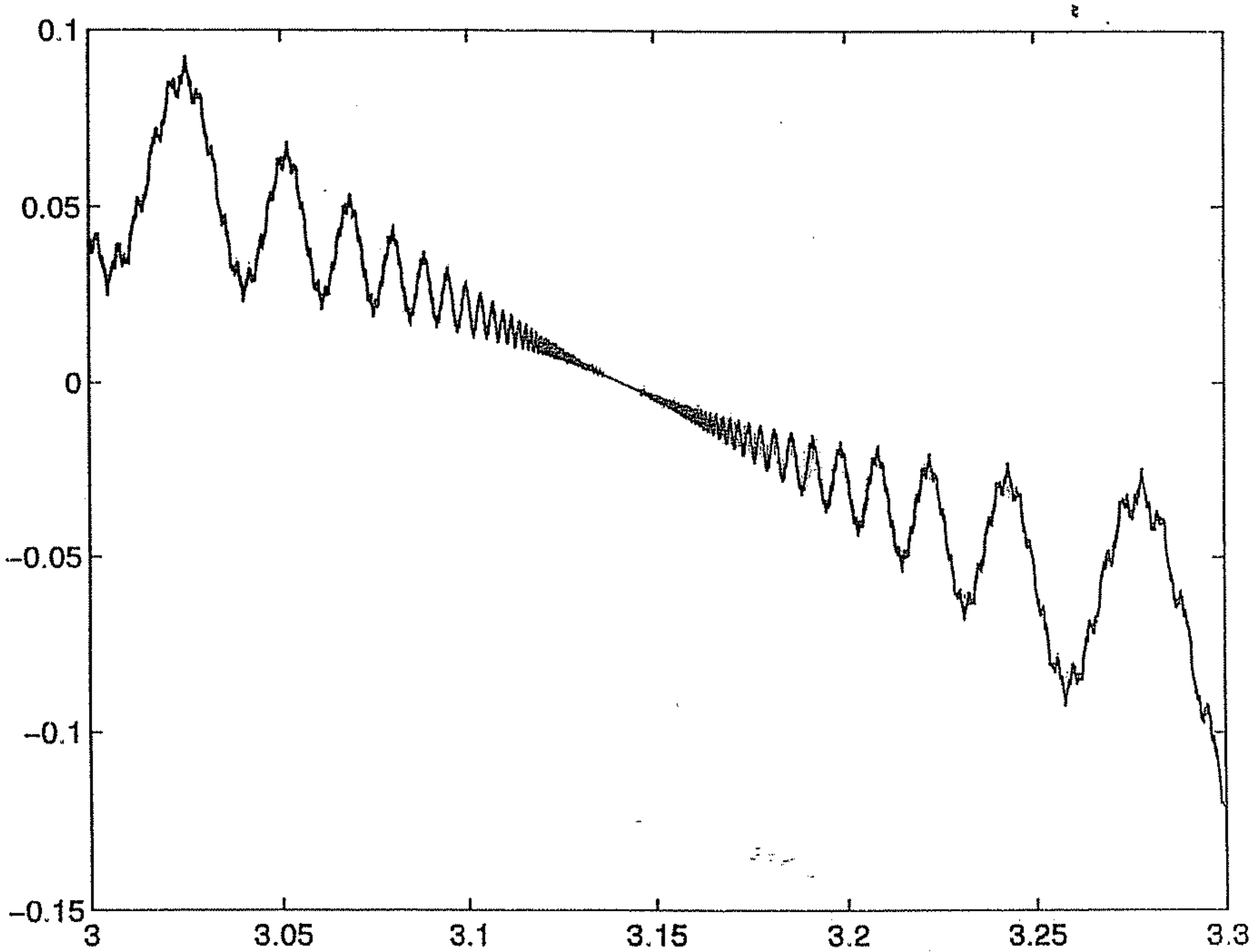


FIGURE E1-3. Graph of a sample of Brownian motion (top), and its white noise increments in units of 1 standard deviation (bottom).

ZOOM

Magnification of $\sum_{1}^{\infty} n^{-2} \sin(n^2 x)$ around π



Fractional Brownian Motion

let $0 < H < 1$; $B_H(t)$ satifies

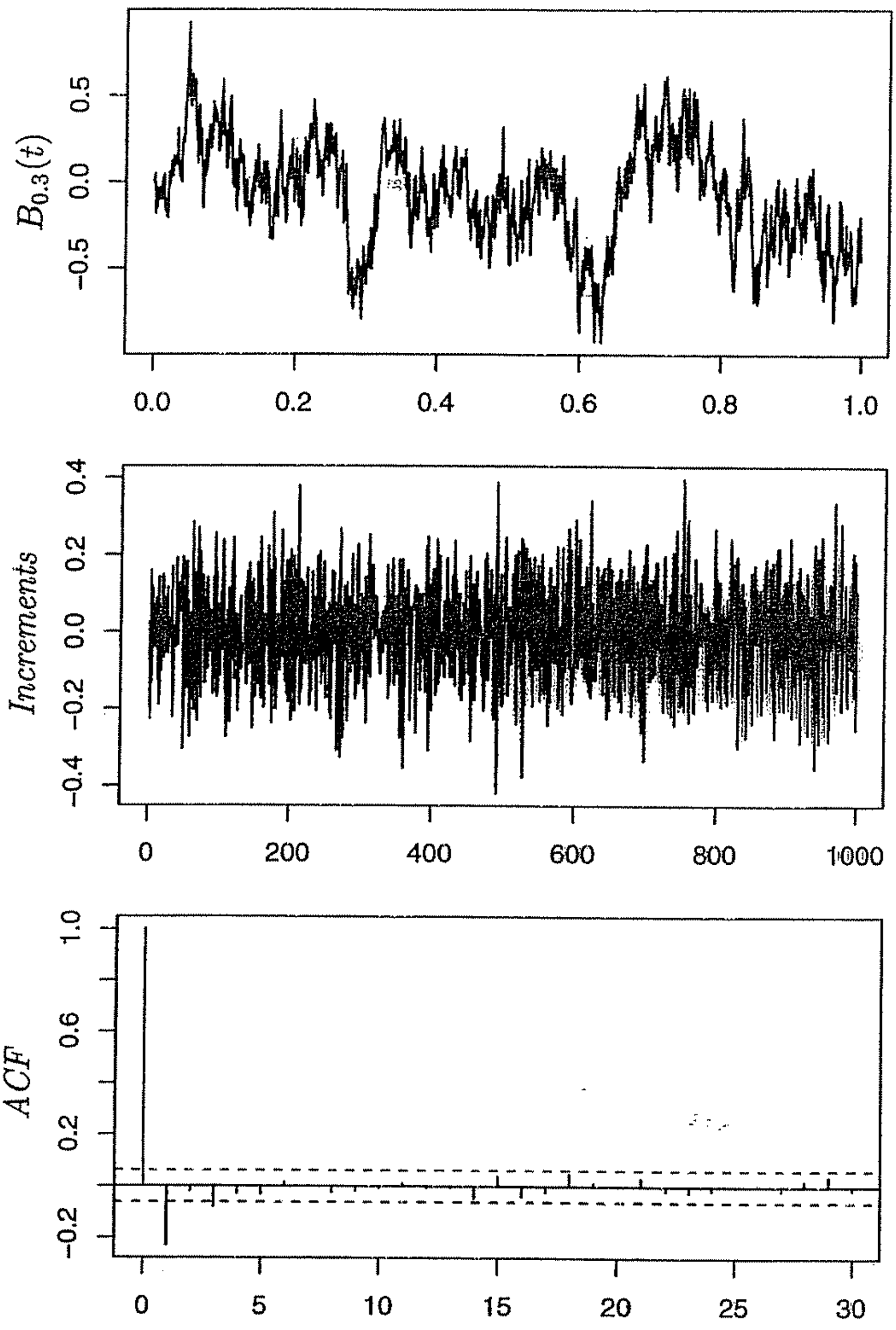
- $B_H(0) = 0$
- $B_H(t)$ is a Gaussian process

($\forall t_1, \dots, t_n$, $\forall \alpha_1, \dots, \alpha_n$, $\sum \alpha_i B_i(t_i)$ has a Gaussian distribution)

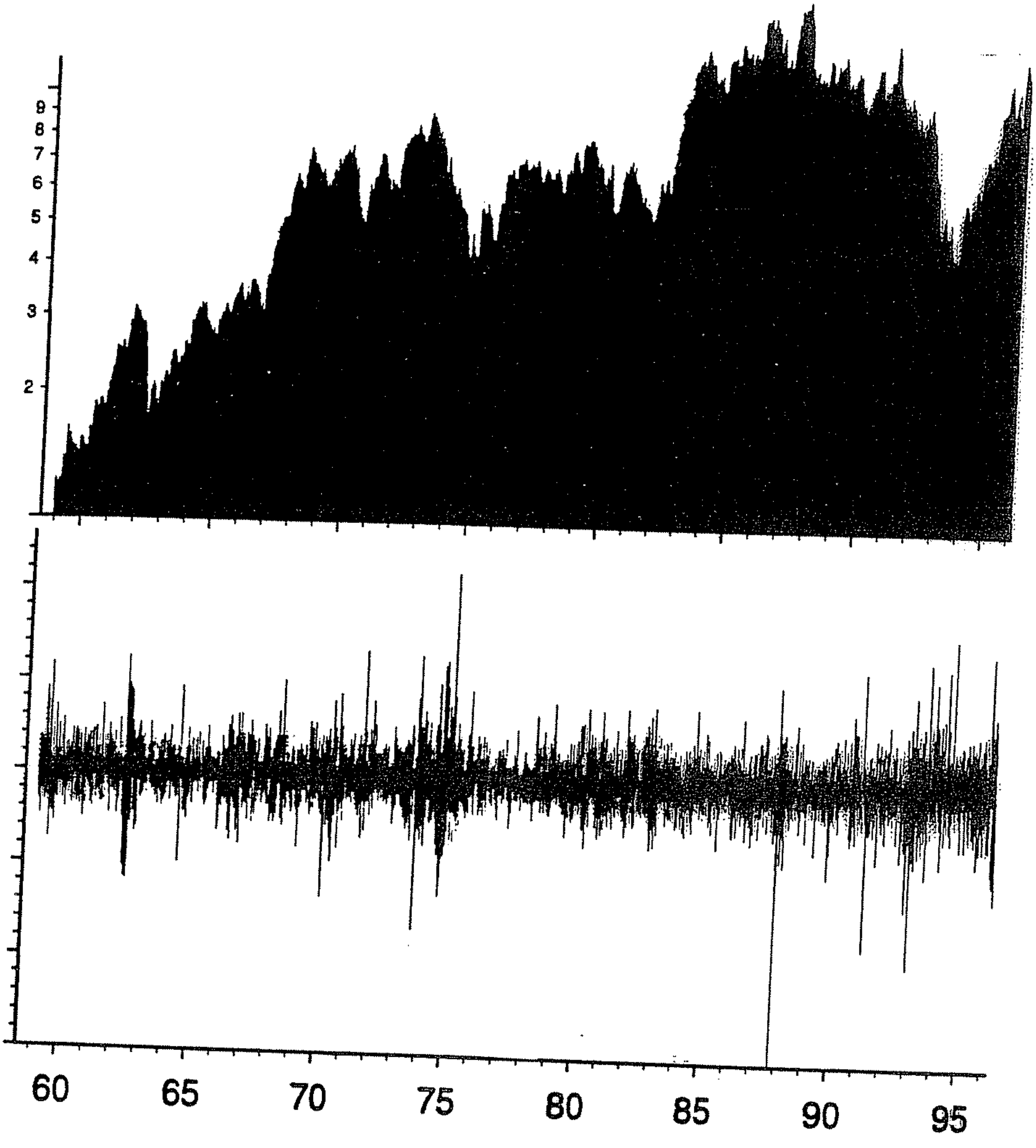
- Its covariance is

$$E(B_H(t) B_H(s)) = |t|^{2H} + |s|^{2H} - |t-s|^{2H}$$

Fractional Brownian motion

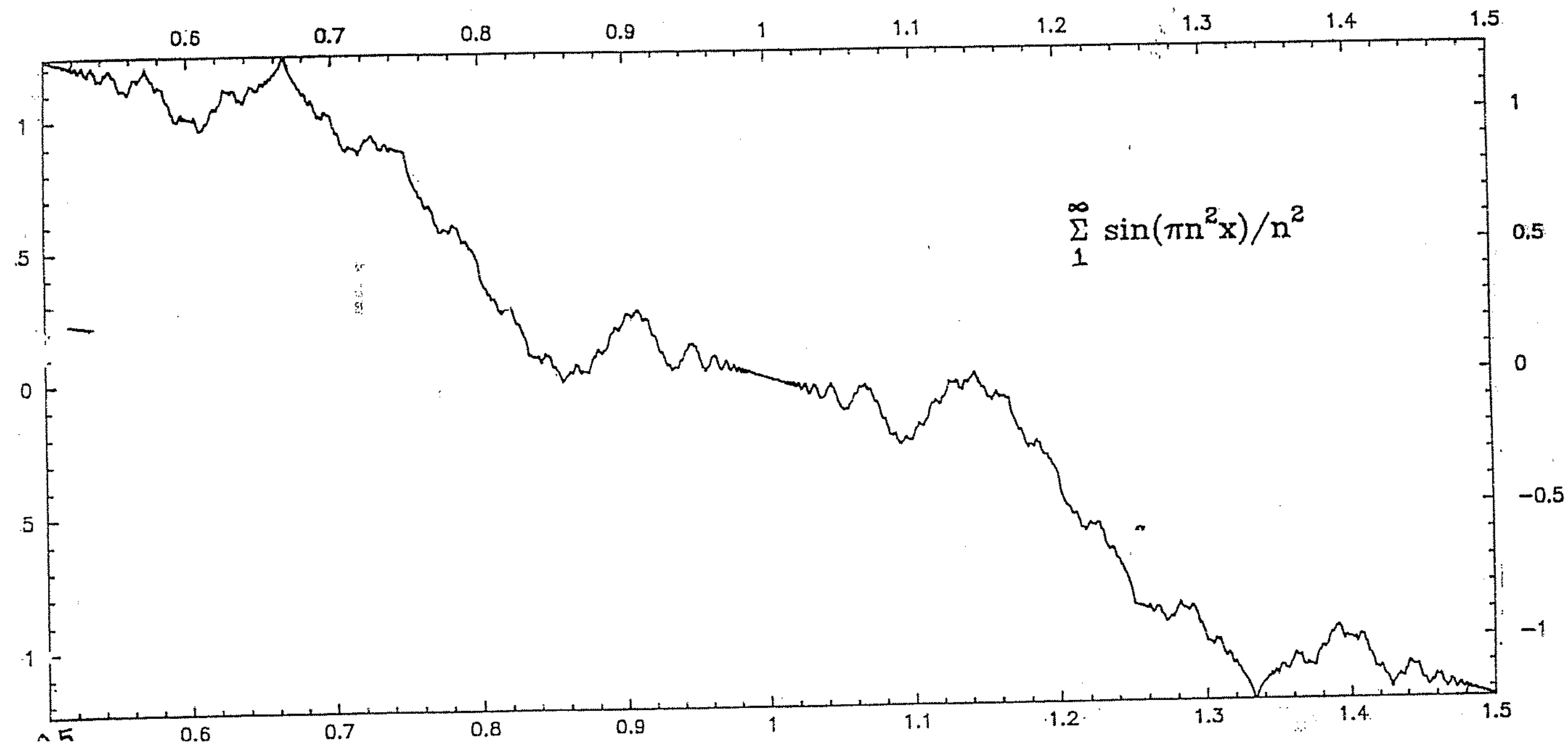


igure 7.4.2 Sample paths of fractional Brownian motion for $H = 0.3$, with the orresponding increment process and sample autocorrelation function. The correlations ecay fast and are negative, as expected.



IGURE E1-1. Top: IBM stock from 1959 to 1996, in units of $10, plotted on logarithmic scale. Bottom: the corresponding relative daily price changes, in units of 1%.

Riemann's "nondifferentiable" function



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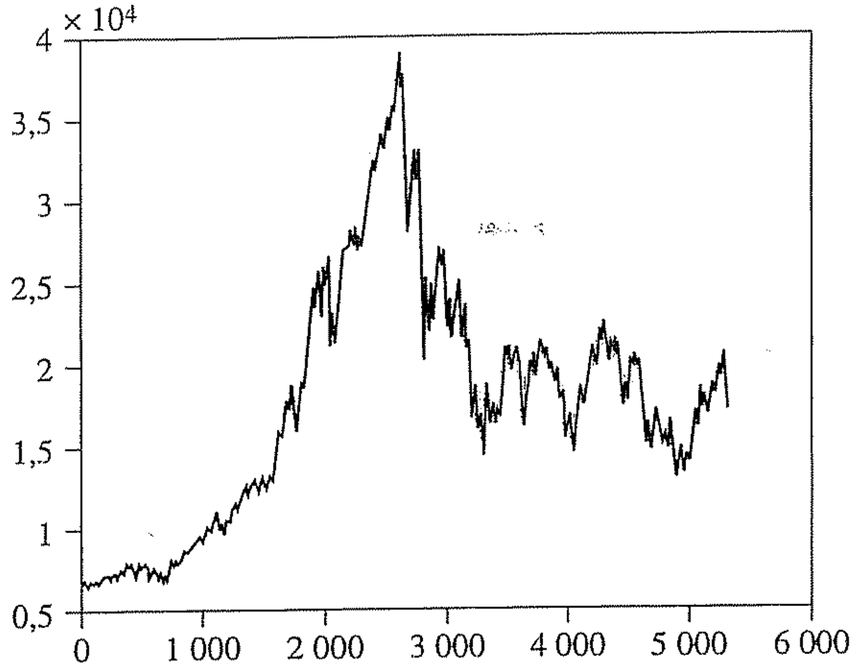
arité que sur la chronique originale, où seul le *krach* apparaît clairement, la subséquente n'étant pas remarquable.



igure 1.5. *L'indice Nikkei entre le premier janvier 1980 et le 5 novembre 2000*

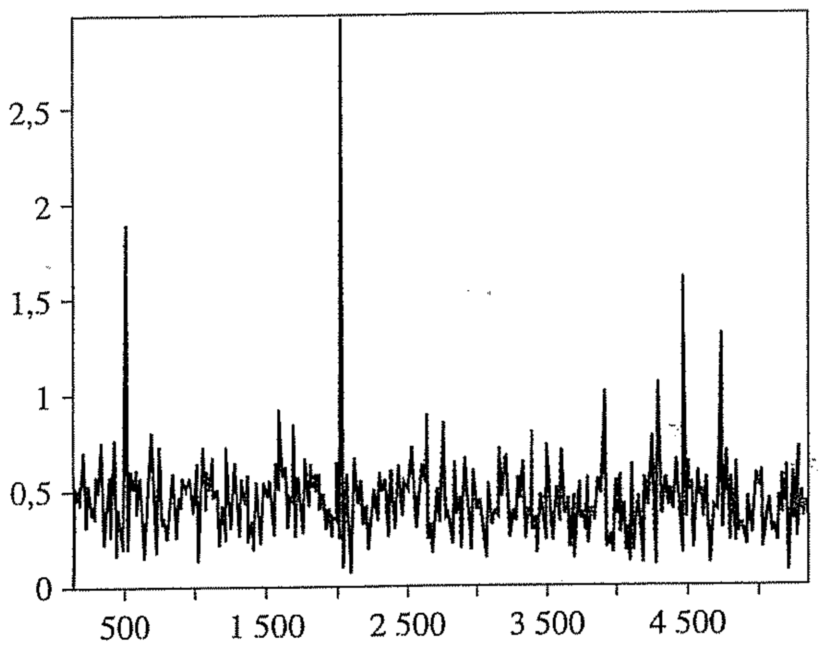


Figure 1.6. *Fonction de Hölder locale de l'indice Nikkei*

sidérons maintenant une autre région qui contient beaucoup de points de faible té avec quelques points réguliers (c'est-à-dire ayant $\alpha_l > 1$) isolés. Il s'agit de comprise entre les abscisses 4 450 et 4 800 : cette période correspond, approximent, à la « crise asiatique », qui a eu lieu entre janvier 1997 et juin 1998 (les

14

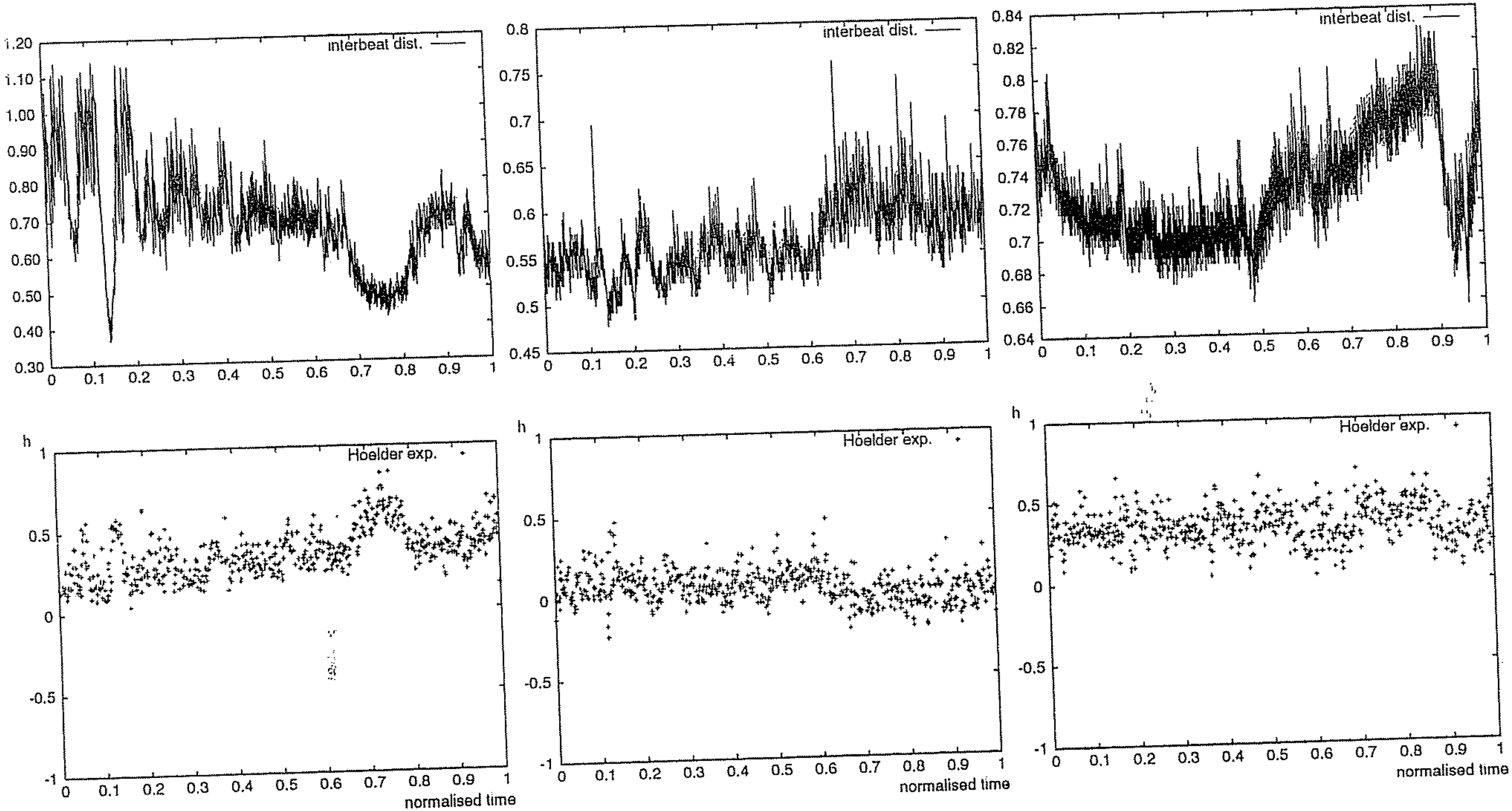


Figure 6: Three example heartbeat interval time series (top row) with their corresponding local effective Hölder exponent (bottom row). Two examples of non-stationarities in local Hölder exponent; they are intrinsic to the local Hölder exponent not to the non-stationarities of the input time series, as is shown in the third example, showing independence of the polynomial trend in the input.

Pointwise Hölder regularity

Definition: Let $\alpha \geq 0$, $x_0 \in \mathbb{R}^d$ and $f \in L^\infty_{loc}$;
f belongs to $C^\alpha(x_0)$ if $\exists\, C > 0$, $\delta > 0$ and a polynomial P of degree $\leq [\alpha]$ such that

if $|x - x_0| \leq \delta$ then $|f(x) - P(x-x_0)| \leq C|x-x_0|^\alpha$.

- The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}$$

- The isohölder sets E_H ($H \geq 0$) are

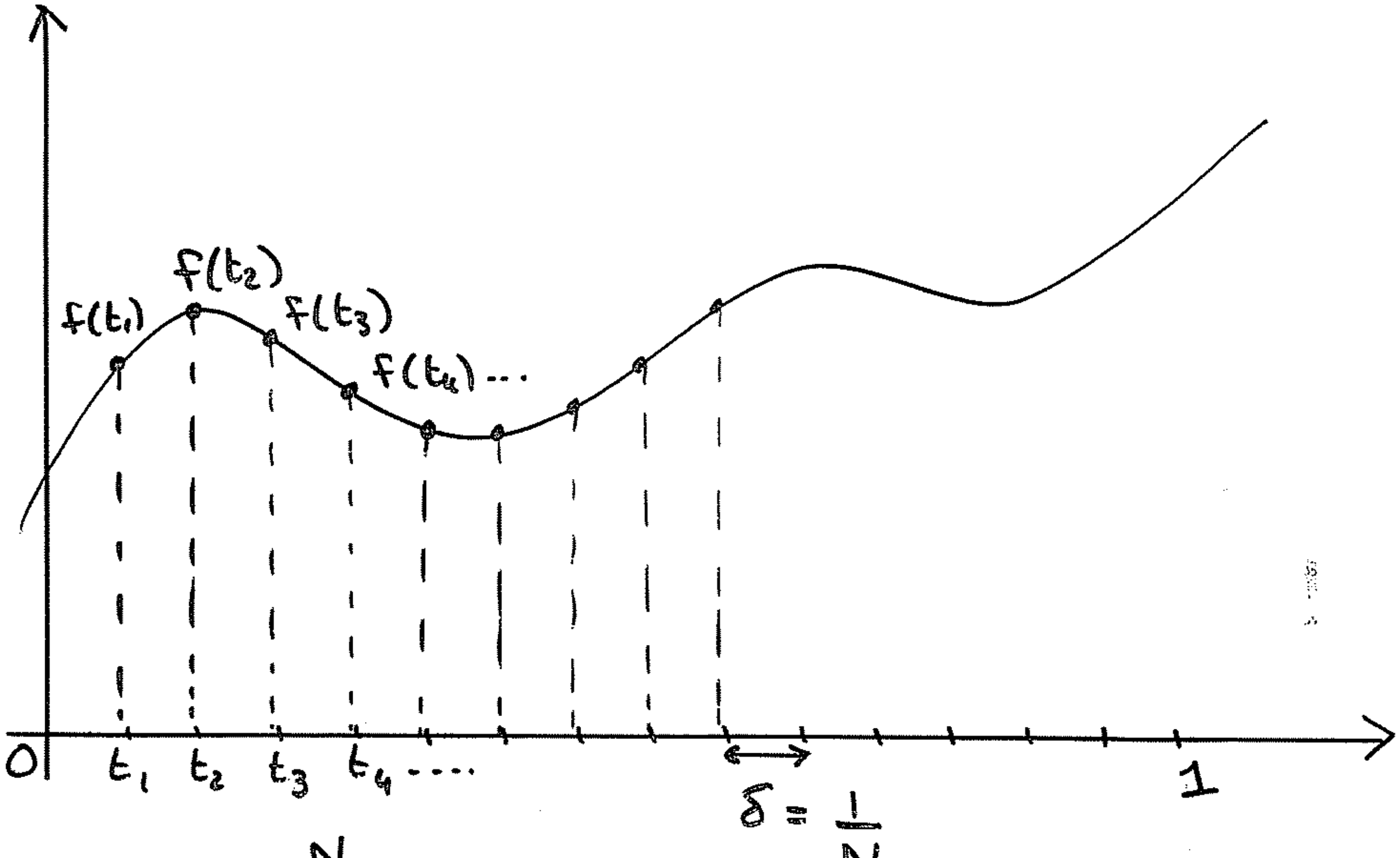
$$E_H = \{x_0 : h_f(x_0) = H\}$$

OPEN PROBLEM

Theorem : Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then $h_f(x)$ is a lim inf of a sequence of continuous function.

Conversely, any liminf of continuous functions is a Hölder exponent

Problem : Characterize the class of the Hölder exponents of functions in L^∞.

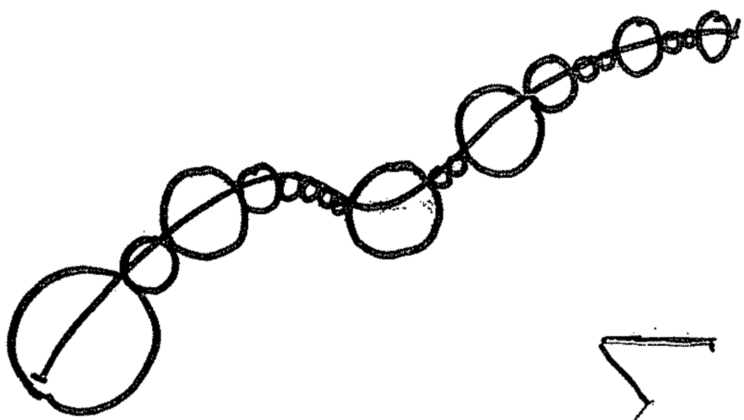


$$S(p,N) = \frac{1}{N} \sum_{i=1}^{N} |f(t_{i+1}) - f(t_i)|^p \sim \left(\frac{1}{N}\right)^{\zeta(p)}$$

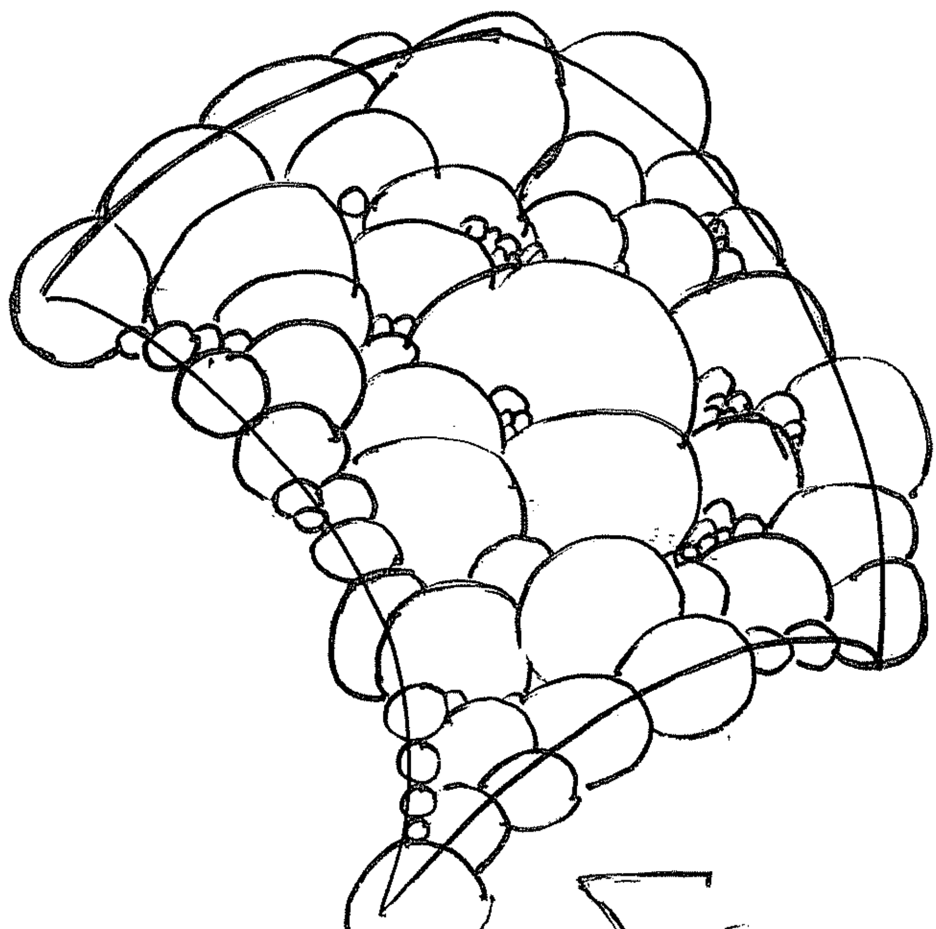
If $|f(t_{i+1}) - f(t_i)| \sim (t_{i+1} - t_i)^\alpha = \left(\frac{1}{N}\right)^\alpha$, then

$$S(p,N) \sim \left(\frac{1}{N}\right)^{\alpha p} \Rightarrow \zeta(p) = \alpha p$$

Hausdorff Dimension



$$\sum \mathrm{diam}(B_i) \sim \ell$$



$$\sum \left[\mathrm{diam}(B_i)\right]^2 \sim S$$

If A is a piece of smooth curve covered with balls B_i of diameter $\leq \varepsilon$

$$\inf_{\{\varepsilon\text{-coverings}\}} \sum (\operatorname{diam} B_i)^d \longrightarrow \begin{cases} +\infty & \text{if } d<1 \\ 0 & \text{if } d>1 \end{cases}$$

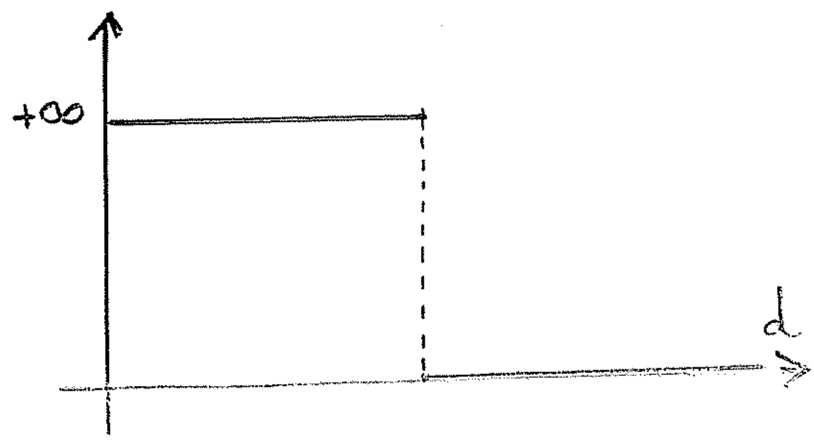
If A is a piece of smooth surface

$$\inf_{\{\varepsilon\text{-coverings}\}} \sum (\operatorname{diam} B_i)^d \longrightarrow \begin{cases} +\infty & \text{if } d<2 \\ 0 & \text{if } d>2 \end{cases}$$

Hausdorff Dimension

If A is a bounded subset of $\mathbb{R}^d$, we consider

$$\inf_{\{\varepsilon\text{-coverings}\}} \sum (\operatorname{diam} B_i)^d$$

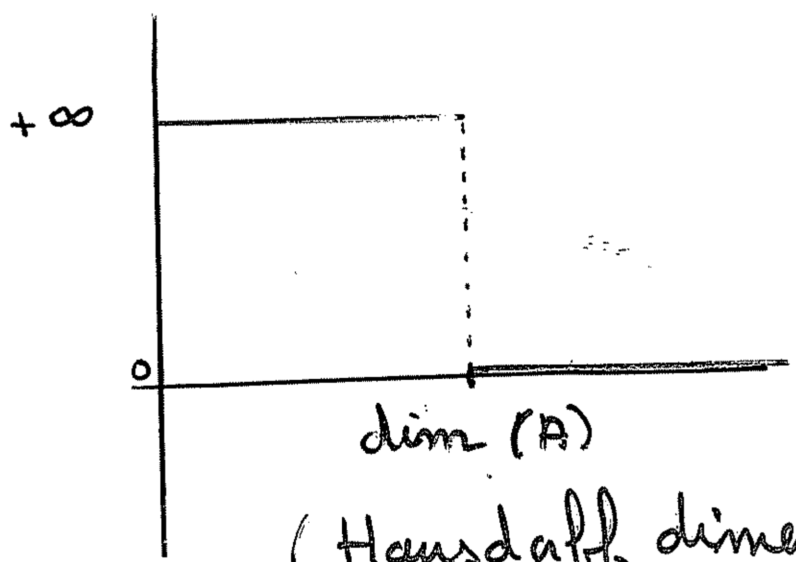


Hausdorff Dimension

Let A be a subset of $\mathbb{R}^d$. An ε-covering of A is a covering by balls of radius less than ε

$$M_d^{\varepsilon}(A) = \inf_{\varepsilon\text{-coverings } C} \sum_{B_i \in C} (\operatorname{diam} B_i)^d$$

$$M_d(A) = \lim_{\varepsilon \to 0} M_d^{\varepsilon}(A)$$



(Hausdorff dimension of A)

Definition: The Hausdorff dimension of A is

$$\dim(A) = \sup\{d : M_d(A) = +\infty\}$$

Spectrum of Singularities

$$E_H = \{ x : h_f(x) = H \}$$

Spectrum of singularities of f:

$$d_f(H) = \dim(E_H)$$

$$S(p,N) = \frac{1}{N} \sum_{i=1}^{N} |f(t_{i+1}) - f(t_i)|^p \sim \left(\frac{1}{N}\right)^{\zeta(p)}$$

"Contribution" of the points of E_H:

- Covered by $\sim N^{d_f(H)}$ intervals of length $\frac{1}{N}$
- On each interval : $|f(t_{i+1}) - f(t_i)| \sim \left(\frac{1}{N}\right)^H$

$$\Rightarrow \quad \frac{1}{N} \cdot N^{d_f(H)} \left(\frac{1}{N}\right)^{Hp} = \left(\frac{1}{N}\right)^{1-d_f(H)+Hp}$$

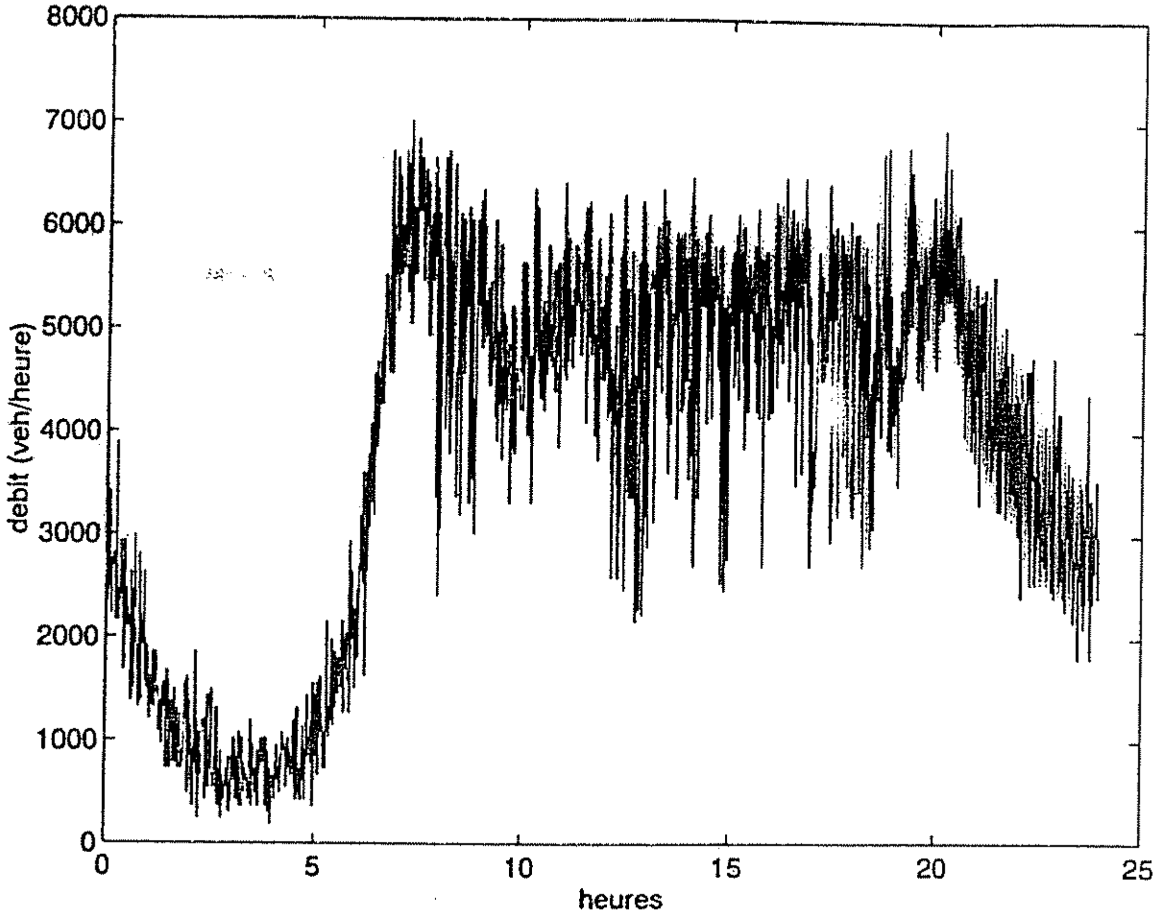
$$\zeta(p) = \inf_H \left(1 - d_f(H) + Hp\right)$$

Multifractal Formalism

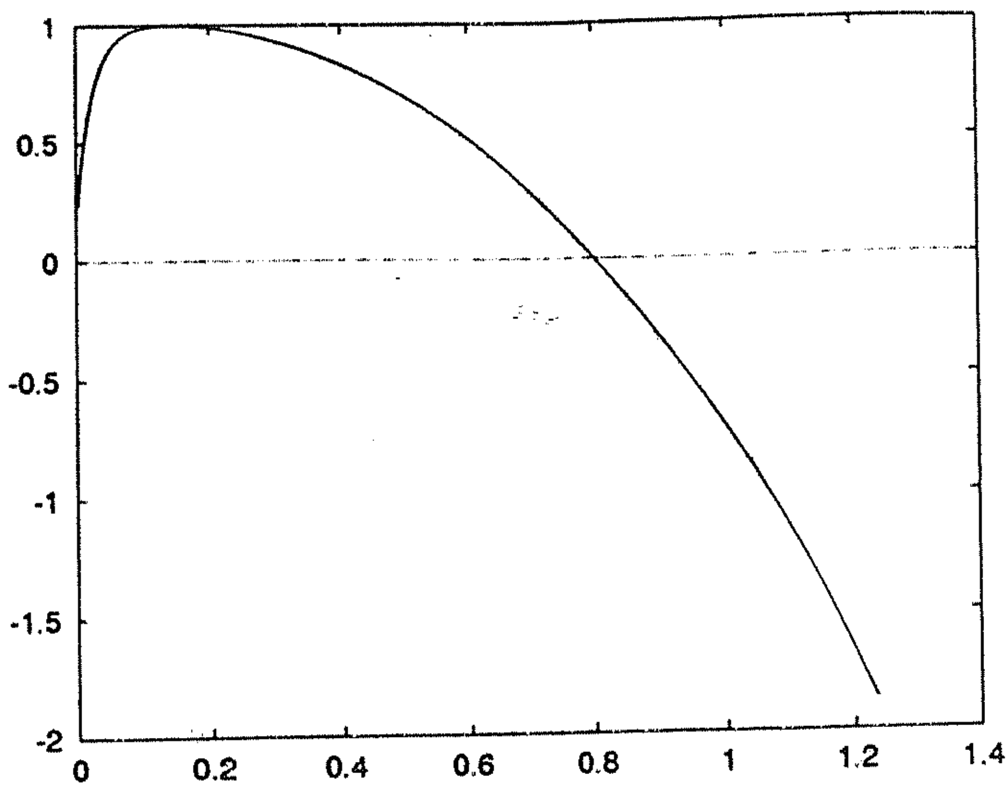
(G. Parisi, U. Frisch)

$$d_f(H) = \inf_p \left(1 - \zeta(p) + Hp \right)$$

P. Kovak, J. Lévy-Véhel DONNÉES DU TRAFIC



Courbe de débit (Porte de Bercy)



Spectre multifractal

OPEN PROBLEM

Let $d(H)$ be a nonnegative continuous function $\mathbb{R}^+ \to \mathbb{R}^+$. Then $d(H)$ is the spectrum of singularities of a function f.

Problem: What is the most general form that a spectrum of singularities can take?

Definition: f is homogeneous if, for any nonempty open set Ω, $d_{f|_\Omega} = d_f$

Problem: Same question if f is homogeneous

Hölder exponent of $W_{A,B}$

lemma : Let $\psi : \mathbb{R} \to \mathbb{C}$ be such that

$$\|\psi(x)\| \leq \frac{c}{1+|x|^2} \quad \text{and} \quad \int_{\mathbb{R}} \psi(x)\, dx = 0$$

If $f \in L^\infty(\mathbb{R})$, $a > 0$ and $b \in \mathbb{R}$, let

$$c(a,b) = \int_{\mathbb{R}} f(x)\, \psi\left(\frac{x-b}{a}\right) \frac{dx}{a}.$$

If $f \in C^\alpha(x_0)$ for $0 < \alpha < 1$, then

$$|c(a,b)| \leq C\left(a^\alpha + |b-x_0|^\alpha\right).$$

We pick ψ such that

- $\hat{\psi}(\xi)$ is C^2
- $\operatorname{supp}(\hat{\psi}) \subset \left[\frac{1}{B}, B\right]$
- $\hat{\psi}(1) = 1$

If $a = B^{-N}$, then

$$C(B^{-N}, b) = \sum_{n=1}^{\infty} A^n \int_{\mathbb{R}} \cos(B^n x)\, \psi\left(\frac{x-b}{B^{-N}}\right) \frac{dx}{B^{-N}}$$

$$= \sum_{n=1}^{\infty} A^n \int \cos(B^{n-N} u + B^n b)\, \psi(u)\, du$$

$$= \sum_{n=1}^{\infty} \frac{A^n}{2} \left[\int e^{iB^{n-N} u} e^{iB^n b} \psi(u)\, du + \right.$$
$$\left. + \int e^{-iB^{n-N} u} e^{-iB^n b} \overline{\psi(u)}\, du \right]$$

$$= \frac{A^N}{2} e^{iB^N b}.$$

Thus, if $a = B^{-N}$, $|c(a,b)| = \frac{1}{2} A^N = \frac{1}{2} \left(B^{-N}\right)^{-\frac{\log A}{\log B}}$.

Continuous wavelet transform

Assumptions on ψ $(\mathbb{R} \to \mathbb{R})$

- $\forall i = 0, \dots, r \quad |\psi^{(i)}(t)| \leq \dfrac{c}{(1+|t|)^{r+2}}$
- $\forall i = 0, \dots, r \quad \displaystyle\int_{\mathbb{R}} \psi(t)\, t^i\, dt = 0$

The wavelet is called "r-smooth".

If $f \in L^2(\mathbb{R})$, then, the wavelet transform of f is

$$C_f(a,b) = \frac{1}{a} \int f(t)\, \overline{\psi}\left(\frac{t-b}{a}\right) dt$$

Theorem: If $\displaystyle\int_{\mathbb{R}^+} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} = \int_{\mathbb{R}^-} |\hat{\psi}(\xi) \frac{d\xi}{|\xi|} = C_\psi$,

then $\displaystyle f(x) = \frac{1}{C_\psi} \iint_{\mathbb{R}^+ \times \mathbb{R}} C_f(a,b)\, \psi\left(\frac{x-b}{a}\right) \frac{da\, db}{a^2}$

We will prove that

$$\left\| f(x) - \frac{1}{C_\psi} \int_{\frac{1}{A} \leq a \leq A} \int_{|b| \leq B} C_f(a,b)\, \psi\!\left(\frac{x-b}{a}\right) \frac{da\, db}{a} \right\|_{L^2} \xrightarrow[A,B \to \infty]{} 0 .$$

$$\|$$

$$\sup_{\|g\|=1} \int \left(f(x) - \frac{1}{C_\psi} \iint_D C_f(a,b)\, \psi\!\left(\frac{x-b}{a}\right) \frac{da\, db}{a} \right) \bar{g}(x)\, dx$$

$$\|$$

$$\sup_{\|g\|=1} \left[\int f(x)\, \bar{g}(x)\, dx - \frac{1}{C_\psi} \iint_D C_f(a,b)\, \overline{C_g(a,b)}\, \frac{da\, db}{a} \right]$$

$$\|$$

$$\sup_{\|g\|=1} \left[\frac{1}{C_\psi} \iint_{(a,b) \notin [\frac{1}{A}, A] \times [-B, B]} C_f(a,b)\, \overline{C_g}(a,b)\, \frac{da\, db}{a} \right]$$

$$\leq \frac{1}{C_\psi} \Bigg(\underbrace{\iint_{\bar{D}} |C_f(a,b)|^2 \frac{da\, db}{a}}_{\substack{\longrightarrow 0 \\ \text{when } A \to +\infty \\ B \to +\infty}} \Bigg)^{1/2} \Bigg(\sup_{\|g\|=1} \underbrace{\iint |C_g(a,b)|^2 \frac{da\, db}{a}}_{=1} \Bigg)^{1/2}$$

lemma: Under the previous hypotheses, $\forall f, g \in L^2$,

$$\iint C_f(a,b)\, \overline{C_g}(a,b) \frac{da\,db}{a} = C_\psi \int f(x)\, \overline{g}(x)\, dx$$

Proof: $C_f(a,b) = \frac{1}{a} \int f(t)\, \overline{\psi}\left(\frac{t-b}{a}\right) dt$

$$= \frac{1}{2\pi} \int \hat{f}(y)\, e^{-iby}\, \overline{\hat{\psi}}(ay)\, dy$$

for an a fixed, $\hat{f}(y)\, \overline{\hat{\psi}}(ay) \xrightarrow{\mathcal{F}} 2\pi C_f(a,b)$

$$\hat{g}(y)\, \overline{\hat{\psi}}(ay) \xrightarrow{\mathcal{F}} 2\pi C_g(a,b)$$

Thus $\int C_f(a,b)\, \overline{C_g}(a,b)\, db = 2\pi \int \hat{f}(y)\, \overline{\hat{\psi}}(ay)\, \overline{\hat{g}(y)}\, \hat{\psi}(ay)\, dy$

$$\Rightarrow \int_{a>0} \int_b C_f(a,b)\, \overline{C_g}(a,b) \frac{da\,db}{a} = \int_{a>0} \int_y \hat{f}(y)\, \overline{\hat{g}}(y)\, |\hat{\psi}(ay)|^2\, dy \frac{da}{a}$$

but $\int_{a>0} |\psi(ay)|^2 \frac{da}{a}$ is independent of y and equal to C_ψ.

EXERCICE

Prove that $\forall f, g \in L^2(\mathbb{R})$

$$\iint C_f(a,b)\, \overline{C_g}(a,b)\, \frac{da\, db}{a} = C_\psi \int f(x)\, \bar{g}(x)\, dx$$

deduce that

$$f(x) - \frac{1}{C_\psi} \int_{\frac{1}{A} \leq a \leq A} \int_{|b| \leq B} C_f(a,b)\, \psi\!\left(\frac{x-b}{a}\right) \frac{da\, db}{a} \longrightarrow 0$$

in $L^2(\mathbb{R})$ when $A, B \longrightarrow \infty$

Selfsimilar functions

Assumption: $\exists \lambda, c > 0: \quad \forall x \quad f(\lambda x) = c\, f(x)$

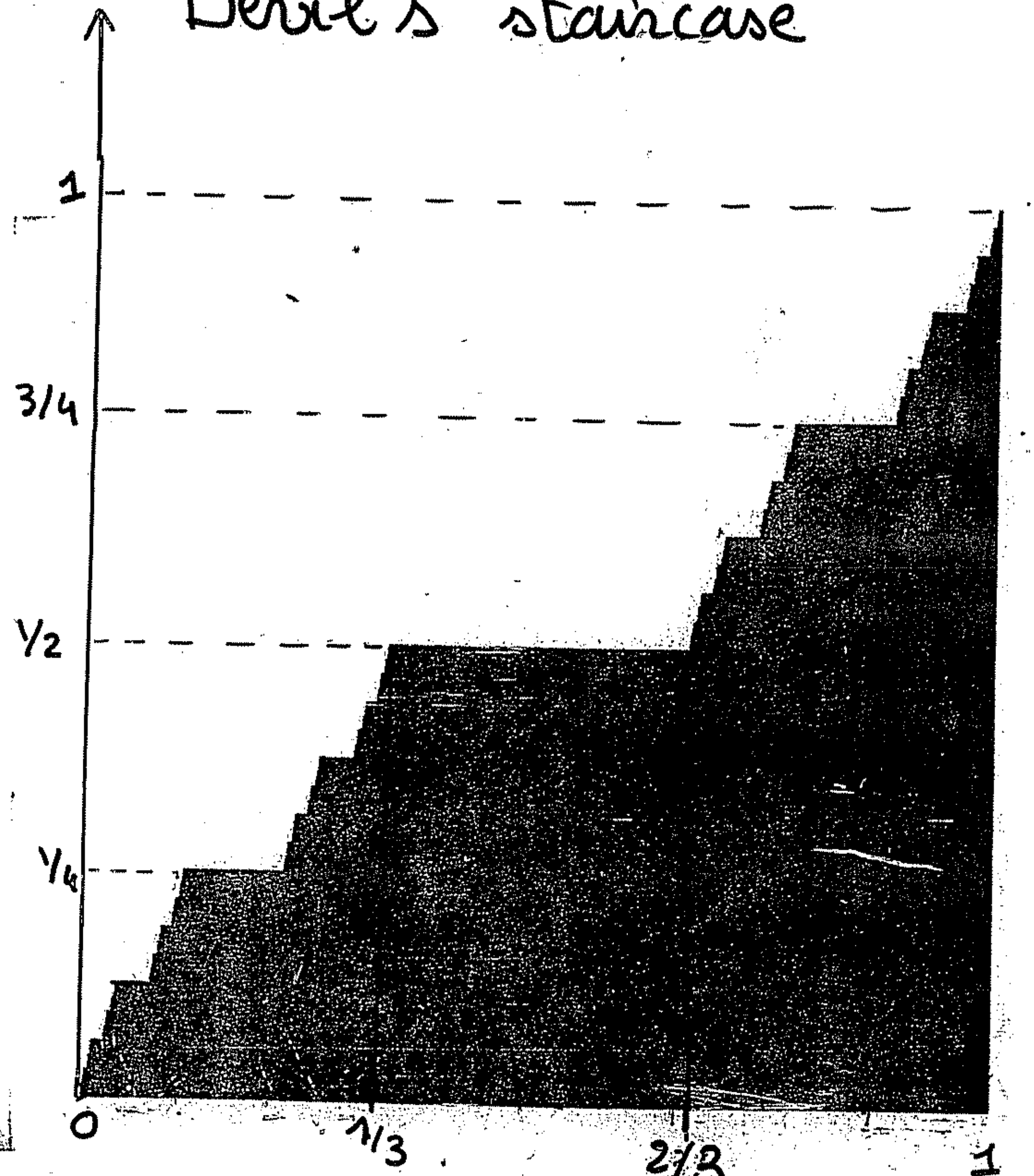
$$C_f(a,b) = \frac{1}{a} \int f(x)\, \psi\left(\frac{x-b}{a}\right) dx$$

$$= \frac{1}{ca} \int f(\lambda x)\, \psi\left(\frac{x-b}{a}\right) dx$$

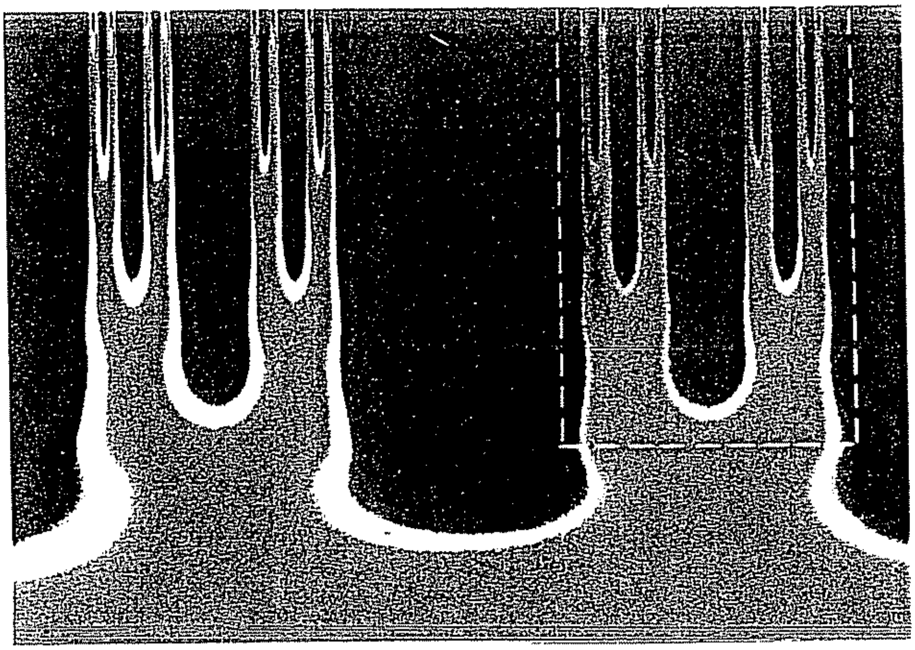
$$= \frac{1}{\lambda c a} \int f(u)\, \psi\left(\frac{\frac{u}{\lambda} - b}{a}\right) du$$

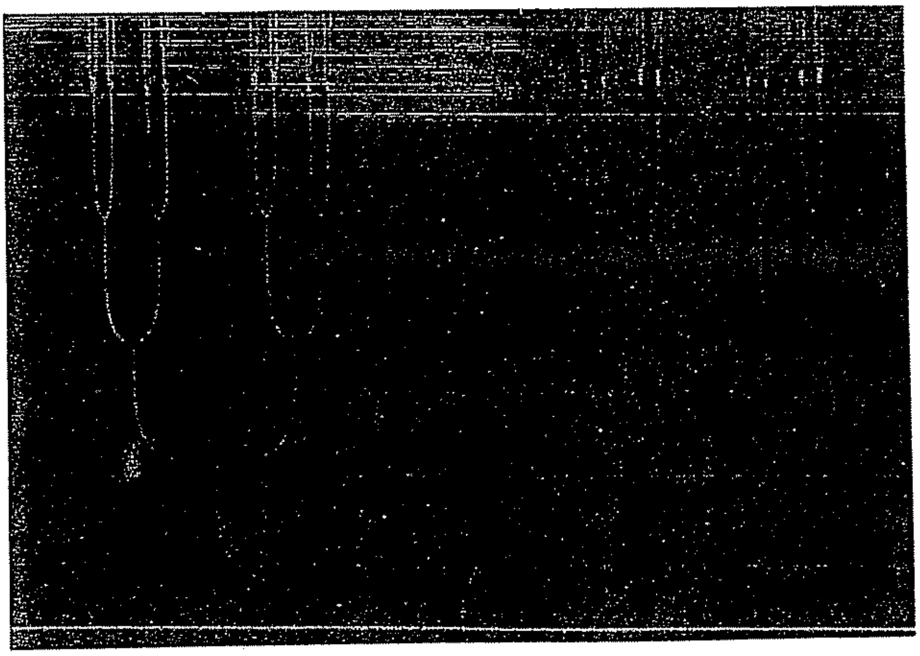
$$= \frac{1}{c}\, C_f(\lambda a, \lambda b)$$

Devil's staircase



$$f\left(\frac{x}{3}\right) = \frac{1}{2} f(x)$$





Multiresolution analysis

Definition : A sequence $(V_j)_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ forms a multiresolution analysis if

- $\forall j, \; V_j \subset V_{j+1}$
- $f(x) \in V_j \iff f(2x) \in V_{j+1}$
- $\exists g \in V_0$ such that the $g(x-k)$ are a Riesz basis of V_0
- $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$
- $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$

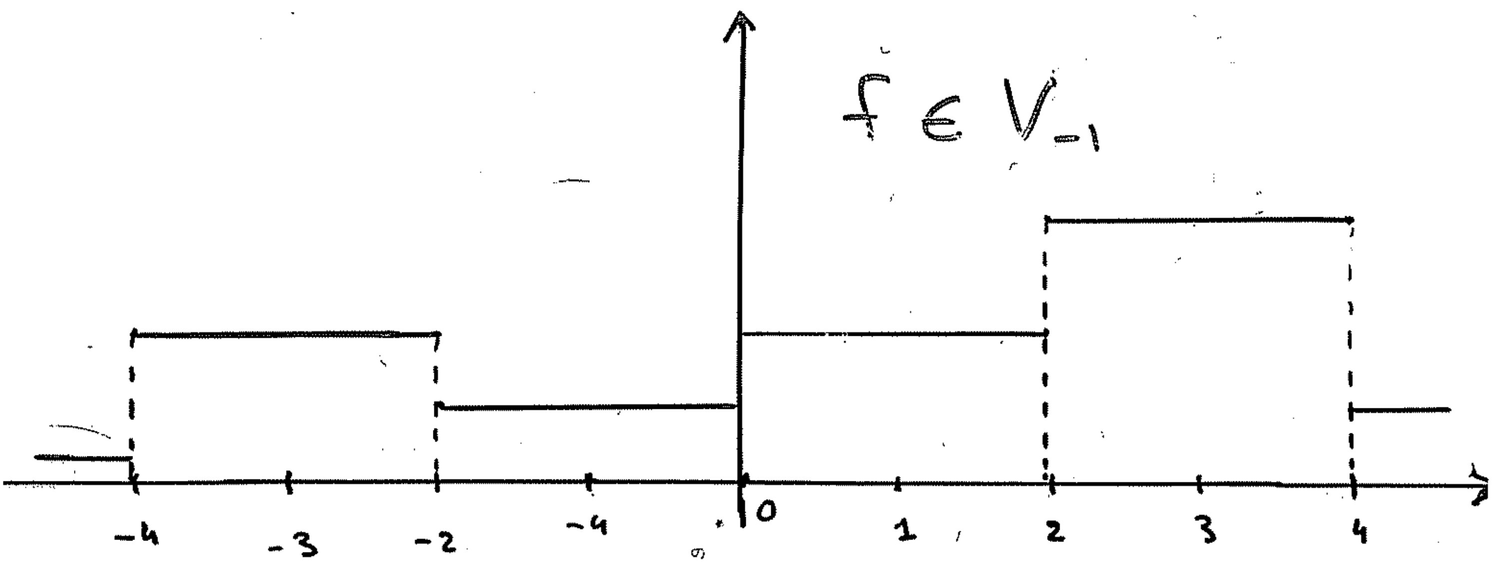
A Riesz basis (e_n) of an Hilbert space H satisfies

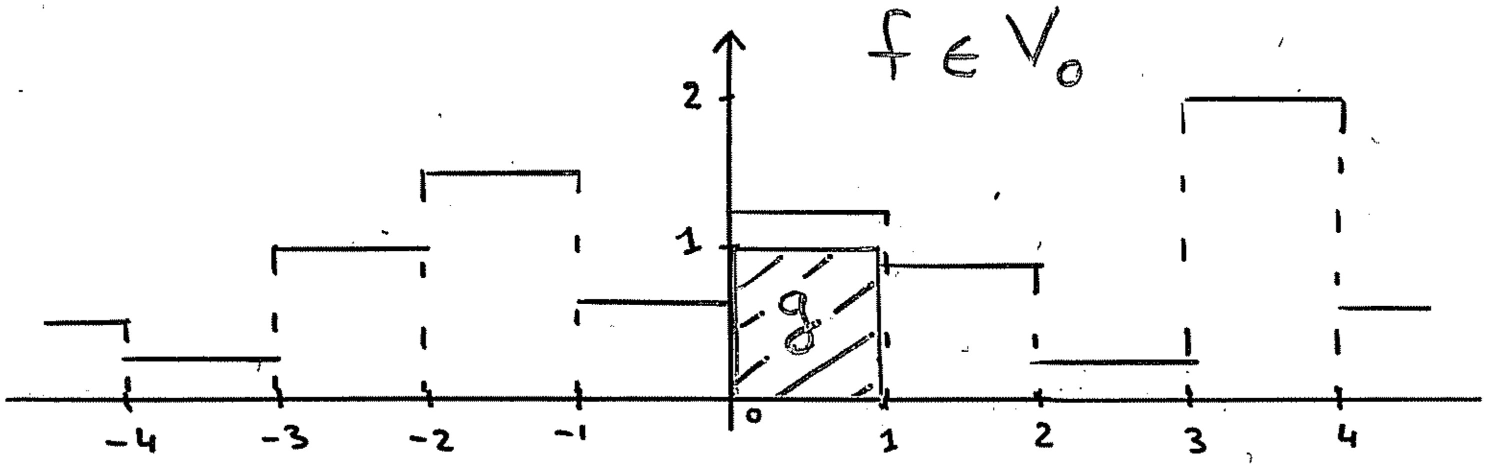
- The e_n span a dense subspace of H
- $\exists c, c' > 0 \qquad c \sum |c_k|^2 \leq \| \sum c_k e_k \|^2 \leq c' \sum |c_k|^2$.

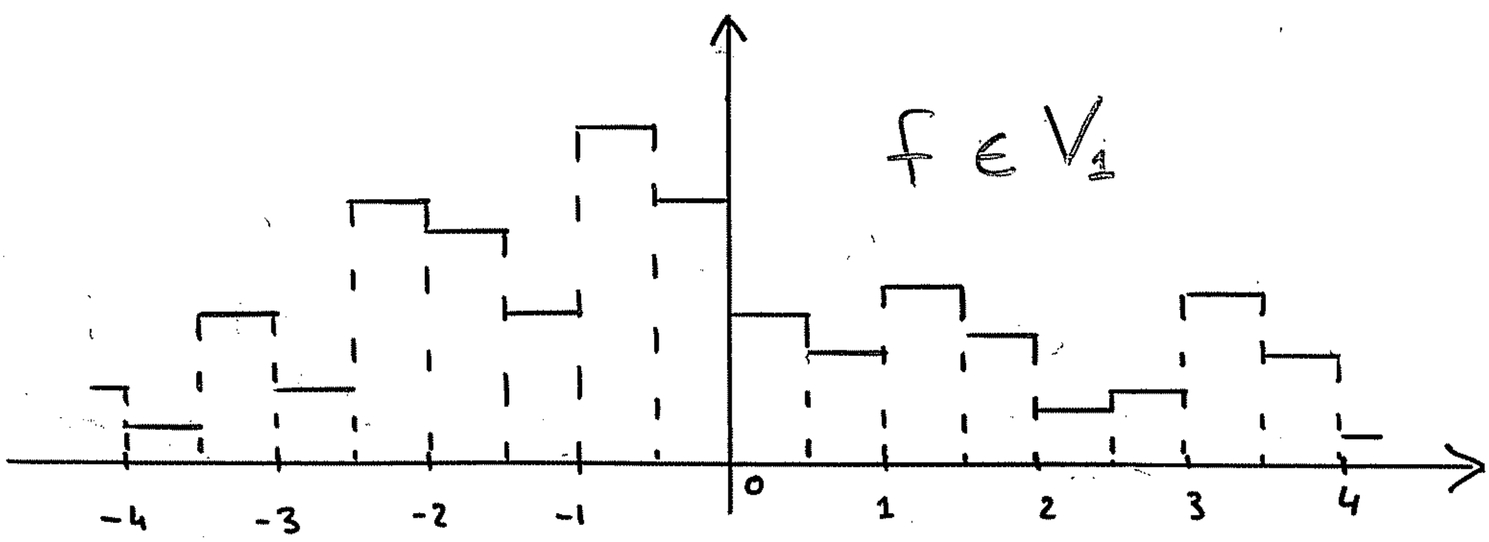
OPEN PROBLEM

Let $c(a,b)$ be the continuous wavelet transform of f. The wavelet maxima are the local maxima of $b \to c(a,b)$. Keeping the values of $c(a,b)$ at the wavelet maxima is not enough to recover f.

Problem: Which additional information can we keep in order to be able to recover f by a stable algorithm.







$f \in V_j \iff \bullet\ f \in L^2$

$\bullet$ f is constant on each dyadic interval $\lambda_{j,k} = \left[\frac{k}{2^j}, \frac{k+1}{2^j} \right)$

lemma : The Riesz basis condition

$$c_1 \sum |a_k|^2 \leq \left\| \sum a_k\, g(x-k) \right\|_{L^2}^2 \leq c_2 \sum |a_k|^2$$

is equivalent to

$$c_1 \leq \sum_{\ell} |\hat{g}(\xi + 2\pi \ell)|^2 \leq c_2$$

Proof : let $f(x) = \sum a_k\, g(x-k)$:

$$\|f\|_{L^2}^2 = \frac{1}{2\pi} \|\hat{f}\|_{L^2}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \Big| \sum_k a_k e^{ik\xi} \hat{g}(\xi) \Big|^2 d\xi$$

$$= \frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}} \int_{2\pi \ell}^{2\pi(\ell+1)} \Big| \sum_k a_k e^{ik\xi} \hat{g}(\xi) \Big|^2 d\xi$$

$$= \frac{1}{2\pi} \sum_{\ell \in \mathbb{Z}} \int_0^{2\pi} \Big| \sum_k a_k e^{ik\xi} \hat{g}(\xi + 2\pi \ell) \Big|^2 d\xi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_k a_k e^{ik\xi} \Big|^2 \Big(\sum_{\ell} |\hat{g}(\xi + 2\pi \ell)|^2 \Big) d\xi$$

Corollary : Let $m(\xi)$ be a 2π-periodic function and

$$\hat{\varphi}(\xi) = \frac{\hat{g}(\xi)\, e^{i m(\xi)}}{\left(\sum_{\ell \in \mathbb{Z}} |\hat{g}(\xi + 2\pi \ell)|^2 \right)^{1/2}} ;$$

then $\varphi \in V_0$ and the $\varphi(x-k)_{k \in \mathbb{Z}}$ form an orthonormal basis of V_0.

Excamples of multiresolution analyzes

- let $g_0(x) = 1_{[0,1]}(x)$

$$g_m(x) = g_0(x) * g_{m-1}(x)$$

g_m is the B-spline of order m. It is a C^{m-1} function and g_m is a polynomial of degree m on each interval $[k, k+1]$.

V_0 : subspace of $L^2(\mathbb{R})$ spanned by the $g_m(x-k)$ $(k \in \mathbb{Z})$.

EXERCICE

Prove that the Riez basis condition

$$c_1 \sum |a_k|^2 \leq \left\| \sum a_k \, g(x-k) \right\|_{L^2}^2 \leq c_2 \sum |a_k|^2$$

is equivalent to

$$\exists\, c_1, c_2 > 0 : \; c_1 \leq \sum_\ell |\hat{g}(\xi + 2\pi\ell)|^2 \leq c_2$$

Deduce that, if $\displaystyle \hat{\varphi}(\xi) = \frac{\hat{g}(\xi)}{\left(\sum_\ell |\hat{g}(\xi + 2\pi\ell)|^2 \right)^{1/2}}$

the $(\varphi(x-k))_{k \in \mathbb{Z}}$ form an othonormal basis of V_0

Orthonormal Basis of V_0

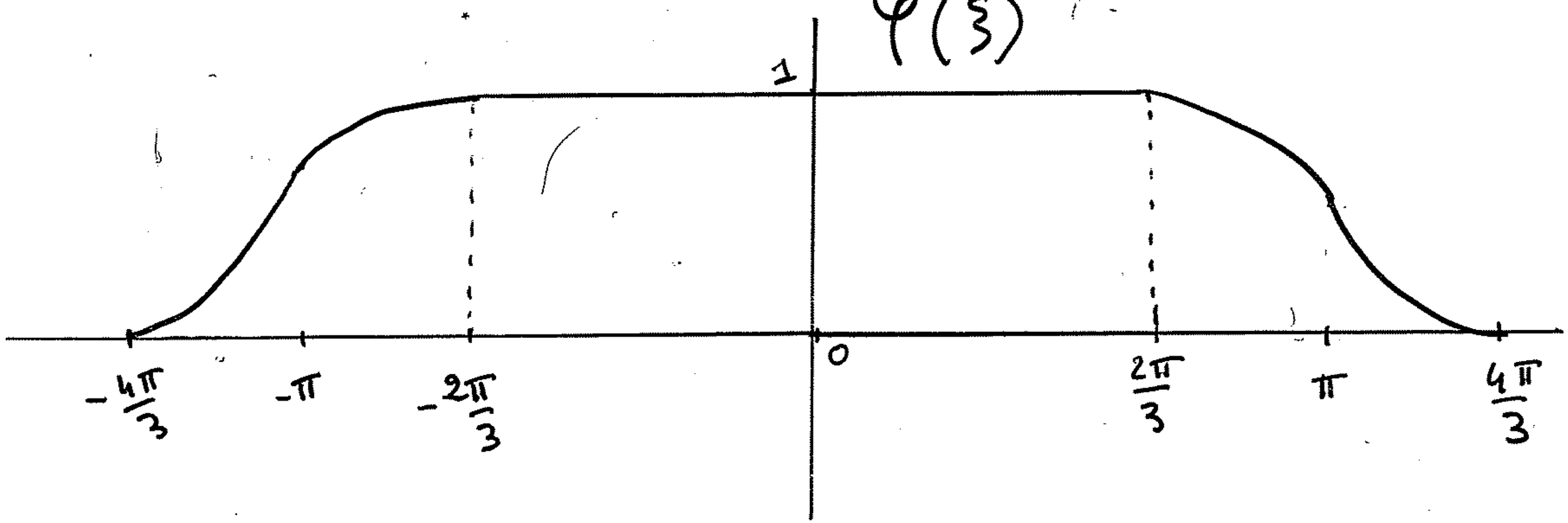
$$\hat{\varphi}(\xi) = \frac{\hat{g}(\xi)}{\left(\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2k\pi)|^2\right)^{1/2}}$$

$\varphi(x-k)$ form an orthonormal basis of V_0

$\Rightarrow \left(2^{j/2}\varphi(2^j x - k)\right)_{k\in\mathbb{Z}}$ form an orthonormal basis of V_j

- $\hat{\varphi}(\xi)$ is C^∞ and supported in $I = \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right]$
- $\hat{\varphi}(\xi) = \hat{\varphi}(-\xi)$
- $\hat{\varphi}(\xi) = 1$ if $\xi \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$
- $\forall \xi \in [0, 2\pi] \quad |\hat{\varphi}(\xi)|^2 + |\hat{\varphi}(\xi - 2\pi)|^2 = 1$

V_0 : Subspace of $L^2(\mathbb{R})$ spanned by the $\varphi(x-k)$.



Construction of ψ

Since $V_j \subset V_{j+1}$, we can define

$$W_j = \{ f \in V_{j+1} : \forall g \in V_j \quad \langle f | g \rangle = 0 \}$$

- $f(x) \in W_j \iff f(2x) \in W_{j+1}$
- $f(x) \in W_0 \iff \forall k \; f(x-k) \in W_0$
- $\forall j \neq l \quad W_j \perp W_l$.

$$\varphi\left(\frac{x}{2}\right) = \sum d_k \varphi(x-k)$$

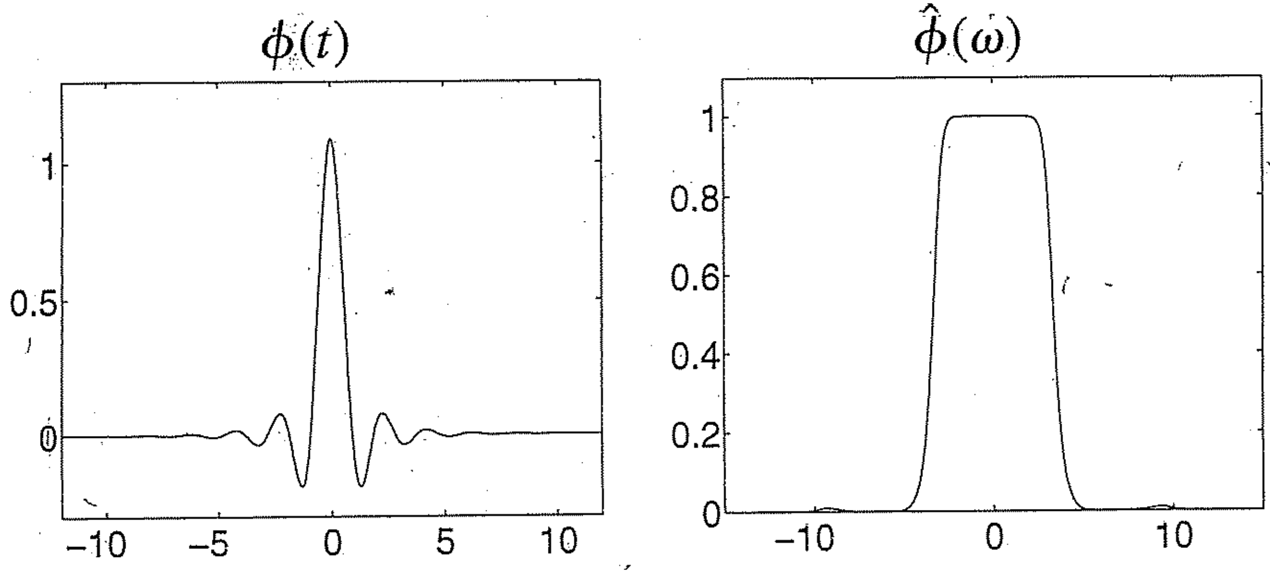
$$\Rightarrow \hat{\varphi}(2\xi) = m_0(\xi)\,\hat{\varphi}(\xi) \qquad m_0 : 2\pi\text{-periodic}$$

$$\hat{\psi}(\xi) = e^{-i\xi/2}\,\overline{m_0}\left(\frac{\xi}{2}+\pi\right)\hat{\varphi}\left(\frac{\xi}{2}\right)$$

(EXERCISE)

$2^j \leq 2^5$.

mple 7.4 For piecewise constant approximations and Shannon multiresolu- approximations we have constructed Riesz bases $\{\theta(t-n)\}_{n\in\mathbb{Z}}$ which are onormal bases, hence $\phi = \theta$.



URE 7.2 Cubic spline scaling function ϕ and its Fourier transform $\hat{\phi}$ computed with 4).

EXERCICE

Compute explicitely $\hat{\varphi}$ and $\hat{\psi}$ for the piecewise linear multiresolution analysis

Piecewise linear wavelet

(Battle - lemarie).

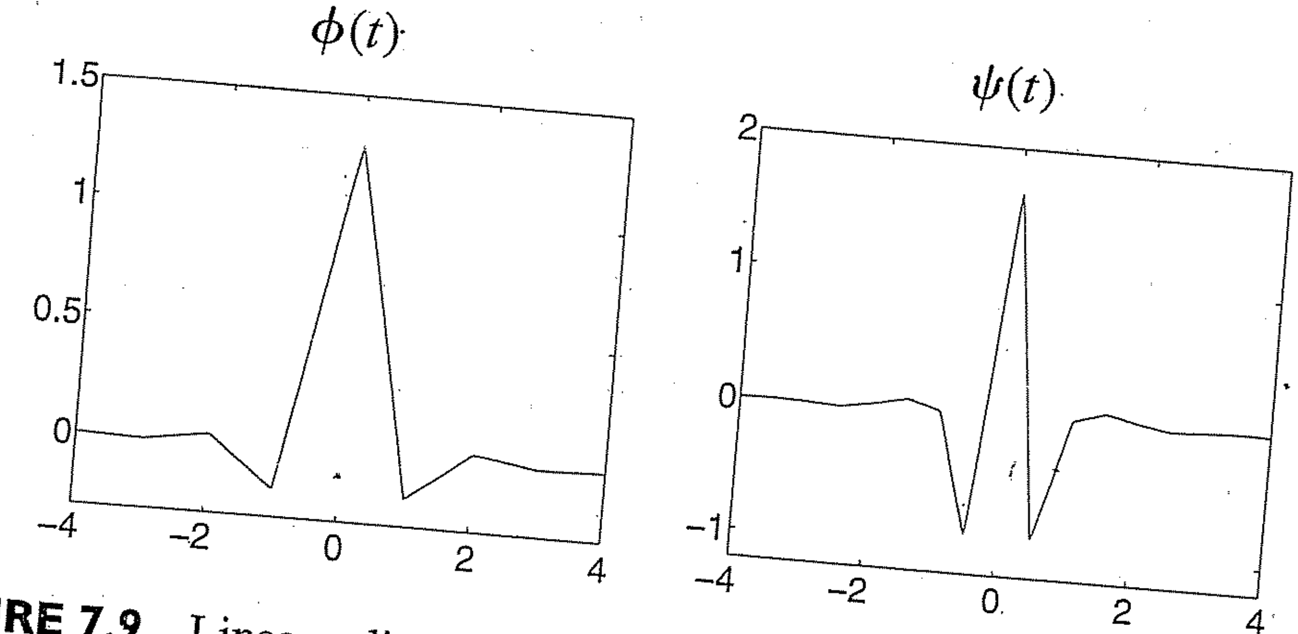


FIGURE 7.9 Linear spline Battle-Lemarié scaling function ϕ and wavelet ψ.

pressions of $\hat{\phi}(\omega)$ and $\hat{h}(\omega)$ are given respectively by (7.24) and (7.56). For lines of degree m, $\hat{h}(\omega)$ and its first m derivatives are zero at $\omega = \pi$. Proposition ? derives that ψ has $m + 1$ vanishing moments. It follows from (7.86) that

$$\hat{\psi}(\omega) = \frac{e^{-i\frac{\omega}{2}}}{\ldots} \sqrt{S_{2m+2}(\frac{\omega}{2} + \pi) \ldots}$$

concentrated in $[-2\pi, -\pi] \cup [\pi, 2\pi]$. For any ψ that generates ...

Piecewise cubic C^2 wavelet

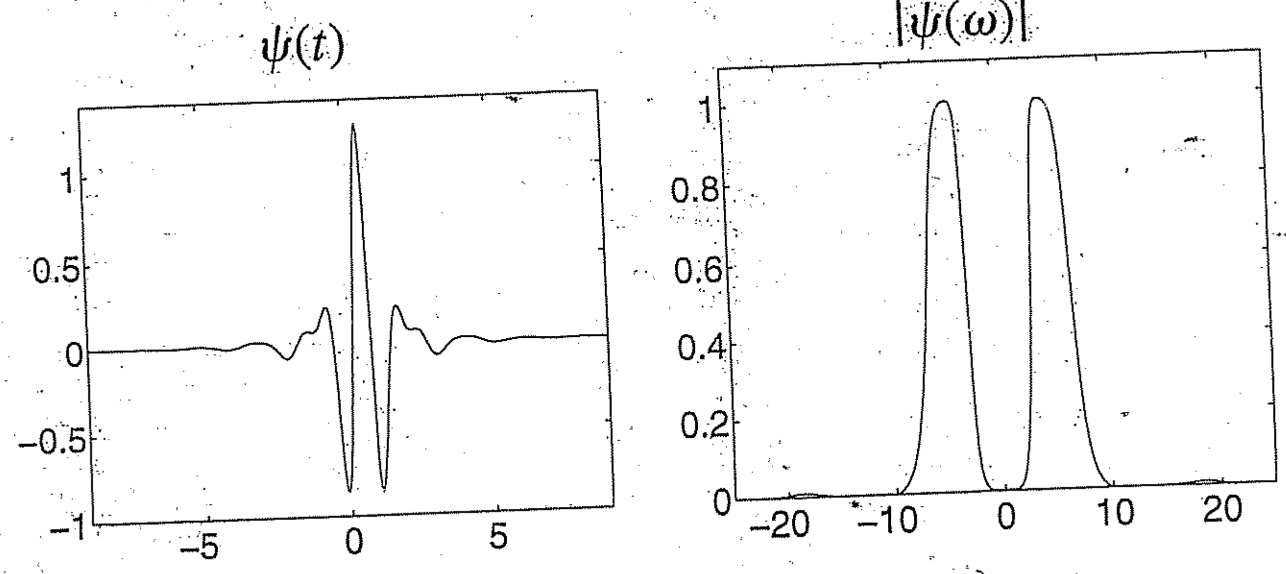
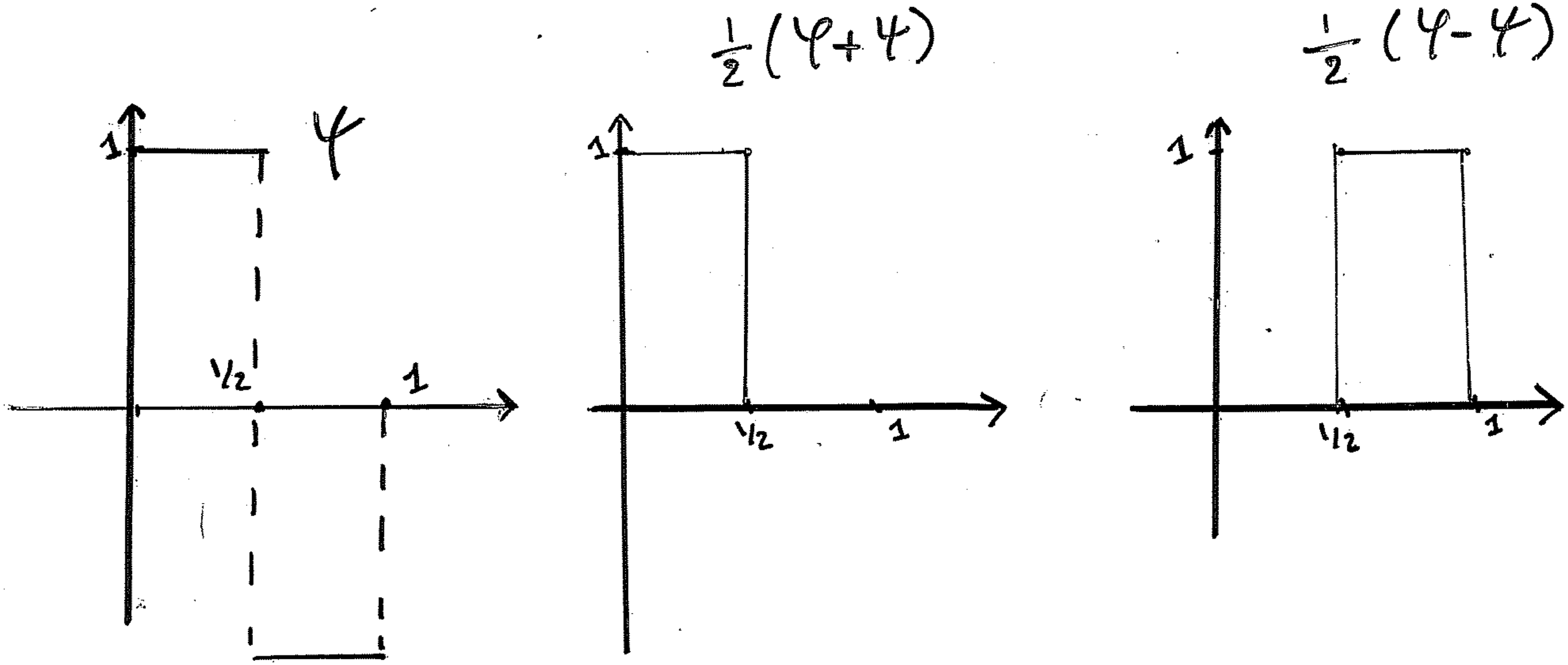


FIGURE 7.5 Battle-Lemarié cubic spline wavelet ψ and its Fourier transform modulu

The Haar wavelet



Compactly supported wavelets
(I. Daubechies)

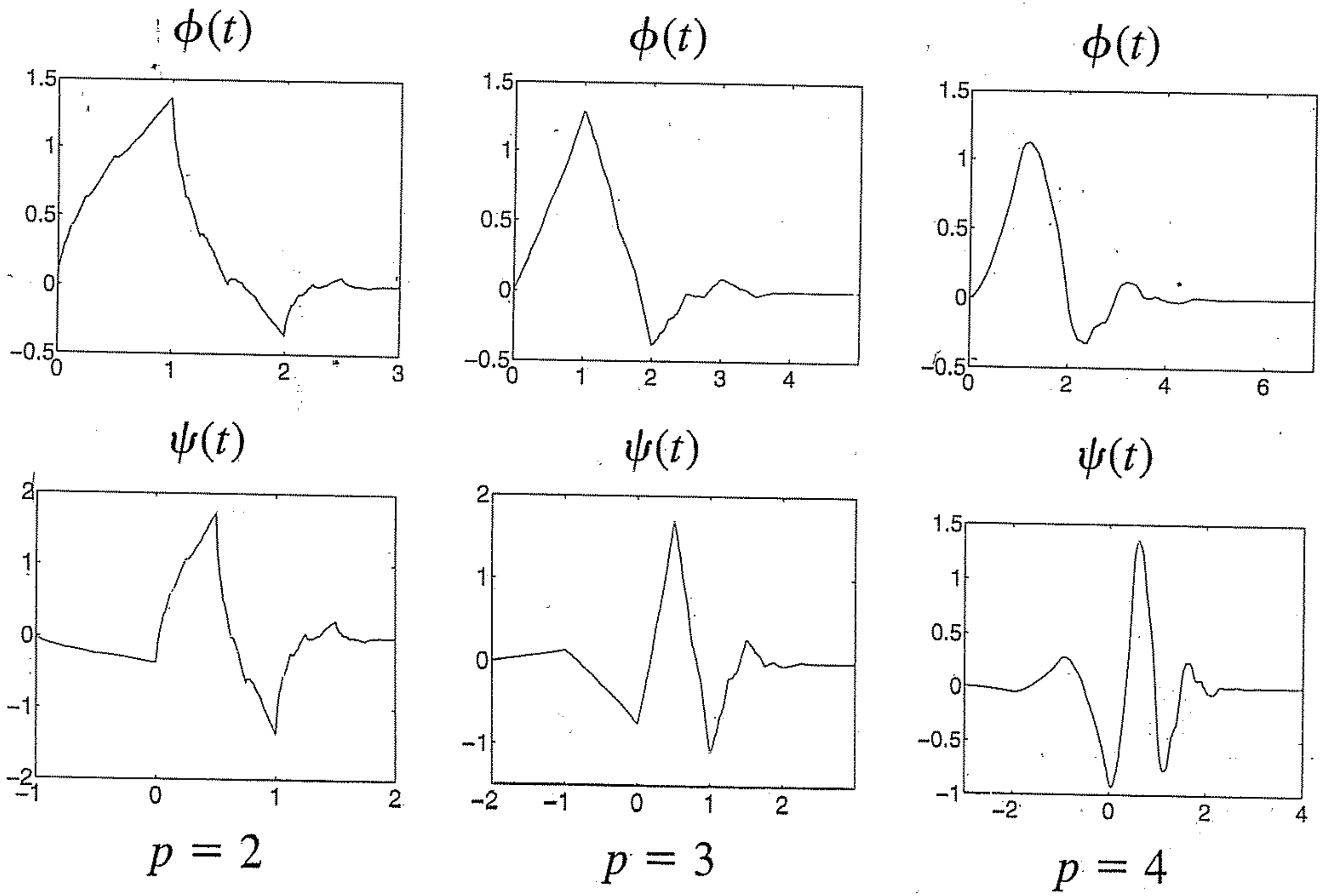


FIGURE 7.10 Daubechies scaling function ϕ and wavelet ψ with p vanishing moments.

Wavelets on $\mathbb{R}$

$\mathcal{V}_0$: Space spanned by the $\left(\varphi(x-k)\,\varphi(y-\ell)\right)_{(k,\ell)\in\mathbb{Z}^2}$

$f \in \mathcal{V}_j$ iff $f\left(\frac{x}{2^j}\right) \in \mathcal{V}_0$.

One thus defines a multiresolution analysis

$\mathcal{W}_j$: orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$.

If $f \in \mathcal{V}_1$, f is of the form

$$\sum a_{k,\ell}\,\varphi(x-k)\,\varphi(y-\ell) + \sum b_{k,\ell}\,\varphi(x-k)\,\psi(y-\ell)$$
$$+ \sum c_{k,\ell}\,\psi(x-k)\,\varphi(y-\ell) + \sum d_{k,\ell}\,\psi(x-k)\,\psi(y-\ell).$$

$\mathcal{W}_0$ is spanned by the translates of

$$\Psi^{(1)}(x,y) = \varphi(x)\,\psi(y)$$
$$\psi^{(2)}(x,y) = \psi(x)\,\varphi(y)$$
$$\psi^{(3)}(x,y) = \psi(x)\,\psi(y)$$

4.0

Wavelets on $\mathbb{R}^d$.

W_0 is spanned by the $\psi^{(i)}(x-k)$ $\quad i = 1, \dots, 2^{d-1}$, $k \in \mathbb{Z}^d$, where :

if $\psi_0(t) = \varphi(t)$, $\psi_1(t) = \psi(t)$

then the $\psi^{(i)}(x) = \psi^{(i)}(x_1, \dots, x_d)$ are the functions

$$\psi_{i_1}(x_1) \dots \psi_{i_d}(x_d)$$

for all d-uples $(i_1, \dots, x_d) \in \{0,1\}^d - (0, \dots, 0)$

(there are $2^d - 1$ such d-uples)

The orthogonal wavelet decomposition

P_j : orthogonal projection on V_j

Q_j : orthogonal projection on W_j

$$V_j = V_{j-1} \oplus W_{j-1} \Rightarrow P_j = P_{j-1} + Q_{j-1} \qquad (j > 0)$$

$$= P_{j-2} + Q_{j-2} + Q_{j-1}$$

$$\cdots\cdots\cdots$$

$$= P_0 + Q_1 + \cdots + Q_{j-1}$$

$$\forall f \in L^2, \quad P_j(f) = P_0(f) + Q_1(f) + \cdots + Q_{j-1}(f)$$

$$f = P_0(f) + Q_1(f) + \cdots + Q_j(f) + \cdots$$

$$\sum d_k \varphi(x-k) \qquad \qquad \sum c^{(i)}_{j,k} \psi^{(i)}(2^j x - k)$$

$$d_k = \int f(x) \varphi(x-k)\,dx \qquad \qquad c^{(i)}_{j,k} = 2^{dj} \int f(x) \psi^{(i)}(2^j x - k)\,dx$$

$\left(\psi(x-k) \right)_{k \in \mathbb{Z}}$: orthonormal basis of W_0

$\Rightarrow \left(2^{j/2} \psi(2^j x - k) \right)_{k \in \mathbb{Z}}$: orthonormal basis of W_j

$\forall j \neq l \quad W_j \perp W_l$

$\bigoplus W_j = L^2(\mathbb{R})$

$\Rightarrow \left\{ 2^{j/2} \psi(2^j x - k) \right\}_{j \in \mathbb{Z},\, k \in \mathbb{Z}}$ orthonormal basis of $L^2(\mathbb{R})$

$L^2(\mathbb{R}) = V_0 \oplus W_1 \oplus W_2 \oplus \cdots$ $\Rightarrow$ Alternative basis

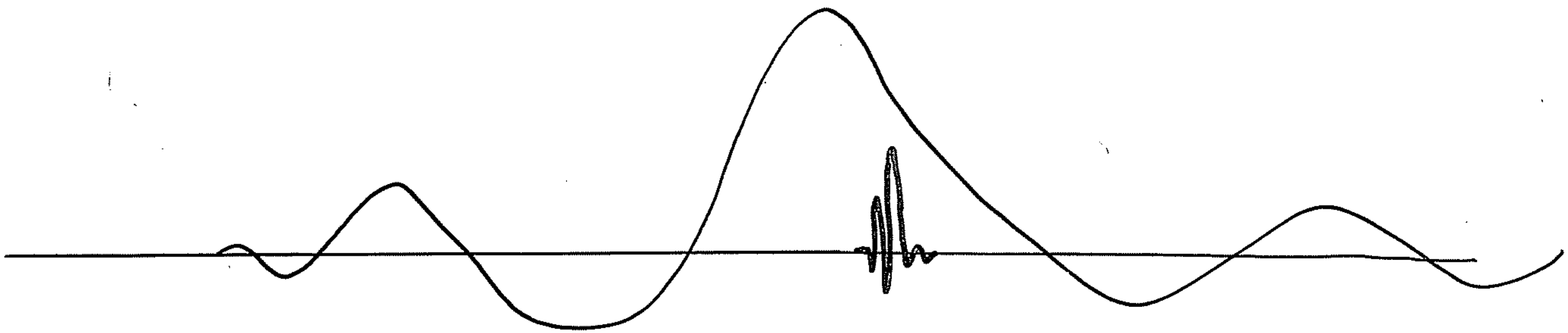
$\left\{ \varphi(x-k) \right\}_{k \in \mathbb{Z}}$, $\left\{ 2^{j/2} \psi(2^j x - k) \right\}_{j \geq 0,\, k \in \mathbb{Z}}$

Vanishing moments

Property: If the multiresolution analysis is r-smooth,

$$\forall \alpha : |\alpha| \leq [r], \qquad \int x^\alpha \psi^{(i)}(x)\, dx = 0$$

(Exercise).



Fast Algorithms

$$f(x) = \sum c_k \varphi(x-k) \in V_0$$

$$= \sum c_\ell^1 \frac{1}{\sqrt{2}} \varphi\left(\frac{x}{2}-\ell\right) + \sum d_\ell^1 \psi\left(\frac{x}{2}-\ell\right)$$

$$c_\ell^1 = \int f(x) \frac{1}{\sqrt{2}} \varphi\left(\frac{x}{2}-\ell\right) dx$$

$$= \sum_k c_k \frac{1}{\sqrt{2}} \int \varphi(x-k)\varphi\left(\frac{x}{2}-\ell\right) dx$$

$$= \sum_k c_k \underbrace{\frac{1}{\sqrt{2}} \int \varphi(u)\, \varphi\left(\frac{u+k-2\ell}{2}\right) dx}_{\omega_{k-2\ell}}$$

Similarly

$$d_\ell^1 = \sum_k c_k\, \theta_{k-2\ell}, \quad \theta_k = \frac{1}{\sqrt{2}} \int \varphi(u)\, \psi\left(\frac{u+k}{2}\right) du$$

Wavelet characterization of $C^{\alpha}(\mathbb{R}^n)$.

Definition : Let $\alpha > 0$ $(\alpha \notin \mathbb{N})$; $f \in C^{\alpha}(\mathbb{R}^d)$ if $\partial^{\beta} f \in L^{\infty}$ for any $\beta = (\beta_1, \dots, \beta_d)$ such that $|\beta| = \beta_1 + \dots + \beta_d \leq [\alpha]$, and, if $|\beta| = [\alpha]$,

$$|\partial^{\beta} f(x) - \partial^{\beta} f(y)| \leq C |x-y|^{\alpha - [\alpha]}.$$

Proposition : If the multiresolution analysis is r-smooth for an $r > [\alpha]+1$; then $f \in C^{\alpha}(\mathbb{R}^d)$ if and only if

$$\exists C > 0 : \quad \forall k \in \mathbb{Z}^d, \ |\alpha_k| \leq C$$

$$\forall j \geq 0 \ \forall i, \ \forall k \in \mathbb{Z}^d, \ |C^i_{j,k}| \leq C\, 2^{-\alpha j}.$$

Z Note that $\alpha \notin \mathbb{N}$

Proof (if $0 < \alpha < 1$ and $f : \mathbb{R} \to \mathbb{R}$)

$$|d_k| \leq \int |f(x)||\varphi(x-k)| dx \leq C\|f\|_\infty .$$

$$\begin{aligned}
|c_{j,k}| &= \left| 2^j \int f(x)\, \psi(2^j x - k)\, dx \right| \\
&= \left| 2^j \int \left(f(x) - f\left(\frac{k}{2^j}\right)\right) \psi(2^j x - k)\, dx \right| \\
&\leq c\, 2^j \int \left| x - \frac{k}{2^j} \right|^\alpha \frac{dx}{(1 + |2^j x - k|)^2} \\
&= c\, 2^{-\alpha j} \int \frac{|u|^\alpha}{(1 + |u|^2)}\, du \\
&\leq c\, 2^{-\alpha j}
\end{aligned}$$

Converse part: If $|d_k| \le C$, $|C_{j,k}| \le C2$;

$$|Q_j f(x)| \le C \sum_k \frac{2^{-\alpha j}}{(1+|2^j x-k|)^2} \le C 2^{-\alpha j}$$

$$|P_0 f(x)| \le C \sum \frac{1}{(1+|2^j x-k|)^2} \le C$$

$\Rightarrow f \in L^\infty$

$$|Q_j' f(x)| \le C \sum_k \frac{2^{-\alpha j} 2^j}{(1+|2^j x-k|)^2} \le C 2^{(1-\alpha)j}$$

$$|P_0' f(x)| \le C$$

let j_0 be defined by $2^{-j_0-1} < |x-y| \le 2^{-j_0}$

$$|f(x)-f(y)| \le \underbrace{\sum_{j \le j_0} |Q_j f(x) - Q_j f(y)|}_{(1)} + \underbrace{\sum_{j > j_0} |Q_j f(x) - Q_j f(y)|}_{(2)}$$

$$(1) \le \sum_{j \le j_0} c|x-y| 2^{(1-\alpha)j} \le c|x-y| 2^{(1-\alpha)j_0} \le c|x-y|^{1-(1-\alpha)}$$

$$(2) \le \sum_{j > j_0} c 2^{-\alpha j} \le c 2^{-\alpha j_0} \le c|x-y|^{\alpha}$$

Wavelet notations

dyadic cubes: $\lambda = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right)$.

$(k_1, \dots, k_d) \in \mathbb{Z}^d$.

$(i) \in \{0,1\}^d - (0, \dots, 0)$

$$\lambda \; (= \lambda(i, j, k)) = \frac{k}{2^j} + \frac{i}{2^{j+1}} + \left[0, \frac{1}{2^{j+1}}\right)^d$$

$$c_\lambda = c^i_{j,k} \quad \text{if } j \geq 0$$
$$= d_k \quad \text{if } j = -1$$

$$\psi_\lambda(x) = \psi^{(i)}(2^j x - k) \quad \text{if } j \geq 0$$
$$= \varphi(x - k) \quad \text{if } j = -1$$

$$\mu_\lambda = \frac{k}{2^j}$$

If $f \in L^\infty$, $|C_\lambda| \leq 2^{dj} \int |f(x)| |\psi_\lambda(x)| \, dx \leq C \|f\|_\infty$

The wavelet leaders are

$$d_\lambda = \sup_{\lambda' \subset \lambda} |C_{\lambda'}|.$$

If $x_0 \in \mathbb{R}^d$, $\lambda_j(x_0)$ is the dyadic cube of width 2^{-j} such that $x_0 \in \lambda_j(x_0)$.

$$d_j(x_0) = \sup_{\lambda' \subset 3\lambda_j(x_0)} |C_{\lambda'}|$$

Proposition: let $f \in L^\infty(\mathbb{R}^d)$. If $f \in C^\alpha(x_0)$, then

$$\exists C > 0 \quad \forall j \geq 0 \qquad d_j(x_0) \leq C 2^{-\alpha j}.$$

Proof: (if $0<\alpha<1$, $f:\mathbb{R}\to\mathbb{R}$)

let $j \geq 0$ and $\lambda' \subset 3\lambda_j(x_0)$

$$c_{\lambda'} = 2^{j'} \int f(x)\, \psi(2^{j'}x - k')\, dx$$

$$= 2^{j'} \int (f(x) - f(x_0))\, \psi(2^{j'}x - k')\, dx$$

$$|c_{\lambda'}| \leq C\, 2^{j'} \int \frac{|x-x_0|^\alpha}{(1+|2^{j'}x-k'|)^2}\, dx$$

$$\leq C\, 2^{j'} \int \frac{2^\alpha \left(\left|x - \frac{k'}{2^{j'}}\right|^\alpha + \left|\frac{k'}{2^{j'}} - x_0\right|^\alpha\right)}{(1+|2^{j'}x-k'|)^2}\, dx$$

$$\leq C\left(2^{-\alpha j'} + \left|\frac{k'}{2^{j'}} - x_0\right|^\alpha\right)$$

$$\lambda' \subset 3\lambda_j(x_0) \Rightarrow j' \geq j-1$$

$$\Rightarrow \left|\frac{k'}{2^{j'}} - x_0\right| \leq 4\cdot 2^{-j}$$

$$\Rightarrow |c_{\lambda'}| \leq C\, 2^{-\alpha j}$$

Definition: A function f is a uniform Hölder function if there exists $\varepsilon > 0$ such that $f \in C^{\varepsilon}(\mathbb{R}^d)$.

Proposition: Let f be a uniform Hölder function. If $d_j(x_0) \leq C\, 2^{-\alpha j}$, then $\exists P$ Polynomial of degree $\leq [\alpha]$ and $C' > 0$ such that, if $|x - x_0| \leq \frac{1}{2}$,

$$|f(x) - P(x - x_0)| \leq C\,|x - x_0|^{\alpha} \operatorname{Log}\left(\frac{1}{|x - x_0|}\right).$$

Exercise: optimal. $\nearrow$ Log

$\searrow$ C^{ε}