

Lecture course of 10 lectures

Palma de Mallorca, Spain, February 2-6

Mather theory and variational
construction of diffusing orbits
in Hamiltonian systems.

1. Examples of Hamiltonian systems: Pendulum, N -body system; Completely integrable Hamiltonian systems; Arnold-Liouville theorem.
2. Convex Hamiltonian and Euler-Lagrange systems, Action functional, Generating function.
3. KAM Theorem, Twist maps, Generating function.
4. Homeomorphisms of the circle, Denjoy sets.
5. Moser Twist Theorem
6. Aubry-Mather theory of twist maps
7. Proof of Aubry-Mather Theorem
8. Peierl's Barrier function
9. Mather Connecting Theorem, its variational proof
10. Mather theory (Aubry-Mather theory in higher dimensions)

lecture 1 Introduction

Examples of Hamiltonian systems:

Pendulum, N -body problem.

Completely Integrable Hamiltonian systems
Arnold-Liouville Theorem.

I Mechanical system $x: \mathbb{R} \rightarrow \mathbb{R}^N$ - motion in $\mathbb{R}^N$

Newton Law $\boxed{\ddot{x} = F(x, \dot{x}, t)}$ (1)

$x(t)$ - position, $\dot{x}(t)$ - velocity at time t

Favorite examples.

1. A falling stone on the earth
 $\ddot{x} = -g$, $g \approx 9.8 \text{ m/s}^2$ Galileo

3. Ideal planar pendulum
 $\ddot{x} = -\sin \pi x$ $x \in S'$ (2)

Exercise 1. Draw trajectories of (2) in the cylinder
 $(x, y) \in S' \times \mathbb{R}$, where $y = \dot{x}$

n-particle motion

$1, \dots, m_n$ - masses $x_1, \dots, x_n$ - positions in $\mathbb{R}^N$

$$J(x_1, \dots, x_n) = \sum_{i < j} \frac{m_i m_j}{|x_i - x_j|} \quad \text{potential energy}$$

$$m_i \ddot{x}_i = - \frac{\partial U}{\partial x_i}$$

Our main interest $N=2$
and $n \leq 3$ (planar motion.)

[Number of degrees of freedom and
integrable systems.

System in 1 degree of freedom

$$\ddot{x} = f(x), \quad x \in \mathbb{R} \text{ (or } S^1) \quad (3)$$

$$T(\dot{x}) = \frac{1}{2} \dot{x}^2 \quad \text{kinetic energy}$$

$$U(x) = - \int_{x_0}^x f(z) dz \quad \text{potential energy}$$

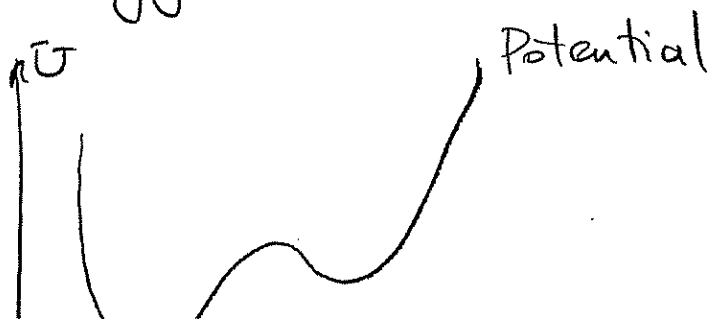
$$E = T + U \quad \text{total energy}$$

Exercise 2. Prove total energy conservation

$$\frac{dE(t)}{dt} = 0.$$

Exercise 3.

Let $U(x)$ have the form



3. System in 2 degree of freedom

$$\ddot{x} = f(x) \quad x \in \mathbb{R}^2 \quad (4)$$

(4) is called conservative if $f(x) = -\frac{\partial U(x)}{\partial x}$

$$\ddot{x} = -\frac{\partial U(x)}{\partial x}$$

Exercise 4. Prove total energy $E = T + U = \frac{1}{2}\dot{x}^2 + U$ conservation.

Example: Spherical pendulum, $x_1, x_2 \in S^1$

$$U(x) = \frac{1}{2}(x_1^2 + \omega^2 x_2^2)$$

If $y_i = \dot{x}_i$ we get

$$\begin{cases} \dot{x}_1 = y_1 \\ \dot{y}_1 = -x_1 \\ \dot{x}_2 = y_2 \\ \dot{y}_2 = -\omega^2 x_2 \end{cases} \quad (5)$$

Exercise 5. Draw trajectories of (5) on (x_1, x_2) plane

2. System in 1.5 degree of freedom

$$\ddot{x} = f(x, t) \quad \begin{array}{ll} t \in S^1 & \text{time periodic} \\ x \in S^1 \text{ or } \mathbb{R}^1 & \text{force} \end{array}$$

Calculation of # of degrees of freedom

$$\begin{cases} \dot{x} = y \\ \dot{y} = f(x, t) \end{cases} \quad \frac{\dim x + \dim y + \dim t}{2} = \frac{3}{2} = 1.5$$

I Hamiltonian System, Integrability.

$$(x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n}$$

$$H: \mathbb{R}^{2n} \rightarrow \mathbb{R} \text{ - Smooth}$$

function called Hamiltonian

$$\dot{x}_i = \partial_{y_i} H(x, y, \cdot) \quad (6)$$

$$\dot{y}_i = -\partial_{x_i} H(x, y, \cdot)$$

$$i=1 \dots n$$

H might periodically depend on time

$$H: \mathbb{R}^{2n} \times S^1 \rightarrow \mathbb{R}$$

(5) is Hamiltonian System (HS)

Mechanical systems

Hamiltonian systems

$$H = T + U$$

check equations (1-5).

def. A function $F: \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is called a first integral if F is constant along trajectories of (6)

$$\frac{d}{dt} F(x(t), y(t)) = 0.$$

rose def. HS (6) is (completely) integrable if it has n "independent first" integrals

Examples of integrable systems. (1)

A 1 degree of freedom is always integrable
Total energy is the first integral.

B. Spherical pendulum is integrable

$F_1(x, y) = \frac{1}{2}(x_1^2 + y_1^2)$
 $F_2(x, y) = \frac{1}{2}(\omega^2 x_2^2 + y_2^2)$ are first integrals
and clearly "independent"

Exercise 6. Planar 2-body problem is integrable.

IV Motion of integrable Hamiltonian systems
is deterministic and simple

Arnold-Liouville Theorem. Suppose HS (6)
is integrable and $F_1, \dots, F_n$ are independent
first integrals. Then for a level set

$$M_a = \{(x, y) \in \mathbb{R}^{2n} : F_i(x, y) = a_i, i=1 \dots n\},$$

where gradients $\nabla F_1, \dots, \nabla F_n$ are independent, we have

M_a is a smooth invariant manifold wrt (6)

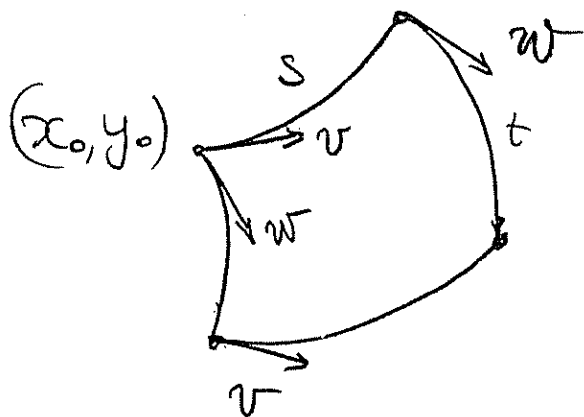
If M_a is compact connected, then M_a is n -torus $\mathbb{T}^n$

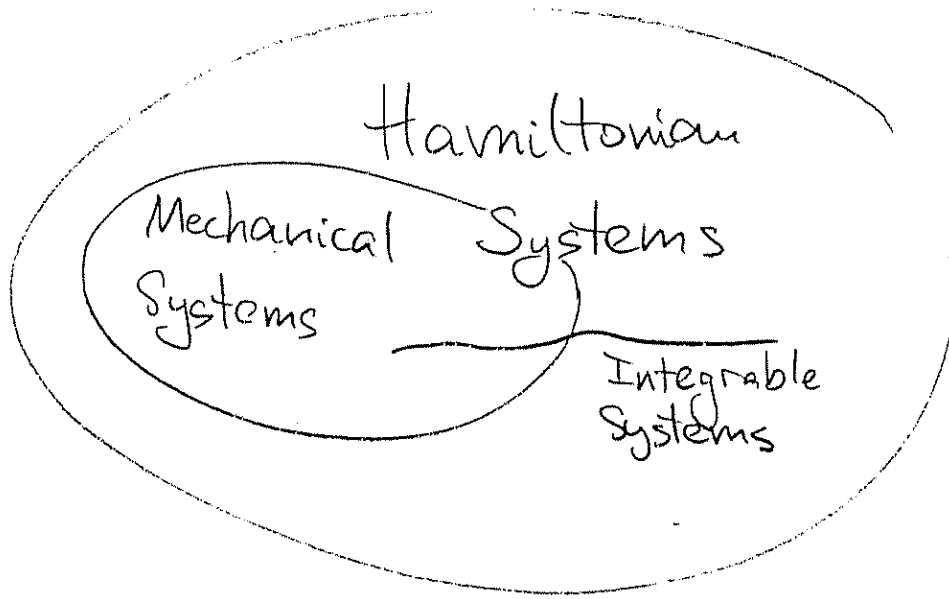
After change of coordinates on $\mathbb{T}^n = \{\varphi = (\varphi_1, \dots, \varphi_n) \in \mathbb{T}^n\}$

$\frac{d\varphi}{dt} = \omega$
 $\omega = \omega(a)$ defines periodic or
almost periodic motion.

of F_i and F_j are independent (or what is called in involution) if Hamiltonian vector field (6) with H replaced by F_i and F_j commute.

Def. Two vector fields, say v and w , commute for any initial condition (x_0, y_0) and any $t \in \mathbb{R}$ moving time s along v then time t along w is the same as time t along w first then time s along v .





Lecture 2. Hamilton $\rightarrow$ Euler-Lagrange

Convex Hamiltonian and Euler-Lagrange systems

Action, Generating Functions, Twist Maps.

I Convex Hamiltonians.

$H: \mathbb{T}^n \times \mathbb{R}^n \times S \rightarrow \mathbb{R}$ smooth Hamiltonian
 $x = (x_1, \dots, x_n) \in \mathbb{T}^n$ position
 $y = (y_1, \dots, y_n) \in \mathbb{R}^n$
 $t \in S$ time
$$\begin{cases} \dot{x}_i = \partial_{y_i} H(x, y, \cdot) \\ \dot{y}_i = -\partial_{x_i} H(x, y, \cdot) \end{cases} \quad (1) \quad i=1, \dots, n$$

Standard Assumptions (SA)

1) (convexity in y) for each $x \in \mathbb{T}^n$ and each $y \in \mathbb{R}^n$
Hessian $\left\{ \partial_{y_i y_j}^2 H \right\}_{1 \leq j, i \leq n}$ is positive definite

2) (superlinearity in y)

$\lim_{|y| \rightarrow +\infty} \frac{H(x, y)}{|y|} \rightarrow +\infty$ for each $x \in \mathbb{T}^n$

3) (completeness) Trajectories of (1) exist for all time (no blow up in finite time)

MetaTheorem Mechanical Systems satisfy

Standard Assumptions.

Abbreviate A, B, C) as Convexity

Examples satisfying SA

Mechanical systems

$$l(x, y) = \frac{y^2}{2} + U(x)$$

$$y^2 = \frac{1}{2} \sum_{i=1}^n y_i^2 \quad - \text{kinetic energy}$$

$$U: \mathbb{R}^n \rightarrow \mathbb{R} \quad - \text{potential energy}$$

Exercise 1. Check assumptions A, B, C)

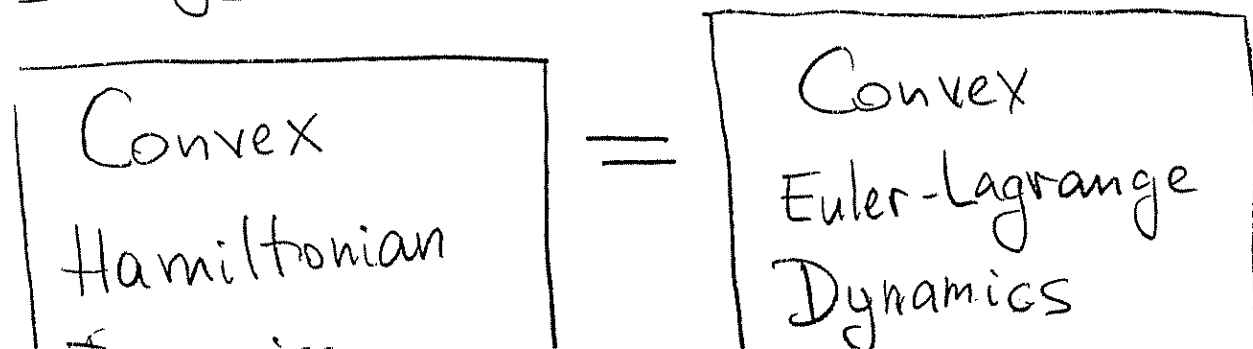
$$H(x, y) = l(y) + U(x),$$

where $l(y)$ is convex, i.e. Hessian $\{\partial_{y_i y_j}^2 l(y)\}_{1 \leq i, j \leq n}$

is positive definite for all $y \in \mathbb{R}^n$,

and superlinear, i.e. $l(y)/|y| \rightarrow +\infty$ as $|y| \rightarrow +\infty$.

From Hamiltonian to Euler-Lagrange systems
via Legendre transform



(3)
 $L: T^*X \times \mathbb{R}^n(x, S) \rightarrow \mathbb{R}$ smooth Lagrangian
 $x = (x_1, \dots, x_n) \in T^*X$ position $t \in S'$ time
 $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ velocity

$$\begin{cases} \frac{d}{dt} \frac{\partial L(x, \dot{x}, t)}{\partial \dot{x}} = \frac{\partial L}{\partial x} \\ \frac{dx}{dt} = \dot{x} \end{cases} \quad (2)$$

$$\frac{\partial L}{\partial x} = \left(\frac{\partial L}{\partial x_1}, \dots, \frac{\partial L}{\partial x_n} \right), \quad \frac{\partial L}{\partial v} = \left(\frac{\partial L}{\partial v_1}, \dots, \frac{\partial L}{\partial v_n} \right)$$

is called
Euler-Lagrange
 system

Legendre transform of smooth $H(x, y, \cdot)$ is

$$L(x, v, \cdot) = \inf_{y \in \mathbb{R}^n} \{ \langle v, y \rangle - H(x, y, \cdot) \},$$

where $\langle \cdot, \cdot \rangle$ is the standard dot product in $\mathbb{R}^n$

Lemma 1. If H is C^1 smooth in (x, y) ,
 convex and superlinear in y , then so is L in v .

The map $L_H: (x, y, \cdot) \rightarrow (x, v, \cdot) = \left(x, \frac{\partial H(x, y)}{\partial y}, \cdot \right)$
 is C^1 diffeomorphism of $T^*X \times \mathbb{R}^n(x, S)$ onto $T^*X \times \mathbb{R}^n_v(x, S)$

Moreover, L_H conjugates (1) and (2), i.e.

Hamiltonian
 System (1) on $(x, y, \cdot)$ $\xleftrightarrow[L_H]{L_H^{-1}}$ Euler-Lagrange
 System (2) on $(x, v, \cdot)$

Examples

Mechanical system

$$H(x, y) = \frac{y^2}{2} + U(x)$$

$$\frac{1}{2}y^2 = \sum_{i=1}^n \frac{y_i^2}{2}$$

kinetic

$$U: \mathbb{R}^n \rightarrow \mathbb{R}$$

potential

$$L(x, v) = \inf_{y \in \mathbb{R}^n} \{ \langle v, y \rangle - \frac{y^2}{2} - U(x) \} = \frac{v^2}{2} - U(x)$$

$$L_H: (x, y) \rightarrow (x, v) = (x, y)$$

for mechanical systems $y = v$ - velocity.

3. Convex Hamiltonians

$H(x, y)$ - convex & superlinear in y

$$L(x, v) = \inf_{y \in \mathbb{R}^n} \{ \langle v, y \rangle - H(x, y) \}$$

Differentiate wrt y and get $v = \frac{\partial H(x, y)}{\partial y}$

$$L(x, \frac{\partial H(x, y)}{\partial y}) = \langle \frac{\partial H(x, y)}{\partial y}, y \rangle - H(x, y)$$

$$L_H: (x, y) \rightarrow (x, v) = (x, \frac{\partial H(x, y)}{\partial y})$$

diff homeomorphism
by convexity

$$L_H^{-1}: (x, v) \rightarrow (x, y) = (x, \frac{\partial L(x, v)}{\partial v})$$

Exercise 2. Calculate Legendre transform of (5)

1. $f(x) = |x|$ (no superlinear growth)

2. $f(x) = |x|^\alpha$, $\alpha > 1$ (— " —)

3. $f(x, y) = x^2 - y^2$ (no convexity)

4. $f(x) = mx^{3/2}$

Conclusion: • Legendre transform is well-defined for convex functions

• Inverse Legendre transform is well-defined for convex & superlinear functions.

Therefore, assumptions A) & B) allow us to conjugate Hamiltonian systems and Euler-Lagrange systems.

Exercise 3. Prove the Lemma. 14

Hint: A bit easier to consider $\mathcal{L}_H^{-1}$ and show that Euler-Lagrange become Hamilton (Compute dH)

□ Hamilton's Principle of Least Action.

let $\gamma: [a, b] \rightarrow \mathbb{T}^n$ smooth $a < b \in \mathbb{R}$
(or absolutely continuous)

$$I(\gamma) = \int_a^b L(d\gamma(t), t) dt \quad - \text{action along } \gamma$$

$d\gamma(t) = (\gamma(t), \dot{\gamma}(t))$ - (position, velocity)
or so-called 1-jet

γ is (action) minimizer subject to fixed
end points if any other $\tilde{\gamma}: [a, b] \rightarrow \mathbb{T}^n$ with
 $\tilde{\gamma}(a) = \gamma(a)$, $\tilde{\gamma}(b) = \gamma(b)$ has

$$I(\gamma) \leq I(\tilde{\gamma}).$$

Lemma 2. Minimizers satisfy Euler-Lagrange.

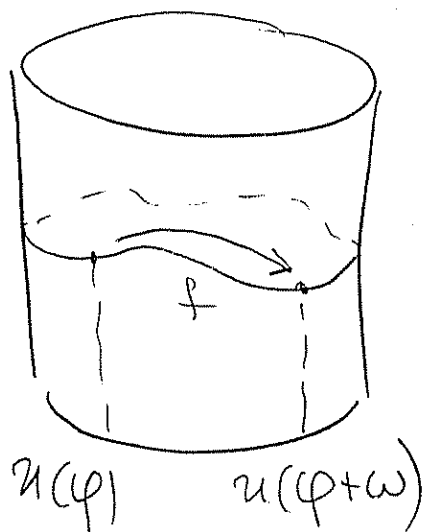
Exercise 4. Prove Lemma 2.

lift $\mathbb{T}^n$ to the universal cover $\mathbb{R}^n$.

Define

$$h(x, x') = \inf \int_a^b L(d\gamma(t), t) dt$$

h is called
a generating
function.



Suppose

$$f(w(\varphi)) = w(\varphi + \omega) \quad (6)$$

Then $f|_Z$ has rotation number ω .

Exercise 4. A C^1 -twist map f on AS has an invariant curve $w(\varphi) = (u(\varphi), v(\varphi))$ parametrized by φ and satisfying (6) if and only if position function $u(\varphi)$ satisfies

$$E(u(\varphi)) \equiv 2_1 h(u(\varphi), u(\varphi + \omega)) + 2_2 h(u(\varphi - \omega), u(\varphi)) = 0$$

Lecture 3 KAM Theorem, Twist Maps, Generating Functions

I KAM Theorem and action-angle coordinates.

$$\begin{cases} \dot{x}_i = \frac{\partial H}{\partial y_i} \\ \dot{y}_i = -\frac{\partial H}{\partial x_i} \end{cases} \quad i=1 \dots n \quad (1)$$

If (1) is integrable by Arnold-Liouville Thm
locally there is a (θ, I) -coordinates

$$\theta = (\theta_1, \dots, \theta_n) \in \mathbb{T}^n \quad \text{angle}$$

$$I = (I_1, \dots, I_n) \in U \subset \mathbb{R}^n \quad \text{action}$$

such that $H(x, y) = H(I(x, y)) : U \rightarrow \mathbb{R}$

$$\begin{cases} \dot{\theta}_i = \frac{\partial H(I)}{\partial I_i} \\ \dot{I}_i = 0 \end{cases} \quad i=1 \dots n \quad (2)$$

(θ, I) -action-angle coordinates.

Integrable system has invariant tori almost everywhere

Suppose $H(I) = \frac{I^2}{2}$ for simplicity

$$I^2 = \sum_{i=1}^n I_i^2, U = B^n \text{ unit ball}$$

KAM Theorem. For any small ε and smooth $H_1(\theta, I; \cdot)$ the Hamiltonian $H_\varepsilon(\theta, I; \cdot) = \frac{I^2}{2} + \varepsilon H_1(\theta, I; \cdot)$.

As most of trajectories in $\pi^n \times B^n(x, \delta)$ belong to invariant (KAM) n (resp $n+1$)-dim'd tori dynamics on each of those tori is smoothly conjugate to a rigid rotation (see Arnold-Liouville statement).
For most of frequencies $\omega \in B^n$ there is (KAM) invariant torus $\pi_{\omega, \varepsilon}^n$ with the flow on it smoothly conjugate to the rigid rotation $\dot{\theta} = \omega$.

In this lecture we are interested in time periodic case and $n=1$, i.e. 1 degree of freedom.

Kota Theorem. For near integrable systems 1 degree system behaves like $(n - \frac{1}{2})$ degree system

1. Energy surface reduces total dimension

II Twist maps as time on maps.

Let $L(\theta, I, t)$ — smooth time periodic Lagrangian $t \in S^1, \theta \in \mathbb{T}^n, I \in \mathbb{R}^n$

Let L satisfy Standard Assumptions (SA), i.e. convex & superlinear in I & $E-L$ flow is complete.

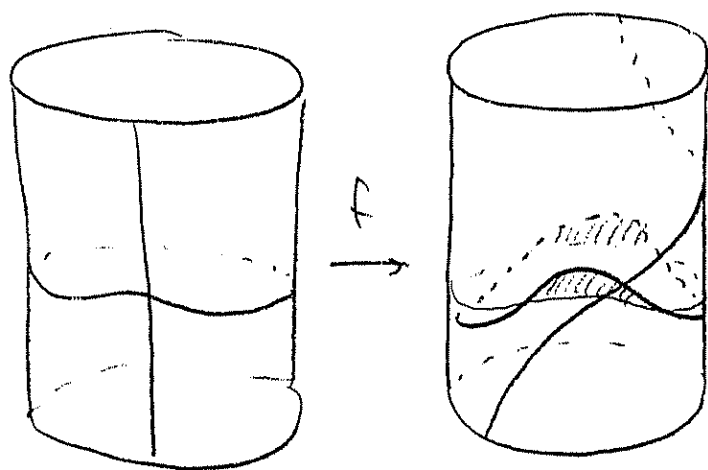
$f: A \rightarrow A$ smooth $A = S^1 \times \mathbb{R} \ni (\theta, I)$

f is called Exact Area-Preserving Twist map or simply twist map if

- f preserves area $d\theta dI$
- f twists, i.e. every vertical line is monotonically twisted
- $\forall \theta_0 = \{\theta = \theta_0\}, \pi: (\theta, I) \rightarrow \theta$ $\pi \circ f(\theta_0, I)$ is monotone in I

f is exact ($\approx$ no vertical drift), i.e. for any noncontractible loop $\gamma \subset A$

area under γ = area under $f\gamma$



Example: Standard Twist map $(\theta, I) \rightarrow (\theta + I \pmod{1}, I)$ = time 1 map of $L(\theta, I) = \frac{I^2}{2}$ $H(\theta, I)$

Lemma 1. Let $L(\theta, I, t) = \frac{I^2}{2} + \varepsilon L_1(\theta, I, t)$.

then time 1 map of L is a twist map.

Exercise 2 Prove Lemma 1.

Theorem (Moser) Let $f: A \rightarrow A$ be smooth twist map. Then there is a smooth convex time periodic Lagrangian $L(\theta, I, t)$ whose time 1 map is f .

examples of twist maps later.

I Generating Functions and Twist Maps.

ift $\pi \ni \theta$ to the universal cover $\mathbb{R}$ - position space, all still θ . Denote $\hat{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $\hat{f}(\theta, t)$

Define
$$h(\theta, \theta') = \inf_{\substack{\gamma(0) = \theta \\ \gamma(1) = \theta'}} \int_0^1 L(\gamma(t), t) dt \quad (3)$$
 minimal action to get from θ to θ' in time 1.

is called generating function.

Example: $L(\theta, I, t) = \frac{I^2}{2}$, $h(\theta, \theta') = \frac{(\theta' - \theta)^2}{2}$

$$\gamma = \theta \quad \int_0^1 \left(\frac{\dot{\gamma}(t)}{2} \right)^2 dt \leq \frac{1}{2} \left(\int_0^1 \dot{\gamma}(t) dt \right)^2 = \frac{1}{2} (\theta' - \theta)^2.$$

$$\boxed{\text{Twist Maps}} = \boxed{\text{Generating function}}$$

Lemma 2. For each c' -twist map $f: \mathbb{A}^S \rightarrow \mathbb{A}^S$ there is a (generating) function $h: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, defined by (3), such that the following properties hold true:

• if $\hat{f}(\theta, I) = (\theta', I')$, then

$$\begin{cases} I' = \partial_2 h(\theta, \theta') \\ I = -\partial_1 h(\theta, \theta') \end{cases} \quad (4)$$

periodicity $h(\theta+1, \theta'+1) = h(\theta, \theta') \quad \forall \theta, \theta' \in \mathbb{R}$

monotonicity $\partial_2 h(\theta, \theta') > 0, \partial_1 h(\theta, \theta') < 0$

twist $\partial_2 h(\theta, \theta') \leq -\delta < 0$

condition at $+\infty$ $h(\theta, \theta') \rightarrow +\infty$ as $|\theta - \theta'| \rightarrow +\infty$

Proof: • Periodicity follows from (3) and

periodicity of L

Monotonicity: As distance between start θ and finish θ' increases action increases and the other way

Same for condition at ∞ .

Twist condition is in the addendum pg. 8 ✓

Condition (4) is important and requires special discussion.

Configuration - $\Theta = \{\theta_n\}_{n \in \mathbb{Z}}$ set of numbers
(think positions at integer times)

$\gamma = \{\theta_n\}_{n \in \mathbb{Z}}$ is (h-)stationary if

$$\partial_2 h(\theta_n, \theta_{n+1}) + \partial_1 h(\theta_{n-1}, \theta_n) = 0 \quad \forall n \in \mathbb{Z} \quad (5)$$

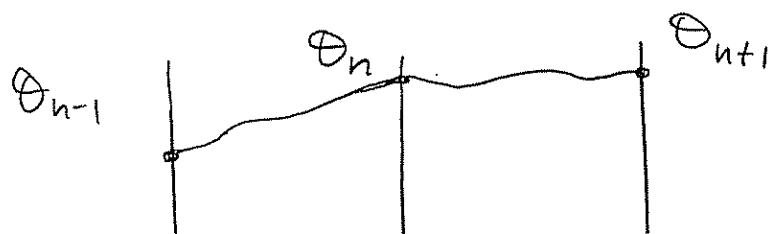
is (h-)minimal if

for any time interval $j < k$ and any $\tilde{\gamma} = \{\tilde{\theta}_n\}_{n=j}^k$ such that $\tilde{\theta}_j = \theta_j, \tilde{\theta}_k = \theta_k$ we have

$$\sum_{n=j}^k h(\theta_n, \theta_{n+1}) \leq \sum_{n=j}^k h(\tilde{\theta}_n, \tilde{\theta}_{n+1})$$

(5) is Euler-Lagrange equation

Trajectories of
EL extremize
action



Exercise 2. Prove that minimal configuration γ is stationary

Exercise 3.

Trajectory of twist map f
 $\{f^n(\theta_0, I_0) = (\theta_n, I_n)\}_{n \in \mathbb{Z}}$

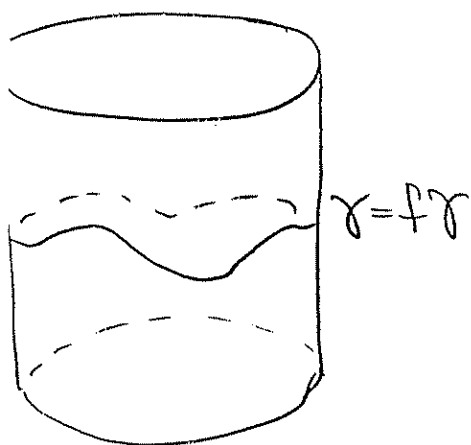
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 $\longleftrightarrow$

Stationary configuration of its generating function h
 $\{\theta_n\}_{n \in \mathbb{Z}}$

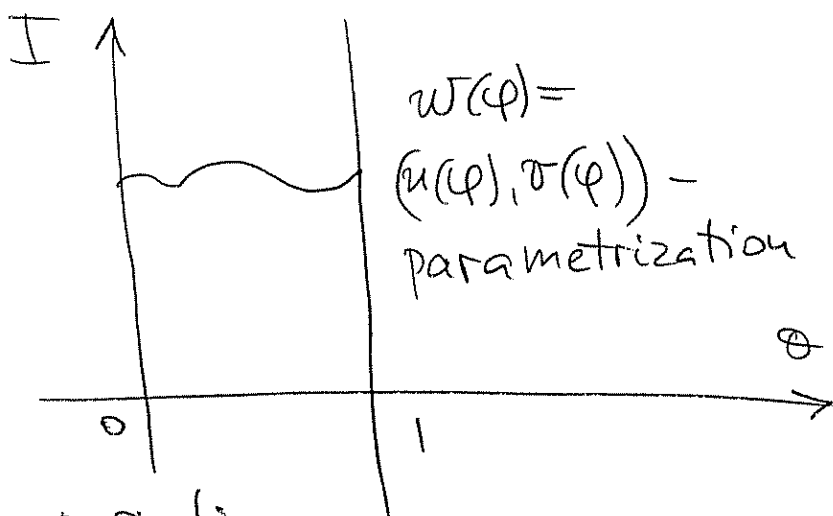
given by $\begin{cases} I_{n+1} = \partial_2 h(\theta_n, \theta_{n+1}) \\ I_n = -\partial_1 h(\theta_n, \theta_{n+1}) \end{cases} \quad (= \partial_2 h(\theta_{n-1}, \theta_n))$
 by stationarity.

I Invariant curves

$f: \mathbb{A} \rightarrow \mathbb{A}$



$\tilde{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ - lift of f

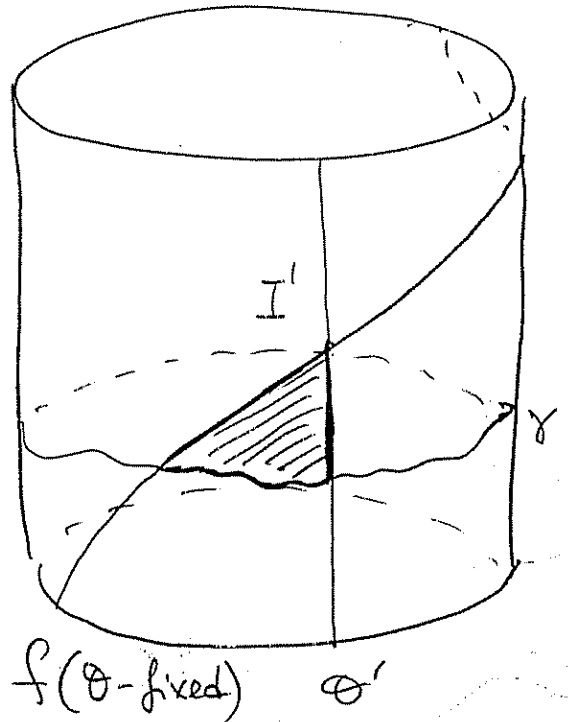


$u(\varphi) - \varphi$ and $v(\varphi)$ periodic functions of period 1.

Proof of twist condition of lemma 2
and geometric definition of
generating function.

Lemma 3. Let $f: A \rightarrow A$ be a C^1 twist map.
Suppose (for simplicity) $\gamma = f\gamma$ is a noncon-
tractible invariant curve. Then

$I(\theta, \theta') =$ shaded area
above γ differs
from h , given by (3), by
a constant.



Proof
We implicitly use the fact
that γ is a Graph
(see Graph Theorem (lect. 7))
Prove by variation of end points

Consider $I(\theta, \theta')$

By definition (3) and Tonelli's Theorem there are trajectories $(\theta, \dot{\theta})(t)$, $t \in [0, 1]$ and $(\theta_\Delta, \dot{\theta}_\Delta)(t)$, $t \in [0, 1]$ such that

$$\theta(0) = \theta_\Delta(0) \text{ and } \theta(1) = \theta', \theta_\Delta(1) = \theta' + \Delta.$$

and they minimize action. Therefore,

$$h(\theta, \theta' + \Delta) - h(\theta, \theta') =$$

$$\int_0^1 \{ L(\theta_\Delta(t), \dot{\theta}_\Delta(t), t) - L(\theta(t), \dot{\theta}(t), t) \} dt =$$

$$= \int_0^1 \left[\left\{ \frac{\partial L(\theta(t), \dot{\theta}(t), t)}{\partial \theta} \Delta(t) + \frac{\partial L(\theta(t), \dot{\theta}(t), t)}{\partial \dot{\theta}} \Delta'(t) \right\} + O(\Delta^2) \right] dt$$

Integration by parts of the second term gives

$$\int_0^1 \left\{ \frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \right\} (t) dt + \frac{\partial L(\theta', \dot{\theta}', t)}{\partial \dot{\theta}} \Delta + \Delta^2 = \Delta \left(\frac{\partial L(\theta', \dot{\theta}', t)}{\partial \dot{\theta}} + \Delta \right),$$

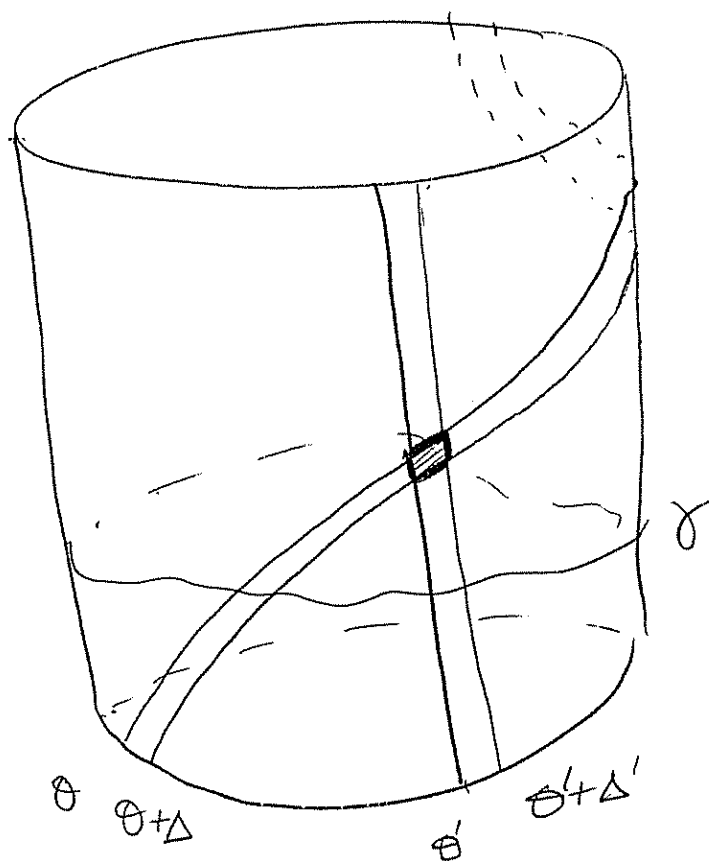
$$\text{therefore } \partial_2 h(\theta, \theta') = \frac{\partial L(\theta', \dot{\theta}', t)}{\partial \dot{\theta}} = I'$$

The second equality by Legendre transform

Corollary

$$\partial_{12} h(\theta, \theta') \leq -\delta \text{ (=minimal twist)} < 0.$$

Proof



$$\begin{aligned} \partial_{12} h(\theta, \theta') &\approx \frac{1}{\Delta \Delta'} (h(\theta + \Delta, \theta' + \Delta') - h(\theta + \Delta, \theta')) + \\ &h(\theta, \theta' + \Delta) - h(\theta, \theta') = \frac{1}{\Delta \Delta'} (\text{-shaded area}) \approx \\ &\text{angle between vertical } \{\theta = \theta'\} \text{ and image} \\ &\text{of vertical } \{\theta\}. \end{aligned}$$

Lecture 4. Homeomorphisms of the circle

Denjoy sets.

Let $f: S^1 \rightarrow S^1$ orientation preserving homeomorphism, i.e. f^{-1} -continuous.

Let $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$, $\hat{f}(\theta+1) = \hat{f}(\theta) + 1$, $\hat{f}(\theta) \equiv f(\theta) \forall \theta \in [0,1)$ be the lift of f to $\mathbb{R}$.

Exercise 1. (Poincaré) For every $\theta \in S^1$ there is a limit
$$p(f) = \lim_{n \rightarrow +\infty} \frac{\hat{f}^n(\theta)}{n}$$
 independent of θ

$p(f)$ is called rotation number.

Exercise 2 If $p(f)$ is rational, f has a periodic point.

$\text{Rec}(f) = \{ \theta \text{ whose trajectory approaches } \theta, \text{ i.e. } \theta \in \{ f^n \theta \}_{n \in \mathbb{Z}_+ \setminus \{0\}} \}$

Equivalently
$$\text{Rec}(f) = \{ \theta^* \in \lim_{k \in \mathbb{Z}_+} f^k(\theta) \}$$
 is called the set of recurrent points

lemma 1 There are two cases:

1) Every orbit is dense in S^1 and $\text{Rec}(f) = S^1$.
 Then there is a continuous change of
 coordinate $g: S^1 \rightarrow S^1$ conjugating f
 with the rigid rotation, i.e.

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ \downarrow & & \downarrow g \\ S^1 & \xrightarrow{\theta \mapsto \theta + p(q)} & S^1 \end{array} \leftarrow \text{is a commutative diagram.}$$

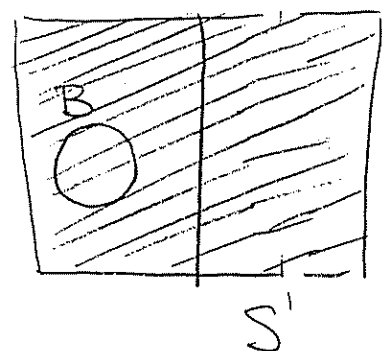
2) $\text{Rec}(f)$ is a Cantor set

Proof is in Bangert Dynamics Reported I
 We shall call $\text{Rec}(f)$ Denjoy-Cantor to
 distinguish from Cantor.

Example of Denjoy-Cantor (Cherry flow)

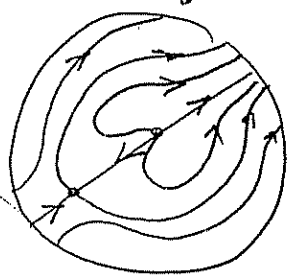
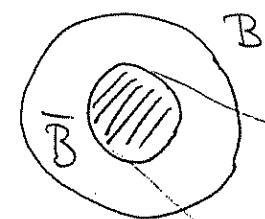
We present a flow on $\mathbb{T}^2$ whose forward Poincaré return map f is defined on $S' \subset \mathbb{T}^2$ and $\text{Rec}(f)$ is a Cantor set.

- Start w rigid irrational flow on $\mathbb{T}^2 = \{(\theta_1, \theta_2) \bmod 1\}$



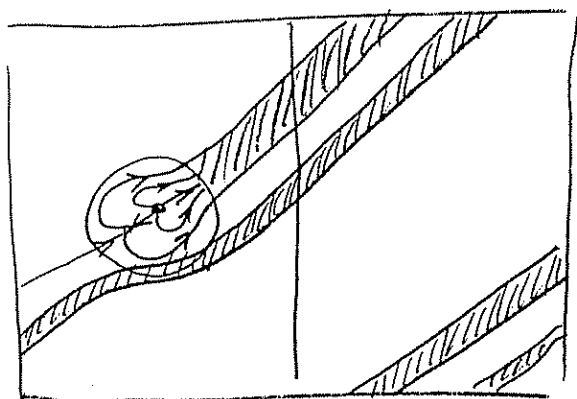
$$\begin{cases} \dot{\theta}_1 = \omega \\ \dot{\theta}_2 = 1 \end{cases} \quad \omega \notin \mathbb{Q} \quad (1)$$

- Pick a transversal $S' = \{\theta_1 = 0\}$
- Cut out a ball $B \cap S' = \emptyset$
- Replace flow in $\overline{B} \setminus B$ by



so that on the boundary ∂B it fits with rigid flow (1)

One could glue in so that resulting flow, called ν , has an irrational rotation vector. The field ν induces forward Poincaré map.



Suppose the source $\frac{1}{2}$ pumps in red paint w π^2 .

• Rotation number is irrational so painted area is dense.

Painted area is bounded by separatrices of the saddle so it is open.

Points inside of the painted channel never come back. Therefore, recurrent points are in the complement.

complement of open dense is a Denjoy-Cantor set.
see Palis, de Melo "Geometric Theory of Dynamical Systems" for details

Lecture 5. Moser Twist

Theorem and its variational proof

Recall that we investigate

$$L(\theta, I, t) = \frac{I^2}{2} + \varepsilon L_1(\theta, I, t) \quad \begin{array}{l} \text{time periodic} \\ \text{perturbation} \end{array}$$

$f: (\theta, I) \rightarrow (\theta', I')$ corresponding time 1 map happen to be twist map

Trajectories of f $\longleftrightarrow$ Stationary Configuration
 $\{ \hat{f}^n(\theta_0, I_0) = (\theta_n, I_n) \}_{n \in \mathbb{Z}}$ $\longleftrightarrow$ $\{ \theta_n \}_{n \in \mathbb{Z}}$ of its generating function h

Invariant curve

$$w(\varphi) = (u(\varphi), v(\varphi)) \text{ of } f$$

$$f(w(\varphi)) = w(\varphi + w)$$

Position $u(\varphi)$ satisfies

$$\begin{array}{|l} E(u(\varphi)) = \partial_2 h(u(\varphi), u(\varphi + w)) \\ \textcircled{1} \quad \partial_1 h(u(\varphi - w), u(\varphi)) = 0 \end{array}$$

Generating function of $L(\theta, I, t) = \frac{I^2}{2}$ is
 $h(\theta, \theta') = \left(\frac{\theta' - \theta}{2} \right)^2$

Small perturbation $L(\theta, I, t) = \frac{I^2}{2} + \varepsilon L_1(\theta, I, t)$ as a generating function

$$h(\theta, \theta') = \left(\frac{\theta' - \theta}{2} \right)^2 + \varepsilon S(\theta, \theta'). \quad \text{Simplify to } h(\theta, \theta') = \left(\frac{\theta' - \theta}{2} \right)^2 + \varepsilon S(\theta)$$

Our goal is solve (1)

$w \in \mathbb{R}$ is diophantine if for some $K, \epsilon > 0$
and all $p, q \neq 0$ we have $|qw - p| > K|q|^{-1-\epsilon}$
or also w is (K, ϵ) -diophantine.

Noser Twist Theorem Let w - diophantine,

then for small ϵ • Euler-Lagrange flow
of L has ^{an} invariant 2-torus with rotation
vector $(1, w) \iff$

Twist map f has an invariant curve

$v(\varphi) = (u(\varphi), v(\varphi)) \iff$

There is a solution $u(\varphi)$

$(u(\varphi)) \equiv \partial_2 h(u(\varphi), u(\varphi + w)) + \partial_1 h(u(\varphi - w), u(\varphi)) = 0$

Now we give precise statement using
the last way to claim invariant curve.

If equivalences are proved in lecture 3

loosely speaking if w - diophantine and

ϵ - small
 $(\dots) \dots + \epsilon S'(u(\varphi)) = 0$

II Analytic topology and exact statement of Moser Twist Theorem

W_r — the set of 1-periodic real analytic functions of θ bounded in the strip $|\operatorname{Im} \varphi| \leq r$. Introduce the maximum norm $\|f\|_r = \sup_{|\operatorname{Im} \varphi| \leq r} |f(\varphi)|$

Example: $f(\varphi) = \cos k\varphi = \frac{1}{2}(\exp(ik\varphi) + \exp(-ik\varphi))$, $f(ir) \sim \frac{1}{2}e^{kr} \rightarrow +\infty$ as $k \rightarrow +\infty$

Moser Twist Theorem Let w be (K, σ) -diophantine for some $K, \sigma > 0$ and $h(\theta, \theta') = \frac{(\theta' - \theta)^2}{2} + \varepsilon S(\theta, \theta')$. There exists $\delta = \delta(r, K, \sigma)$ such that if $\|E(\theta)\|_r < \delta$, then there exists a unique solution $u^*(\varphi)$ near $u_0(\varphi) = \varphi$ of $E(u) = 0$ with $u^*(\varphi) - \varphi \in W_{r/2}$ and mean value of $u^*(\varphi) - \varphi$ is zero.

We use Levi-Moser paper based June 1986 lecture of Moser at ETH

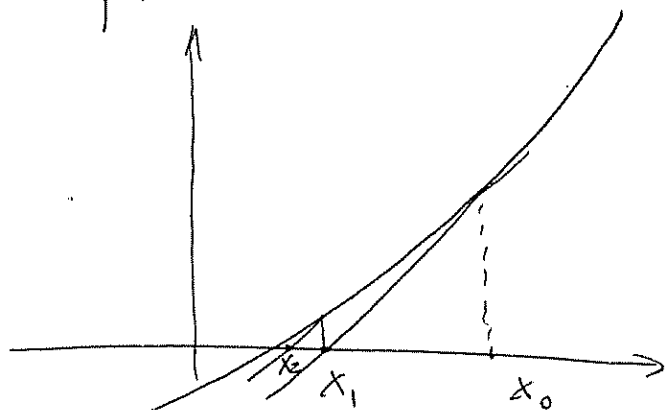
Kolmogorov-Newton Method of solving equations (4) via converging approximations.

$F: B \rightarrow \mathbb{R}^n$ smooth map.

$\|F(x_0)\|$ - small

Find solution x^*
 $F(x^*) = 0$ near x_0

suppose $F(x_0)$
is non-degenerate, $F'(x_0) \neq 0$



$\|F(x_0)\| = \delta_0$ - small

step 1. Solve linearized equation

$$F(x_0) + F'(x_0)(x - x_0) + \mathcal{O}(x - x_0)^2 = 0 \quad (1)$$

Denote solution x_1 . By Taylor formula

$$F(x) = F(x_0) + F'(x_0)(x - x_0) + \mathcal{O}(x - x_0)^2$$

at $x = x_1$, and get $|F(x_1)| \lesssim |x_1 - x_0|^2 \approx |F(x_0)|^2 \quad (2)$

step 2 Solve linearized equation

$$F(x_1) + F'(x_1)(x - x_1) + \mathcal{O}(x - x_1)^2 = 0$$

Denote solution x_2 . Argue $|F(x_2)| \lesssim |F(x_1)|^2$

iv Proof of Moser Twist Theorem using Kolmogorov-Newton Method & a trick

Goal $|E(u_0)|_r < \delta$ small

Find solution u^* near u_0

$$E(u^*) = 0.$$

By Taylor formula

$$E(u+v) = E(u) + E'(u)v + Q$$

Notations: $u^\pm(\varphi) = u(\varphi \pm \omega)$

$$\nabla f(\varphi) = f(\varphi + \omega) - f(\varphi) = f^+(\varphi) - f(\varphi)$$

$$\nabla^* f(\varphi) = f(\varphi) - f^-(\varphi)$$

Recall

$$E(u(\varphi)) = \partial_1 h(u^-(\varphi), u(\varphi)) + \partial_2 h(u(\varphi), u^+(\varphi))$$

Abbreviate

$$E(u) = \partial_1 h(u, u^+) + \partial_2 h(u^-, u)$$

$z, 1$

Direct differentiation shows

$$\begin{aligned} E'(u) &= (\partial_{11} h(u, u^+) + \partial_{22} h(u^-, u))v + \partial_{12} h(u, u^+)v^+ + \partial_{12} h(u^-, u)v^- \\ &= (\partial_{11} h + \partial_{22} h^-)v + (\partial_{12} h)v^+ + (\partial_{12} h^-)v^- \end{aligned}$$

Olmogorov-Newton Step. Need to solve

$$E'(u) v = -E(u) \quad (\text{analog of (1)})$$

Add quadratic term!

and multiply by u_φ

$$u_\varphi E'(u) v - \underbrace{v E'(u)}_{\text{small}} \underbrace{u_\varphi}_{\text{small}} = -u_\varphi E(u)$$

the left hand side has the form

$$\partial_{12} h(u_\varphi v^+ - u_\varphi^+ v) - \partial_{12} h(u_\varphi v^- - u_\varphi^- v)$$

introduce $w = \frac{v}{u_\varphi}$ and get

$$\begin{aligned} \partial_{12} h(u_\varphi u_\varphi^+ (w^+ - w)) - \partial_{12} h(u_\varphi^- u_\varphi (w - w^-)) = \\ = \nabla^* (\partial_{h_{12}} u_\varphi u_\varphi^+ \nabla w) \end{aligned}$$

We get

$$\nabla^* (\partial_{h_{12}} u_\varphi u_\varphi^+ \nabla w) = -u_\varphi E(u) \quad (3)$$

Standard Lemma. Let w be

(K, σ) -diophantine for some $K, \sigma > 0$

and let $g \in W_r$ with mean zero

Then

$$\nabla \psi = g \quad (4)$$

has a unique solution $\psi \in W_r$, with mean zero for any $0 < r' < r$ and

$$|\psi|_{r'} \leq c(K, \sigma) \frac{|g|_r}{(r-r')^{\sigma+1}}$$

Proof: Expand in Fourier series

$$g(\varphi) = \sum_{k \in \mathbb{Z} \setminus 0} g_k \exp(ik\varphi)$$

$$\psi(\varphi) = \sum_{k \in \mathbb{Z} \setminus 0} \psi_k \exp(ik\varphi)$$

Exercise 1, A) Prove $\psi_k = \frac{g_k}{\exp(2\pi i k w) - 1}, \forall k \neq 0, \psi_0 = c$

B) Prove (K, σ) -diophantine condition (pg. 2)

imply $|\exp(2\pi i k w) - 1| > \frac{c(k)}{|k|^{1+\sigma}}$

c) Prove $g \in W_r$ implies $|g_k| \leq |g|_r e^{-2\pi |k| r}$

1) Check

$$e^{-a|k|} |k|^{-b} \leq e^{-b} (b/a)^b \text{ for all } k.$$

application of A), B), and c) gives

$$|\psi_k| \leq |g|_r \frac{|k|^{1+\sigma}}{C(k) e^{2\pi r|k|}} = |g|_r e^{-2\pi s|k|} \frac{|k|^{1+\sigma}}{C(k) e^{2\pi(r-s)|k|}}$$

at $a = 2\pi(r-s)$ and $b = 1+\sigma$ in D) Get

$$|\psi_k| \leq c_1(k, \sigma) |g|_r \frac{e^{-2\pi s|k|}}{(r-s)^{1+\sigma}}$$

from this

$$\begin{aligned} |\psi|_r &\leq \sum_k |\psi_k| e^{2\pi r'|k|} \leq \frac{2c_1 |g|_r}{(r-s)^{1+\sigma}} \left(1 - e^{-2\pi(s-r')}\right)^{-1} \\ &\leq \frac{2c_1 |g|_r}{(r-s)^{1+\sigma} (s-r')} \end{aligned}$$

Put $s = \frac{r+r'}{2}$. Lemma is proven.

Lemma (on estimating increment) Assume $|u_\varphi|_r < N, |u_\varphi^{-1}|_r < N$ for $|\operatorname{Im} \varphi| \leq r$

Then (3) has a unique solution $w \in W_p$ with zero mean for any $0 < p < r$ and the correction $v = u_\varphi w$ satisfies the estimates

$$|w|_p \leq \frac{C(K, \sigma, N)}{(r-p)^{2(1+\sigma)}} |E(u)|_r, |v_\varphi|_p \leq \frac{C(K, \sigma, N)}{(r-p)^{5+2\sigma}} |E(u)|_r$$

Lemma (on estimating error)
Let $\tilde{u} = u + v$ with u and v as above
Then

$$|E(\tilde{u})|_p \leq \frac{C(K, \sigma, N)}{(p-r)^{4(1+\sigma)}} |E(u)|_r^2.$$

See Levi-Moser for some of missing details. Total length of the proof is 8 pages don't be afraid.

Salomon-Zehnder extended this to any dimension.

likely smooth case.

$E(u^*) = 0$ need to solve.

~~after u is analytic~~

$$E(u^*) = \partial_2 h(u^-, u) + \partial_1 h(u, u^+)$$

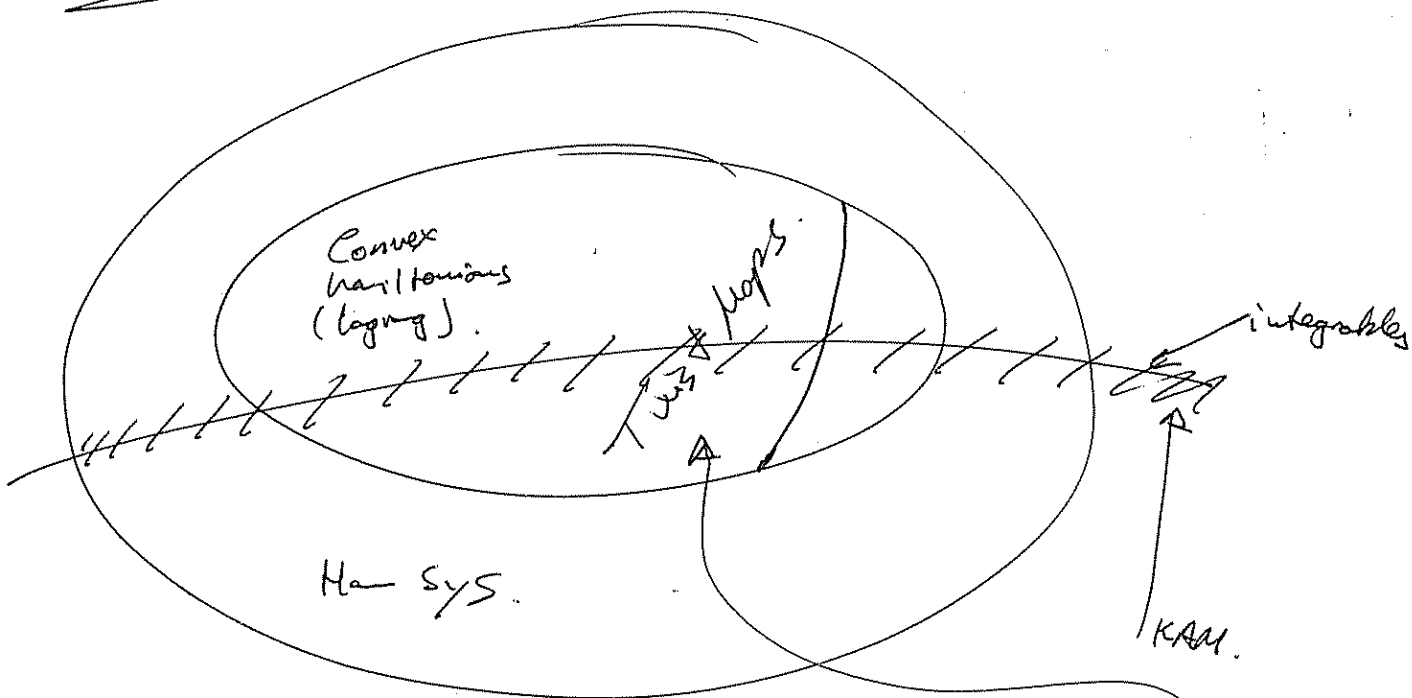
approximate h by analytic h_n .

$$\text{solve } \partial_2 h_n(u^-, u) + \partial_1 h_n(u, u^+) = 0 \rightarrow u_n$$

As $h_n \rightarrow h$
 $u_n \rightarrow u$

Knowing speed $u_n \rightarrow u$ you get smoothness of u .

=====



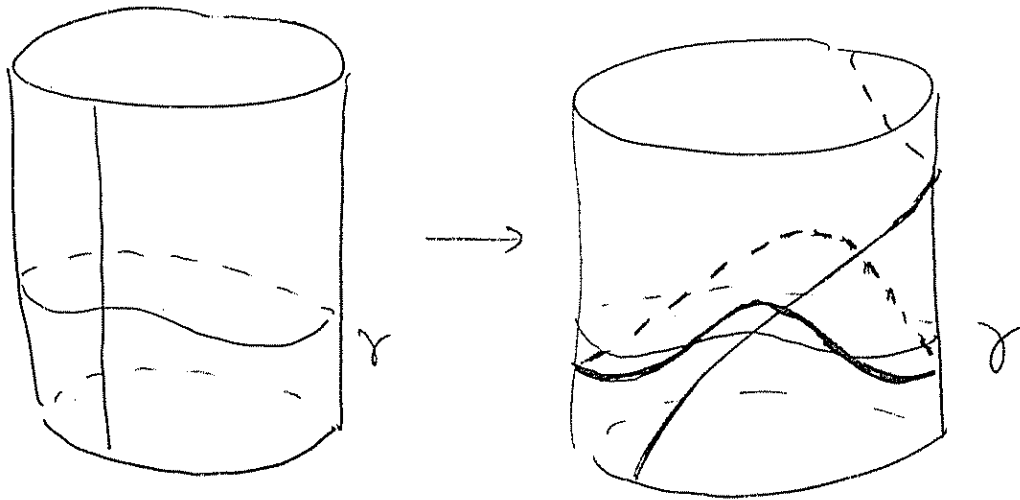
Mather Theory takes place in Convex hamiltonians. (twist maps)

Aubry-Mather theory in 1's deg. of freedom.

Lecture 6. Examples of Twist Maps, Generating functions and Aubry-Mather Theory.

Recall $f: \mathbb{A}^2 \rightarrow \mathbb{A}^2$ is a Twist Map if

- area-preserving
 - twist (every vertical line monotonically twisted)
 - exact (no average drift)
- area under γ =
area under $f\gamma$
 $\forall \gamma$ (1)



Remark If (1) holds for one γ , then for all
By Moser then twist maps come from
time periodic convex Lagrangians.

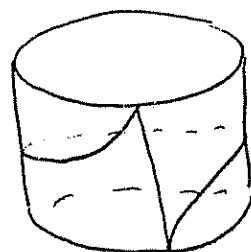
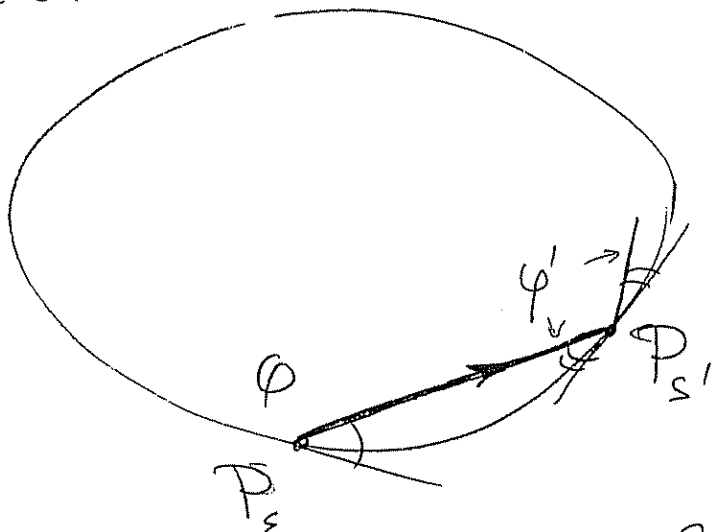
Billiard maps are Twist maps.

Let $R \subset \mathbb{R}^2$ be a convex bounded region
 with C^2 boundary ∂R parametrized
 by length, $P_s \in \partial R$ $s \in S'$

$(s, \varphi) \rightarrow (s', \varphi')$
 $s \in S', \varphi \in [0, \pi]$
 and a ball

from P_s at
 angle φ

It hits ∂R at some $P_{s'}$ and reflects
 so angle of incidence =
 angle of reflection.



Exercise 1. f is a twist map preserving
 smooth area form $\omega = \sin \varphi ds \wedge d\varphi$.
 $i(s, s') = -\text{dist}_{\mathbb{R}^2}(P_s, P_{s'})$ is a generating
 function.

1. ... perimeter

1-dimensional crystals and the Standard map

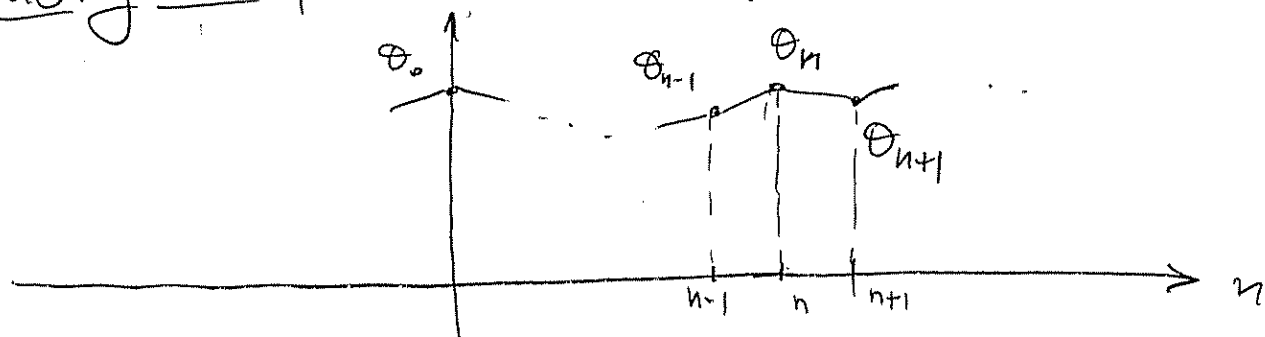
1-dim'l crystal is a bi-infinite sequence of real numbers (compare with configuration lect. 3) $\Theta = (\dots \Theta_n \dots) = \{\Theta_n\}_{n \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$

$$h(\Theta, \Theta') = \frac{1}{2} (\Theta' - \Theta)^2 + u(\Theta) \quad u(\Theta) - \text{interaction function}$$

1-dim'l crystals minimize action, i.e. if $\{\Theta_n\}_{n \in \mathbb{Z}}$ is 1-dim'l crystal, then for any $j < k \in \mathbb{Z}$ and any $\{\tilde{\Theta}_n\}_{n=j}^k$ with $\tilde{\Theta}_k = \Theta_k$ and $\tilde{\Theta}_j = \Theta_j$

$$\sum_{n=j}^{k-1} h(\Theta_n, \Theta_{n+1}) \leq \sum_{n=j}^{k-1} h(\tilde{\Theta}_n, \tilde{\Theta}_{n+1})$$

Aubry Graph of a configuration $\{\Theta_n\}_{n \in \mathbb{Z}}$



• a piecewise linear function which coincides with Θ_n at integer n and nearly interpolated in between.

Recall that in terms of Lagrangian mechanics Θ_n is a position at time n . Aubry graph is a linear approximation of the trajectory of E-L flow.

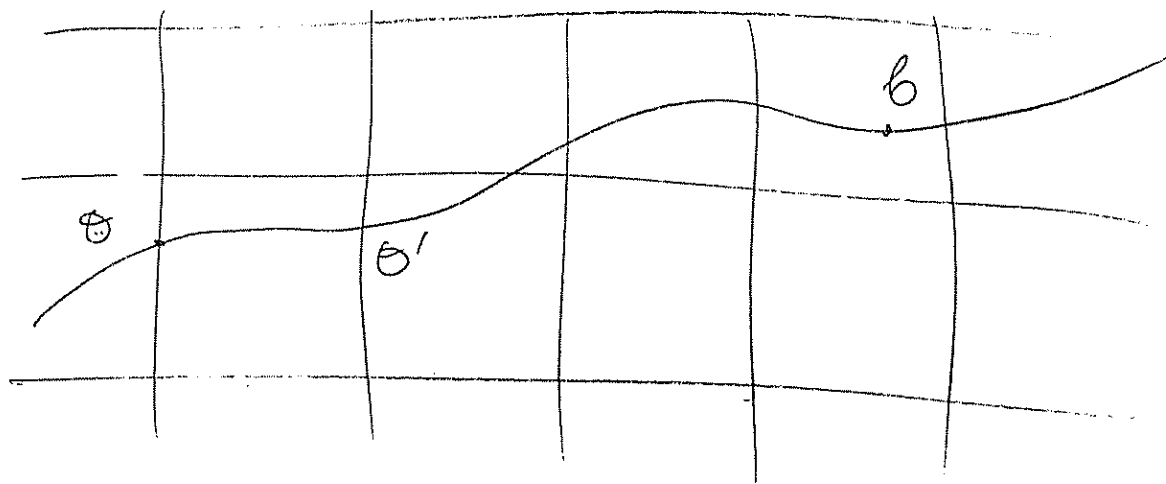
Exercise 2 The generating function $h(\theta, \theta') = \frac{1}{2}(\theta' - \theta)^2 + u(\theta)$ defines a twist map
$$\begin{pmatrix} \theta \\ I \end{pmatrix} \rightarrow \begin{pmatrix} \theta' \\ I' \end{pmatrix} = \begin{pmatrix} \theta + I - u'(\theta) \\ I - u'(\theta) \end{pmatrix}.$$

Remark Use formula (4) of Lecture 3

$$\begin{aligned} I' &= \partial_2 h(\theta, \theta') \\ I &= -\partial_1 h(\theta, \theta') \end{aligned}$$

We shall investigate minimizers!

IV Geodesic flows on $\mathbb{T}^2$ fit into variational scheme for Twist Maps



γ is a geodesic if for any $a, b \in \mathbb{R}$ distance between a, b is the shortest along γ .

Suppose a geodesic is a graph.

Define $h(a, a') = \text{dist}((0, a), (1, a'))$.

Then ^{such} geodesics minimize

$$\sum_{n=j}^K h(\theta_n, \theta_{n+1})$$

Aubry-Mather Theorem, For any
 rotation number ω and any
 lift map $f: \mathbb{A} \rightarrow \mathbb{A}$ there is
 an (action minimizing) invariant set $\Sigma_\omega \subset \mathbb{A}$ such
 that every point (θ, I) in Σ_ω has
 rotation number ω , i.e. $f^n(\theta, I) = (\theta_n, I_n)$

$$\lim_{|n| \rightarrow \infty} \frac{\theta_n - \theta_0}{n} = \omega.$$

Σ_ω is called Aubry-Mather set

Bourgoin: nicest exposition of Aubry-Mather th.

Lecture 7 Aubry - Mather Theory.

I Results

Twist Maps
 $f: A \rightarrow A$
 $\theta \in \mathbb{T}, I \in \mathbb{R}$

Moser $\rightarrow$

Time periodic convex
Lagrangians $L(\theta, I, t)$
 $\theta \in \mathbb{T}, I \in \mathbb{R}, t \in S^1$

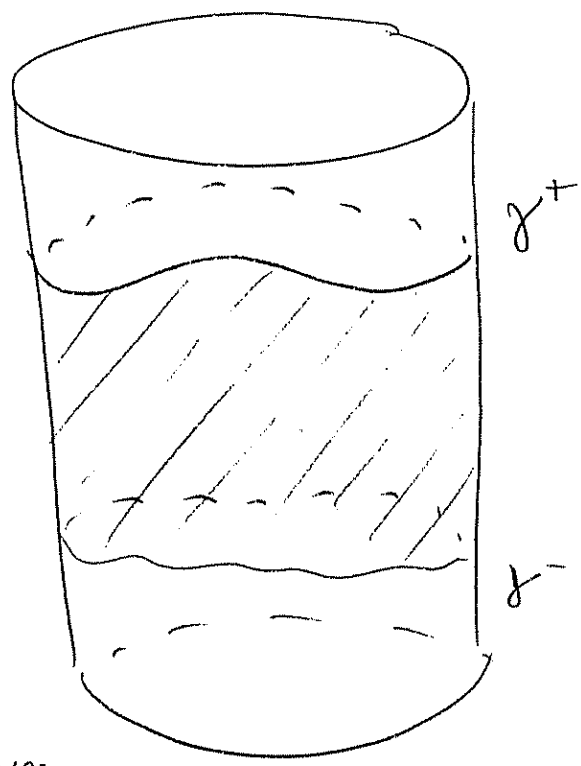


Stationary Configurations
of generating functions
 $\{\theta_n\}_{n \in \mathbb{Z}}$

Let $f: A \rightarrow A$ be a smooth
twist map

Suppose γ^+, γ^- invariant
curves / open, bounded
def. $U \subset A$ homeomorphic
to A , f -invariant,
 $\partial U = \gamma^+ \cup \gamma^-$

U is called Birkhoff Region
of Instability (BRI) if
no other noncontractible curves
in U .



$\hat{f}: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$ lift of f

$$\hat{f}(\theta+1, I) = \hat{f}(\theta, I) + (1, 0)$$

$$(\theta, I) \equiv f(\theta, I) \quad \theta \in [\theta, \theta+1]$$

point (θ, I) has rotation number ω
for $(\theta_n, I_n) = \hat{f}^n(\theta, I)$ we have

$$\lim_{n \rightarrow +\infty} \frac{\theta_n - \theta}{n} = \omega$$

$\gamma^\pm$ has rotation number $\omega^\pm$
By twist $\omega^- < \omega^+$

Aubry-Mather Theorem For any $\omega \in [\omega^-, \omega^+]$
there is an (action minimizing) invariant
set $\Sigma_\omega \subset A$ such that every point
 $(\theta, I) \in \Sigma_\omega$ has rotation number ω .
(weaker)
in terms of configurations, for every $\omega \in [\omega^-, \omega^+]$
there is a minimal configuration $\{\theta_n\}_{n \in \mathbb{Z}}$
such that $\lim_{|n| \rightarrow +\infty} \frac{\theta_n}{n} = \omega$

Structure Theorem A) If ω is irrational,
 then $\Sigma_\omega^{\text{rec}}$ is either a Denjoy-Cantor set
 or an invariant curve

$\Sigma_\omega^{\text{rec}}$ believed to coincide Σ_ω generically.

B)
 If ω is rational, then generically
 Σ_ω is a finite union of periodic
 points of period q if $\omega = p/q$

Graph Theorem. A) Let $\pi: S' \times \mathbb{R} \rightarrow S'$ be the natural
 projection. Then Σ_ω is a Lipschitz graph
 over $\pi \Sigma_\omega$ or $\pi^{-1}|_{\pi \Sigma_\omega}: \pi \Sigma_\omega \rightarrow \Sigma_\omega$ is Lipschitz

B) There is a map $\phi_\omega: S' \rightarrow S'$ such that
 it conjugates $f|_{\Sigma_\omega}$ w rigid rotation by ω , i.e.

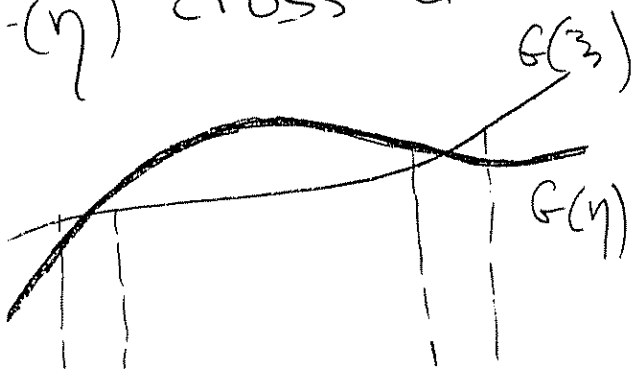
$$\begin{array}{ccc} \Sigma_\omega & \xrightarrow{f} & \Sigma_\omega \\ \phi_\omega \uparrow & & \uparrow \phi_\omega \\ S' & \longrightarrow & S' \\ \theta & \mapsto & \theta + \omega \pmod{1} \end{array} \quad \text{is commutative.}$$

There are two approaches to prove Aubry-Mather theorem.

The (geometric) is due to Aubry
the other (abstract minimization) is due to Mather

We shall discuss twist maps using Aubry's approach and later multidim'l case using Mather's approach.

Aubry Crossing Lemma If $\gamma, \eta \in \mathbb{R}^Z$ are minimal configurations, then their Aubry graphs $G(\gamma)$ and $G(\eta)$ cross at most once.



Proof: Assume there are two noninteger intersections in $[j, j+1]$

Define

$$\max(z, \eta) = \{\max(z_n, \eta_n)\}_n$$

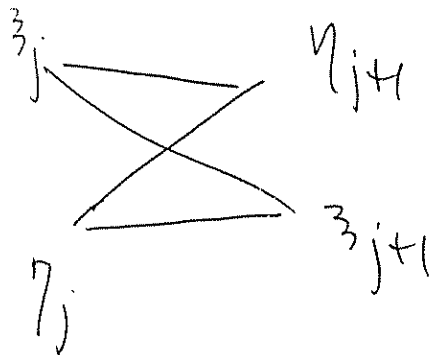
$$\min(z, \eta) = \{\min(z_n, \eta_n)\}_n$$

Twist condition $\exists \delta h \leq -\delta < 0$ implies

$$\begin{aligned} & \sum_{n=j}^k h(\max(z_n, \eta_n), \max(z_{n+1}, \eta_{n+1})) + \\ & \sum_{n=j}^k h(\min(z_n, \eta_n), \min(z_{n+1}, \eta_{n+1})) - \\ & \sum_{n=j}^k h(z_n, z_{n+1}) - \sum_{n=j}^k h(\eta_n, \eta_{n+1}) \end{aligned}$$

Proof of it boils down to

$$h(z_j, \eta_{j+1}) + h(\eta_j, z_{j+1}) < h(z_j, z_{j+1}) + h(\eta_j, \eta_{j+1})$$



and the same
for j replaced by k .

QED

Construction of minimal configurations w rational rotation number $\omega = P/Q$

translation operator

$\exists a, b \in \mathbb{Z}$

$$T_{a,b} \{\theta_n\}_{n \in \mathbb{Z}} = \{\theta'_n\}_{n \in \mathbb{Z}} \quad \text{s.t.} \quad \theta'_{n+b} = \theta_n + a$$

by periodicity^{any} translation $T_{a,b}$ of minimal configuration is minimal.
 $\exists \{\theta_n\}_{n \in \mathbb{Z}}$ is periodic and has rotation number P/Q , then $\theta_{n+Q} = \theta + P$ or

$$T_{P,Q} \{\theta_n\}_{n \in \mathbb{Z}} = \{\theta_n\}_{n \in \mathbb{Z}}$$

Fundamental Lemma (see e.g. Bangert)
 $\{\theta_n\}_{n \in \mathbb{Z}}$ is minimal and $p, q \in \mathbb{Z}$, then
every graphs $G(\theta)$ and $G(T_{p,q}\theta)$ are
either disjoint or coincide.

Lecture 8 Aubry-Mather theory and Peierls Barrier

Continuation of Lecture 7.

To construct minimal "rational" configuration we need two steps.

Step 1. Consider variational problem
Finite piece $\theta_0 = \theta, \theta_q = \theta + p$

Find $\min_{\theta_0, \theta_{q-1}} \sum_{h=0}^{q-1} h(\theta_h, \theta_{h+1})$.

Exercise 1. Prove that minimum should occur in a compact region (condition at $+\infty$ for h lecture 3 lemma 2). Then use the fact $\sum_{h=0}^{q-1} h(\theta_h, \theta_{h+1})$ is continuous.

Let $\theta_0^*, \dots, \theta_q^* = \theta_0^* + p$ realize minimum.

Step 2 Periodic extension of finite piece and minimality

Consider periodic continuation

$$\vartheta_{k+nq}^* = \vartheta_k^* + np \quad \text{for any } 0 \leq k < q, n \in \mathbb{Z}.$$

Lemma 1. $\{\vartheta_m^*\}_{m \in \mathbb{Z}}$ is minimal.

Proof. Suppose it is not minimal.

Then for some $j < k \in \mathbb{Z}$ we have

$\{\vartheta_m'\}_{m=j}^{k-1}$ such that $\vartheta_j' = \vartheta_j^*$, $\vartheta_k' = \vartheta_k^*$, and

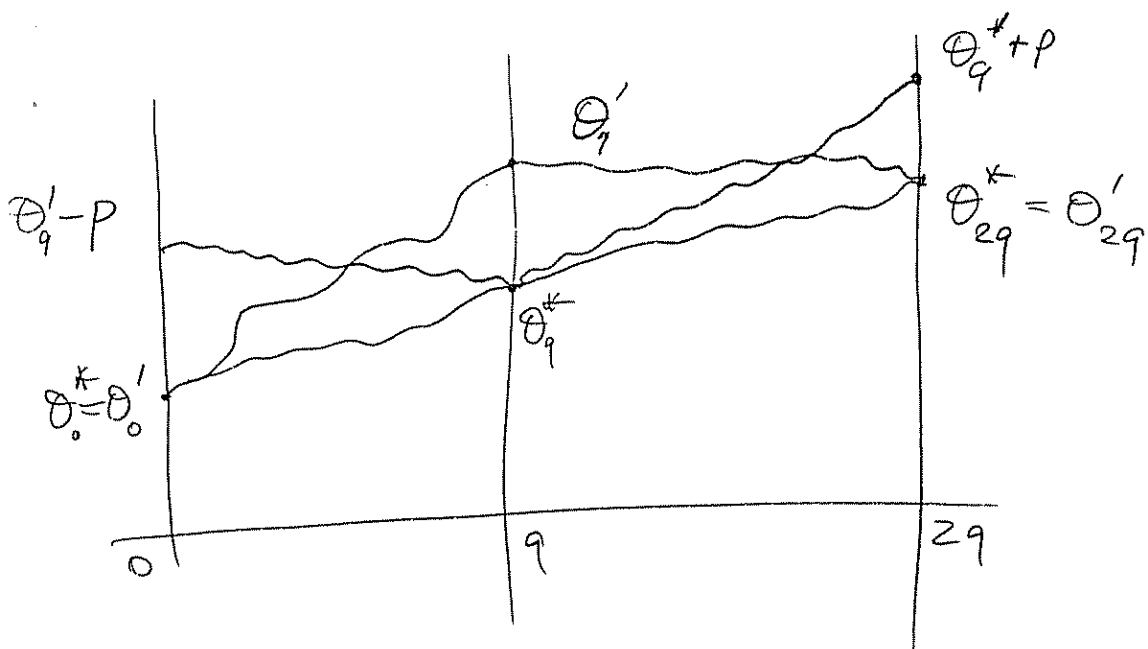
$$\sum_{n=j}^{k-1} h(\vartheta_n', \vartheta_{n+1}') < \sum_{n=j}^{k-1} h(\vartheta_n^*, \vartheta_{n+1}^*)$$

By periodicity assume $0 \leq j < q$ and

$1) q \leq k < s q$.

For simplicity we put $s=2$.

For general s the proof is the same



Translate $[0, q]$ part to $[q, 2q]$
and $[q, 2q]$ part to $[0, q]$

$$\tilde{\theta}_n = \theta'_{n+q} - p \quad 0 \leq n \leq q$$

$$\tilde{\theta}_n = \theta'_{n-q} + p \quad q \leq n \leq 2q$$

Translation does not change action.

If $\{\theta'_n\}_{n=0}^{2q}$ were minimal between any configuration with $\theta'_{2q} = \theta'_0 + 2p$, then

$\{\tilde{\theta}'_n\}_{n=0}^{2q}$ is minimal too.

Their Aubry graphs intersect twice.
Contradiction. This proves Lemma 1.

Exercise 2. Extend the proof above
to the case $s (= \text{number of periods}) > 2$.

Construction of minimal configurations
"irrational" rotation and
fundamental property of planar
minimizers

Exercise 3. Limit of minimizers
 $\{\theta_n^s\}_{n \in \mathbb{Z}} \rightarrow \{\theta_n\}_{n \in \mathbb{Z}}$ is minimizer

proof by contradiction.

let ω be irrational, $p_s/q_s \rightarrow \omega$.

let $\{\theta_n^s\}_{n \in \mathbb{Z}}$ $\theta_{k+m q_s} = \theta_k + m p_s$ be

periodic minimal $\theta_0^s \in [0, 1]$.

then there is a converging subsequence.

Exercise 4 Using Fundamental Lemma

Fundamental Theorem. Every minimizer $\{\theta_n\}$ has rotation number.

Proof is left as Exercise 5

Hint: Define $\omega = \min \{P_q : T_{P,q}\theta \text{ has Aubry graph above the one of } \theta\}$

Similarly

$\omega = \max \{P_q : T_{P,q}\theta \text{ --||-- below the one of } \theta\}$

III Peierl's Barrier (rational case)

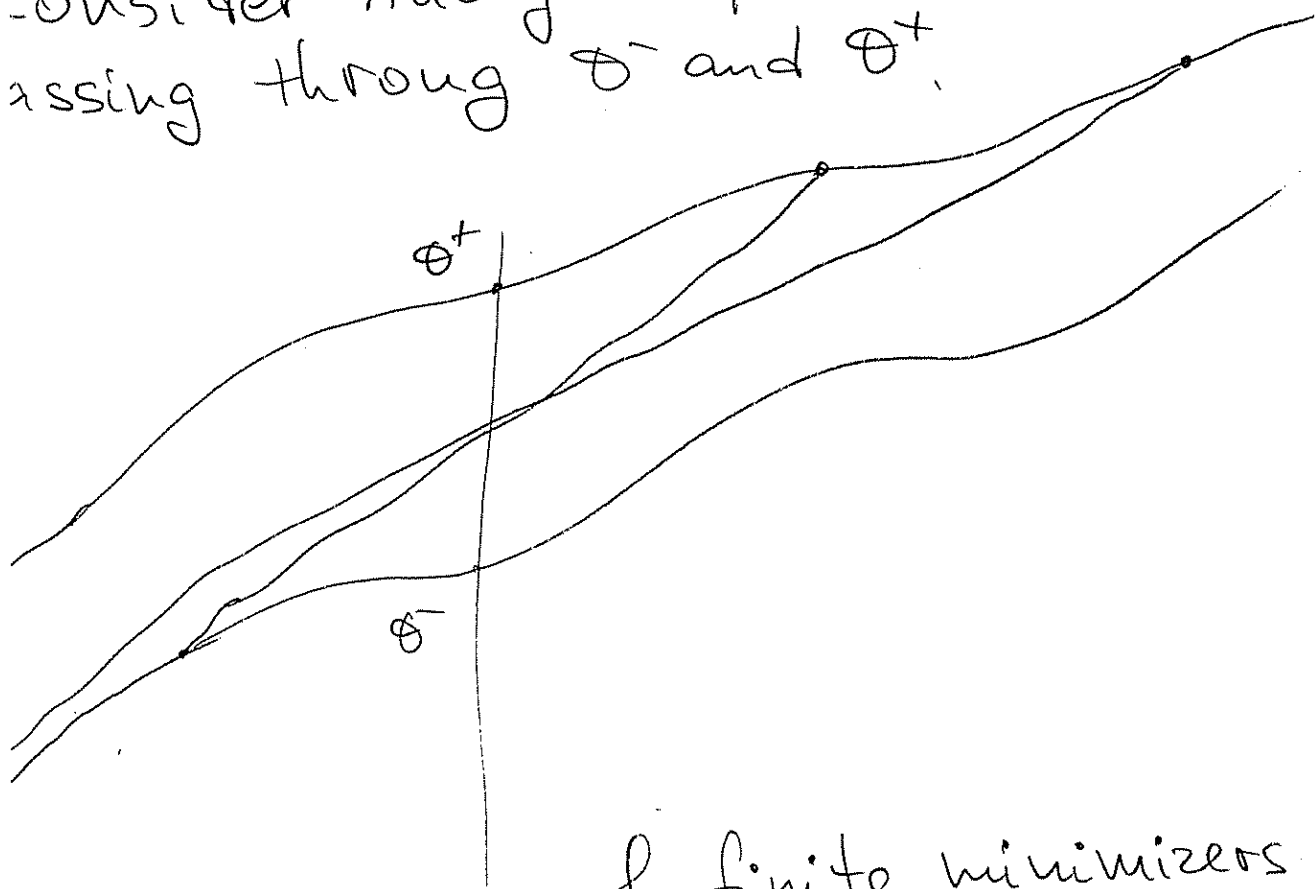
Let $\Sigma_{P/q}$ be the set of minimal periodic points. Suppose for simplicity $\Sigma_{P/q}$ - one periodic orbit.

By Graph Theorem $\pi \Sigma_{P/q}$ has exactly q points $\theta_0 \dots \theta_{q-1} \in S'$.

Pick two neighbors, denote them by θ^- and θ^+ so no $\theta_k \in (\theta^-, \theta^+)$

Construction of homoclinic orbit
 or class A geodesic, or $(P_q)^\pm$
 minimizer.

Consider Aubry Graphs $G(\theta^-)$ and $G(\theta^+)$
 passing through θ^- and θ^+ .



Consider sequence of finite minimizers

$$\min_{\theta_n = \theta_{-n}^-} \sum_{m=-n}^{n-1} h(\theta_m, \theta_{m+1})$$

Denote minimizing one $\{\theta_m^n\}_{m=-n}^n$

Slightly more general framework

Consider a configuration $\theta^a = \{\theta_n^a\}_{n \in \mathbb{Z}}$

• $\theta_0^a = a$

• $\{\theta_n^a\}_{n \geq 0}$ is minimal and $|\theta_n^a - \theta_n^+| \rightarrow 0$
as $n \rightarrow +\infty$

• $\{\theta_n^a\}_{n \leq 0}$ is minimal and $|\theta_n^a - \theta_n^-| \rightarrow 0$
as $n \rightarrow -\infty$

Define

$$P_{P/q}(a) = \lim_{n \rightarrow +\infty} \left(\sum_{i=-n}^{n-1} h(\theta_i^a, \theta_{i+1}^a) - \inf_{\substack{\theta_{\pm n} = \theta_{\pm n}^a}} \left\{ \sum_{i=-n}^{n-1} h(\theta_i, \theta_{i+1}) \right\} \right)$$

$P_{P/q}$ is called Peierl's Barrier

Lemma 3. The limit exists, continuous in a .

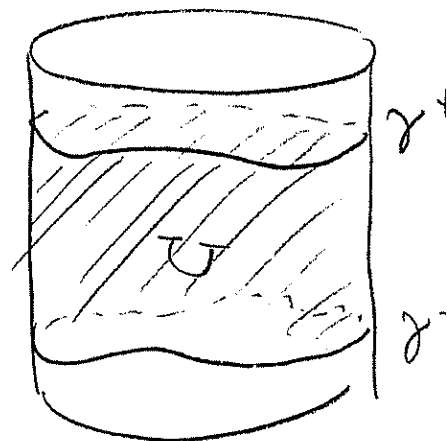
Lemma 4. At a point $\min_{\theta^- < a < \theta^+} P_{P/q}(a) = P_{P/q}(a^*)$

there is a trajectory with ω -limit θ^+
and α -limit θ^- .

Lecture 3 Mather Connecting Theorem and its variational proof using Barrier functions

Let $f: A \rightarrow C'$ twist map.

Suppose γ^-, γ^+ invariant curves of f and no invariant curves in between.



Region U between γ^- and γ^+ is BRI (Birkhoff Region of Instability).

Let $w^\pm = \rho(f|_{\gamma^\pm})$ - rotation number by twist w (MCT). Mather Connecting Theorem. For any pair of rotation numbers $w, w' \in [w^-, w^+]$ there is a (connecting) trajectory $\{f^n(\theta, I)\}_{n \in \mathbb{Z}}$ st.

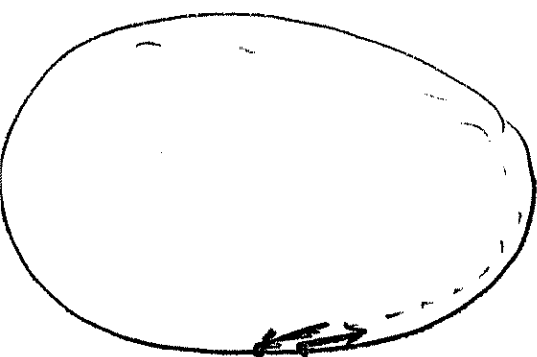
as $n \rightarrow +\infty$ $f^n(\theta, I) \rightarrow \Sigma_{w'}$

as $n \rightarrow -\infty$ $f^n(\theta, I) \rightarrow \Sigma_w$

Moreover, for any sequence of $\varepsilon_n > 0$, $w_n \in [w^-, w^+]$, and $T_n > 0$ there is a trajectory $\{f^n(\theta, I)\}_{n \in \mathbb{Z}}$ and moments of time $t_1 + T_1 < t_0 < t_0 + T_0 < t_1 < t_1 + T_1$ so that during $n \in [t_0, t_0 + T_0]$ we have $f^n(\theta, I)$ is

Application to Convex Billiards

(2)



$\{R_\theta\}_{\theta \in S^1}$ - length
parametrization of
the Boundary

$f: (\theta, \varphi) \rightarrow (\theta', \varphi')$ -
Billiard Map.

R -convex

$\partial R - C^2$ smooth

curvature of ∂R vanishes
at one point.

Lemma (Mather see Mather-Forni Lecture Notes in Math) 1587?

If curvature of ∂R vanishes at one point
the billiard map has no invariant curves except
the boundaries.

Corollary of Mather Connecting Theorem. Note $w^- = 0, w^+ = 1$.

There is a (connecting) trajectory $f^h(\theta, I)$
such that $f^h(\theta, I) \rightarrow \Sigma_0 = \{\varphi = 0\}$ as $h \rightarrow -\infty$

$f^h(\theta, I) \rightarrow \Sigma_1 = \{\varphi = \pi\}$ as $h \rightarrow +\infty$.

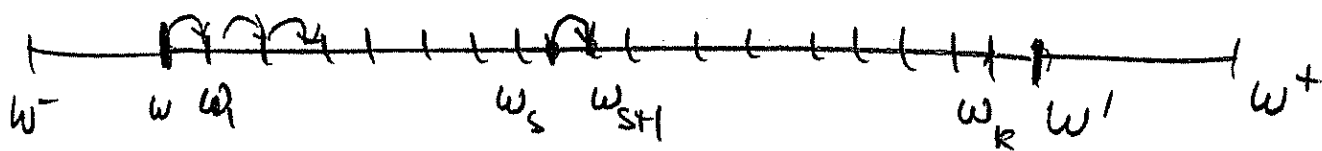
We shall sketch the proof of the first part of Mather's
Variational construction (barriers).

Idea: Divide the interval $[w, w']$ into
many small parts $w < w_1 < w_2 < \dots < w_k < w'$.

Suppose we can construct a (connecting)
trajectory for Σ_{w_s} and $\Sigma_{w_{s+1}}$, i.e.

$$f^n(\theta^s, I^s) \rightarrow \Sigma_{w_{s+1}} \quad \text{as } n \rightarrow +\infty$$

$$f^n(\theta^s, I^s) \rightarrow \Sigma_{w_s} \quad \text{as } n \rightarrow -\infty$$



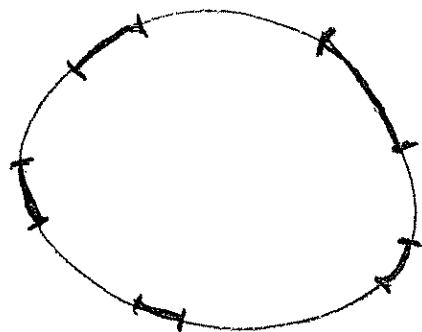
If we could "jump" from Σ_{w_s} to $\Sigma_{w_{s+1}}$
for any pair w_s and w_{s+1} sufficiently
close one to the other we can "connect"
any two Aubry-Mather sets Σ_w and $\Sigma_{w'}$

Dual definitions of Barriers

Direct Barriers (Continuation of last lecture)

Let $\omega \notin \mathbb{Q}$ irrational $\pi_1 \Sigma_\omega \subset \mathbb{T}^1$
 projected Aubry-Mather set. By Graph Theorem and Structure Theorem $\pi_1 \Sigma_\omega$ is a Denjoy-Cantor set. Pick a maximal interval (a^-, a^+)

in the complement of $\pi_1 \Sigma_\omega$.



$P_\omega: \mathbb{T}^1 \rightarrow \mathbb{R}$ is a Peierls' barrier function

defined as follows

$$P_\omega(a) = 0 \quad \text{if } a \in \pi_1 \Sigma_\omega.$$

if $a \notin \pi_1 \Sigma_\omega$ choose $(a^-, a^+) \ni a$

Pick a configuration $\{\theta_n^a\}_{n \in \mathbb{Z}}$

$$\theta_0^a = a$$

$$\{\theta_n^a\}_{n \geq 0} \text{ minimal}$$

$$|\theta_n^a - \theta_n^+| \rightarrow 0 \text{ as } n \rightarrow +\infty$$

where $\{\theta_n^\pm\}_{n \in \mathbb{Z}}$ minimal configuration
with $\theta_0^\pm = a^\pm$ and rotation number ω .

$$P_\omega(a) = \lim_{N \rightarrow +\infty} \left[\sum_{i=-N}^{N-1} h(\theta_i^a, \theta_{i+1}^a) - \inf_{\substack{\theta_{\pm N} = \theta_{\pm N}^a}} \sum_{i=-N}^{N-1} h(\theta_i, \theta_{i+1}) \right]$$

Lemma. The limit exists and continuous
in a .

Exercise 1. Prove it using proof in rational
case.

Similar definition can be given for
so-called (P_a^+) and (P_a^-) rotation number

II Dual Barrier or Barriers with control

Recall

$$h(\theta, \theta') = \inf_{\substack{\gamma(0) = \theta \\ \gamma(1) = \theta'}} \int_0^1 L(\gamma(t), \dot{\gamma}(t), t) dt$$

minimal
action
required
to get
from θ to
 θ'
in time 1

$$h_c(\theta, \theta') = \inf_{\gamma} \int_0^1 (L - \langle c, \dot{\gamma}(t) \rangle) dt =$$

Fix $\theta_0 = a$. Let's minimize

$$(*) \sum_{i=0}^{N-1} h_c(\theta_i^{a,N}, \theta_{i+1}^{a,N}) \quad \text{with } \theta_0^{a,N} = a \text{ fixed.}$$

Example. For $L(\theta, I, t) = \frac{I^2}{2}$

$$h(\theta, \theta') = \frac{(\theta' - \theta)^2}{2} \Rightarrow h_c(\theta, \theta') = \frac{(\theta' - \theta)^2}{2} - c(\theta' - \theta) = \frac{1}{2} (\theta' - \theta - c)^2 - \frac{c^2}{2}$$

$$\min \sum_{i=0}^{N-1} h_c(\theta_i^{a,N}, \theta_{i+1}^{a,N}) \geq -\frac{c^2}{2} \cdot N$$

and minimum is achieved at $\theta_i = a + ic$ $i=0 \dots N$

Exercise 2. Limit of minimizers is a minimizer.

Exercise 3. Minimizers of (*) is a minimizer between 0 and N

Define $\{\theta_n^{a,c}\}_{n \in \mathbb{Z}}$ limiting configuration

$$\theta_0^{a,c} = a$$

$\{\theta_n^{a,c}\}_{n \geq 0}$ a minimizer obtained as a limit $N \rightarrow +\infty$ of minimizers of (*)

$$\{\theta_n^{a,c}\}$$

$$\| \cdot \|$$

$$N \rightarrow +\infty$$

Lemma 3. There is a constant $\alpha(c)$ such that

$$P^c(a) = \lim_{N \rightarrow +\infty} \sum_{i=-N}^{N-1} \{h_c(\theta_i^{a,N}, \theta_{i+1}^{a,N}) - \alpha(c)\}$$

exists and is continuous where $\{\theta_n^{a,N}\}_{n=0}^N$ minimizes (*)

$\{\theta_n^{a,N}\}_{n=0}^{-N}$ minimizes (*) with N replaced by $-N$.

Remark. $\alpha(c)$ - ergodic average.

Example (continuation of the above)

$$h_c(\theta, \theta') = \frac{1}{2} (\theta' - \theta - c)^2 - \frac{c^2}{2}$$

$$\text{then } \alpha(c) = -\frac{c^2}{2}.$$

Exercise. Check that if $P^c(a) = \min_{\theta \in S'} P^c(a)$

then the limiting configuration $\{\theta_n^{a,c}\}_{n \in \mathbb{Z}}$ is stationary and, therefore, represents a trajectory of a twist map under consideration.

$\{\theta_n^{a,c}\}_{n \in \mathbb{Z}}$ - minimize

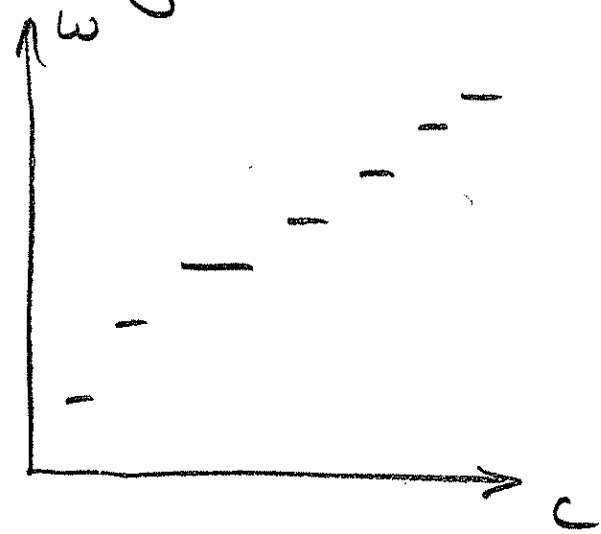
Idea of connecting orbits

(8)

Since every minimal configuration has rotation number, there is a relation between c and ω .

Roughly, for each ω there is c such that c -minimizers have rotation number ω .

Fubry Devil's Staircase For every rational $\omega = p/q$ there is an interval $[c^-, c^+]$ such that every $c \in [c^-, c^+]$ has every c -minimizers with rotation number ω .



To change
rotation
number ω

Change
control (parameter)
 c

Lemma. Let $[b^-, b^+] \subset (a^-, a^+)$

be an interval, where (a^-, a^+) is the maximal interval in the complement of projected Aubry-Mather set $\pi_1 \Sigma_\omega$.

Let c be control so that c -minimizers have rotation number ω . If $[b^-, b^+]$ properly chosen then

$$\sum_{i \geq 0} \{h_c(\theta_i^b, \theta_{i+1}^b) - \alpha(c)\} +$$

$$\sum \{h_{c'}(\theta_i^b, \theta_{i+1}^b) - \alpha(c')\} = \mathcal{P}_{c,c'}(b)$$

has minimum in interior of $[b^-, b^+]$

$\{\theta_n^b\}_{n \geq 0}$ c -minimal

$\{\theta_n^{b'}\}_{n \leq 0}$ c' -minimal.

$\mathcal{D}(b)$ - so called restricted barrier.

(10)

Since $P_{c,1}(b)$ has a minimum
 $\{\theta_n^b\}_{n \in \mathbb{Z}}$ is a stationary configuration
and, therefore, a trajectory (local
minimal).

Slightly more general framework.

Consider a configuration $\theta^a = \{\theta_n^a\}_{n \in \mathbb{Z}}$

- $\theta_0^a = a$

- $\{\theta_n^a\}_{n \geq 0}$ is minimal and $|\theta_n^a - \theta_n^+| \rightarrow 0$ as $n \rightarrow +\infty$

- $\{\theta_n^a\}_{n \leq 0}$ is minimal and $|\theta_n^a - \theta_n^-| \rightarrow 0$ as $n \rightarrow -\infty$

Define

$$P_{P/q}(a) = \lim_{h \rightarrow +\infty} \left(\sum_{i=-h}^{h-1} h(\theta_i^a, \theta_{i+h}^a) - \inf_{\theta_{\pm h} = \theta_{\pm h}^a} \left\{ \sum_{i=h}^{h-1} h(\theta_i, \theta_{i+h}) \right\} \right)$$

$P_{P/q}$ is called Peierl's Barrier

Lemma 3. The limit exists, continuous in a

Lemma 4. At a point a^* : $\min_{\theta^- < a < \theta^+} P_{P/q}(a) = P_{P/q}(a^*)$

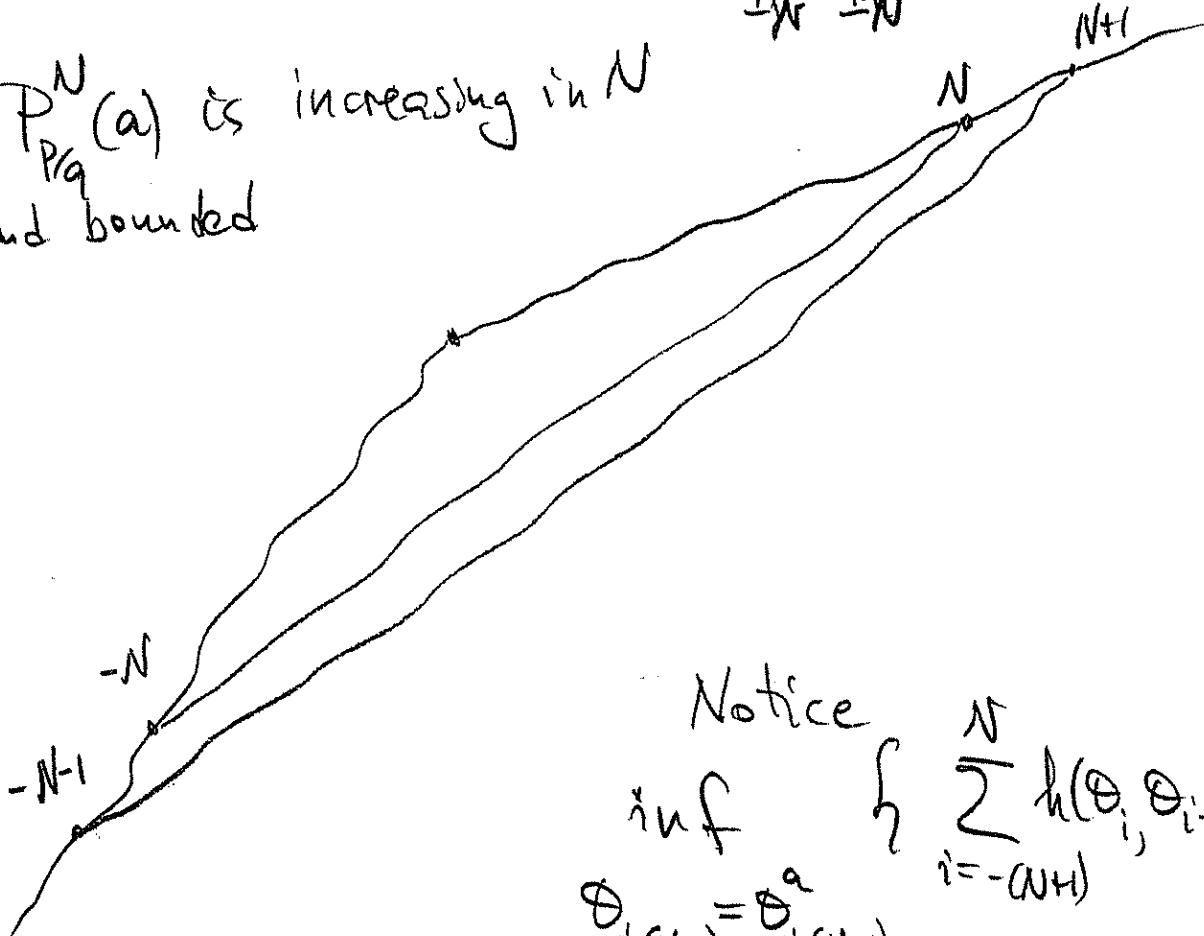
there is a trajectory with ω -limit θ^+ and α -limit θ^- .

Proof of Lemma 3.

Denote

$$\Phi_{P/q}^N(a) = \sum_{i=-N}^{N-1} h(\theta_i^a, \theta_{i+1}^a) - \inf_{\substack{\theta_{\pm N} = \theta_{\pm N}^a}} \left\{ \sum_{i=-N}^{N-1} h(\theta_i, \theta_{i+1}) \right\}$$

$\Phi_{P/q}^N(a)$ is increasing in N
and bounded



Notice

$$\inf_{\substack{\theta_{\pm(N+1)} = \theta_{\pm(N+1)}^a}} \left\{ \sum_{i=-N}^N h(\theta_i, \theta_{i+1}) \right\} \leq$$

$$h(\theta_{-N-1}^a, \theta_{-N}^a) + h(\theta_N^a, \theta_{N+1}^a) + \inf_{\substack{\theta_{\pm N} = \theta_{\pm N}^a}} \left\{ \sum_{i=-N}^{N-1} h(\theta_i, \theta_{i+1}) \right\}$$

indeed, the first sum has less restrictions on $\{\theta_i\}$
and, therefore, realizes at least as small as
the second sum value.

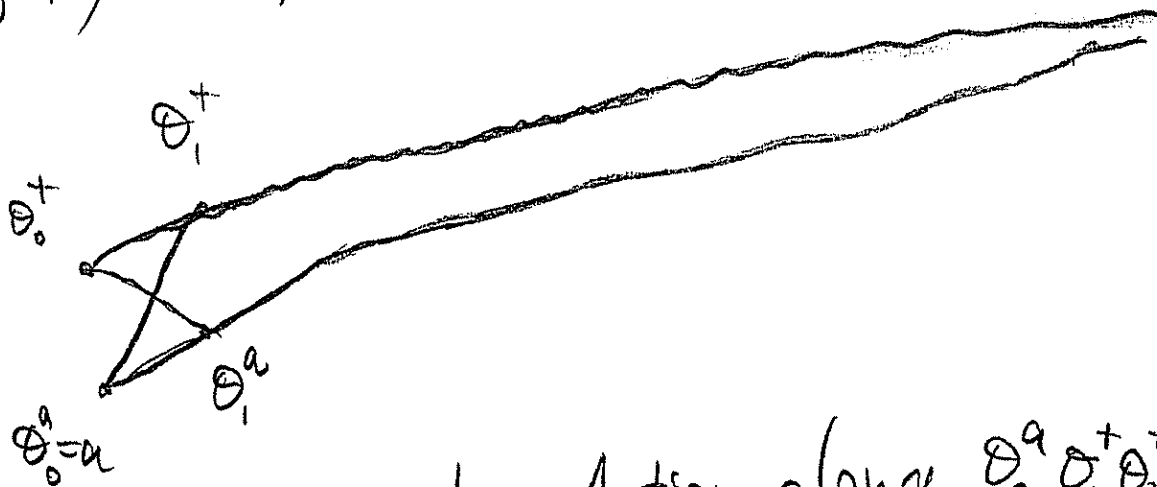
To see that $P_{P/q}^N(a)$ is bounded
 Consider the following configuration

$$\Theta_n^* = \Theta_n^+ \quad n \geq 0$$

$$\Theta_n^* = \Theta_n^- \quad n < 0.$$

$$\left| \sum_{i \geq 0} h(\Theta_i^a, \Theta_{i+1}^a) - \sum_{i \geq 0} h(\Theta_i^+, \Theta_{i+1}^+) \right| \leq$$

$$h(\Theta_0^+, \Theta_1^a) + h(\Theta_0^a, \Theta_1^+) - h(\Theta_0^+, \Theta_1^+) - h(\Theta_0^a, \Theta_1^a)$$



Comparison Arguments. Action along $\Theta_0^a \Theta_1^+ \Theta_2^+$
 is not smaller than along $\Theta_0^a \Theta_1^a \Theta_2^a$ by minimality
 (have in mind that $|\Theta_n^a - \Theta_n^+| \rightarrow 0$ as $n \rightarrow +\infty$)
 Action along $\Theta_0^+ \Theta_1^a \Theta_2^a$ is not smaller than along
 $\Theta_0^+ \Theta_1^+ \Theta_2^+$ by minimality of $\{\Theta_n^+\}_{n \in \mathbb{Z}}$
 Combine these actions.

