

FREQUENCY ANALYSIS AND THE TROJAN ASTEROIDS PROBLEM

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Sieben Heldenköpfe vor 1800 aus Homerwerk. G. Morghen nach J. H. W. Tischbein. Photo © Maicar Förlag - GML
Menelaus, Paris, Diomedes, Odysseus, Nestor, Achilles and Agamemnon.

Recent Trends in Nonlinear Science

Castelló, 24–28 January 2005

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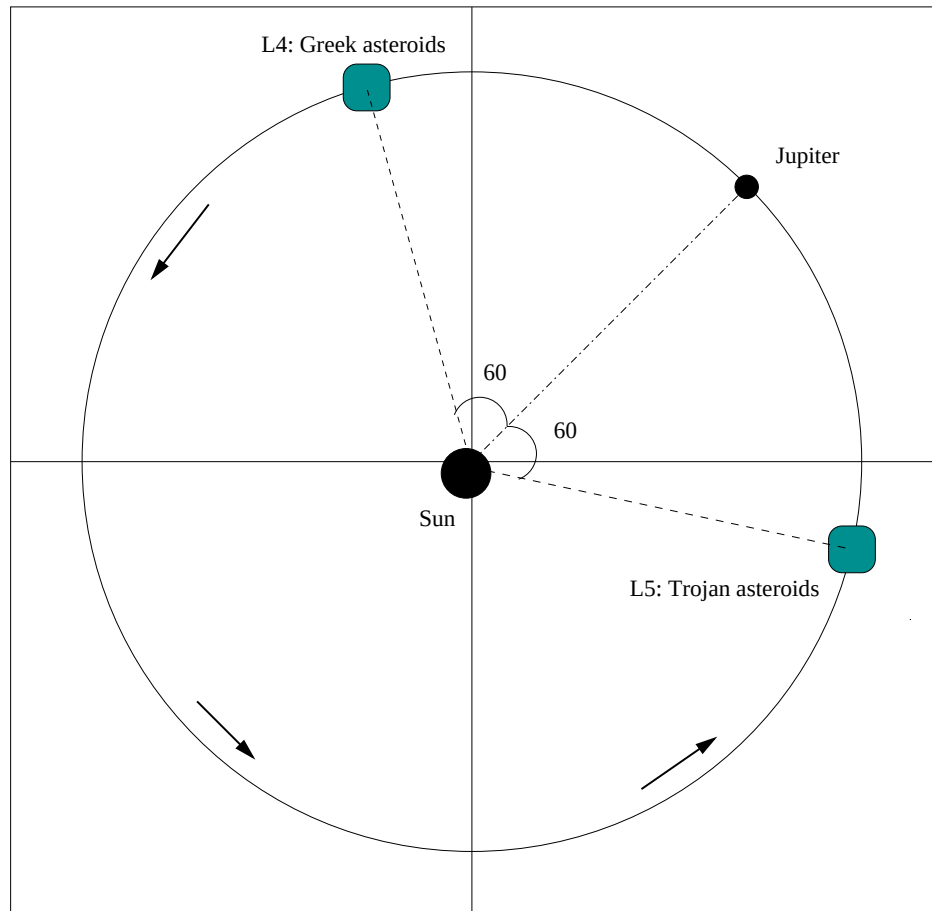
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Overview

- Introduction.
 - What are the Trojan asteroids?
 - Restricted Three Body Problem.
- Frequency Analysis.
 - Close-to-integrable systems and KAM tori.
 - Numerical approximation of the fundamental frequencies.
- Global dynamics of the Trojans phase space.
 - The Sun–Jupiter–Saturn model.
- Computer Lab:
 - RTBP equations.
 - Numerical integrator of ODE: Taylor method.
 - FURIANE: Frequency Analysis program.
 - Plotting the results: PGPLOT.

What are the Trojan asteroids? (I)



L_4 :

Achilles



L_5 :

Patroclus



Hector



Aeneas



Nestor



Memnon



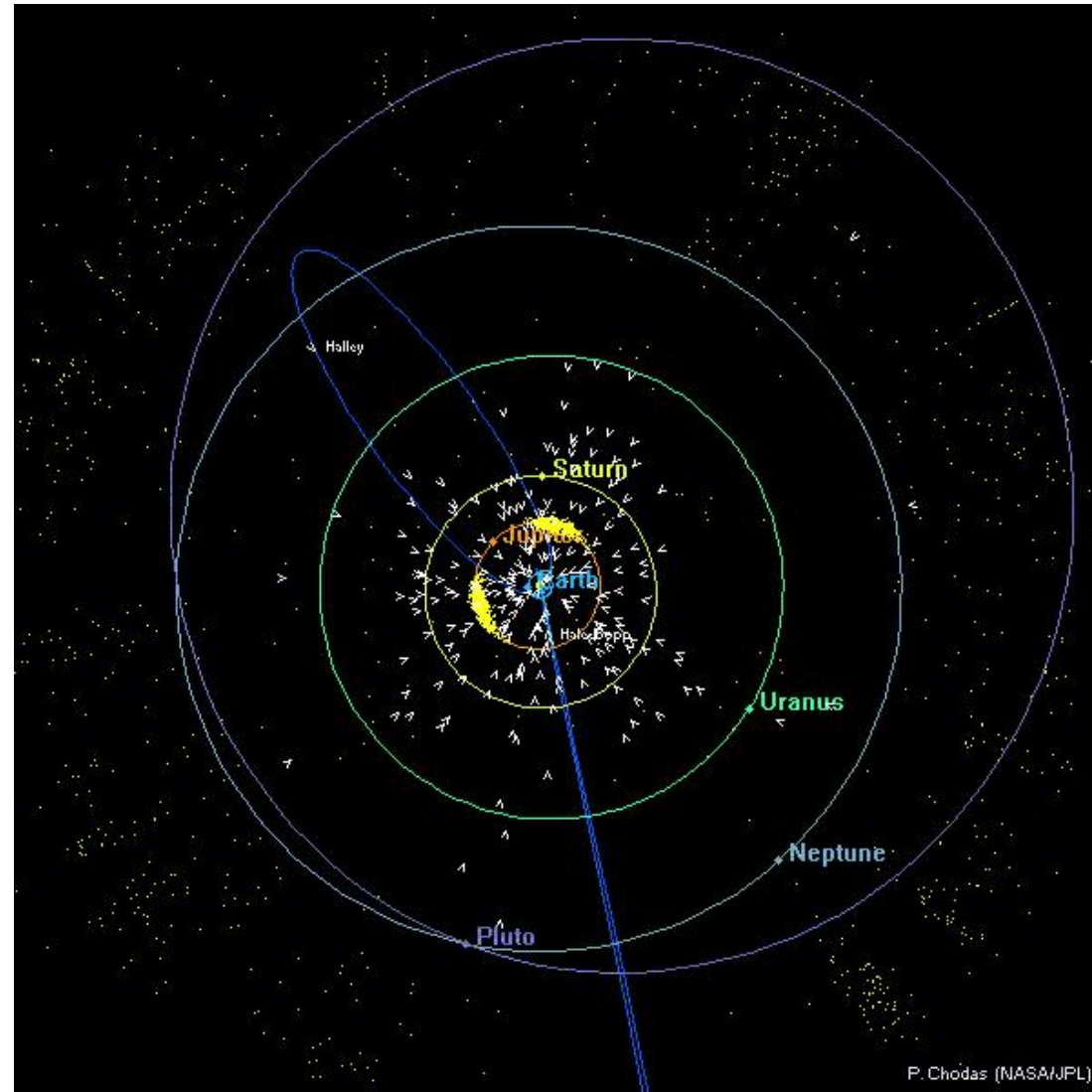
Agamemnon



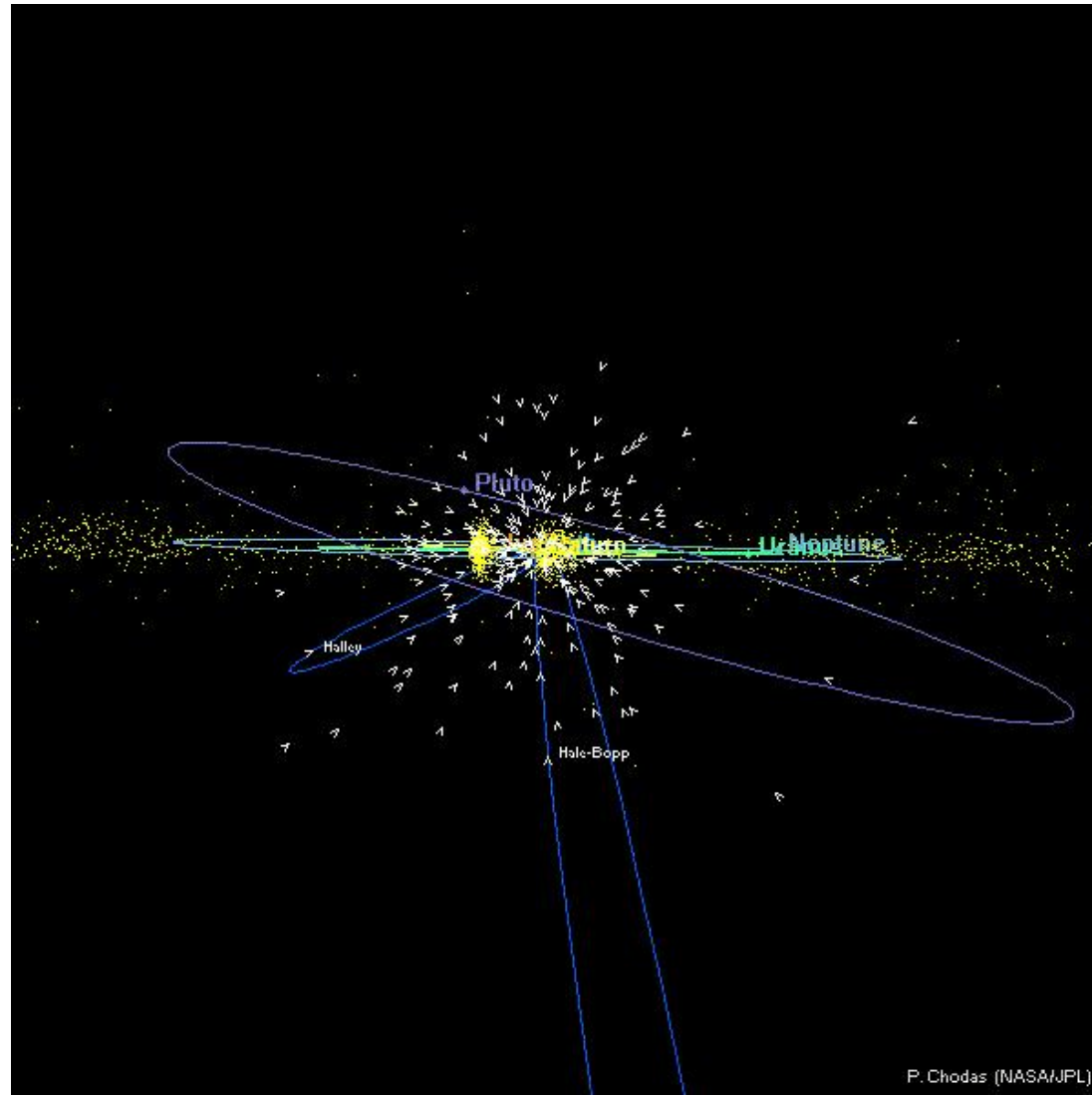
Paris



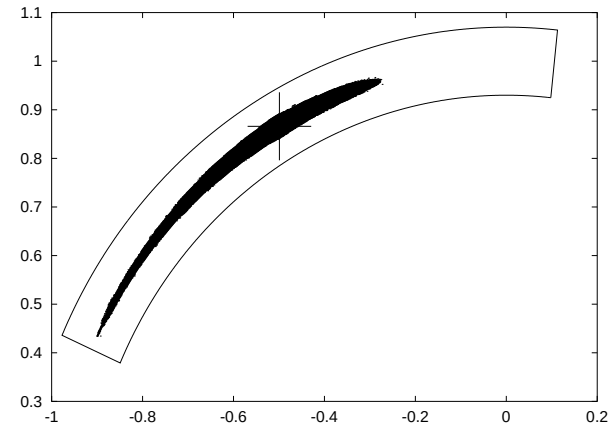
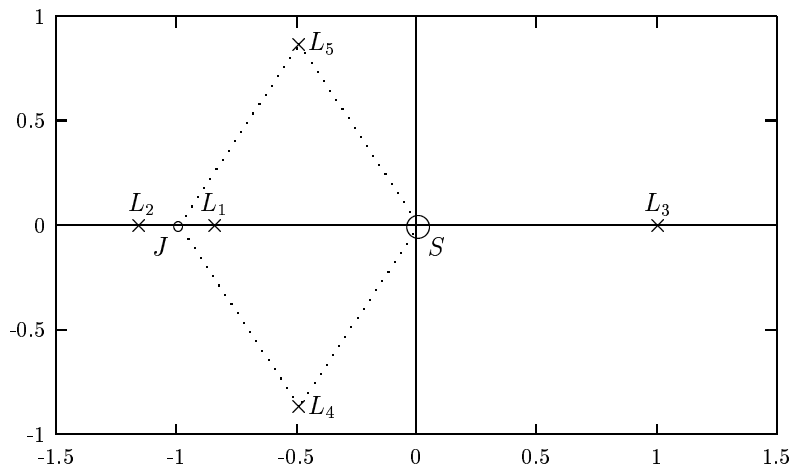
What are the Trojan asteroids? (II)



What are the Trojan asteroids? (III)

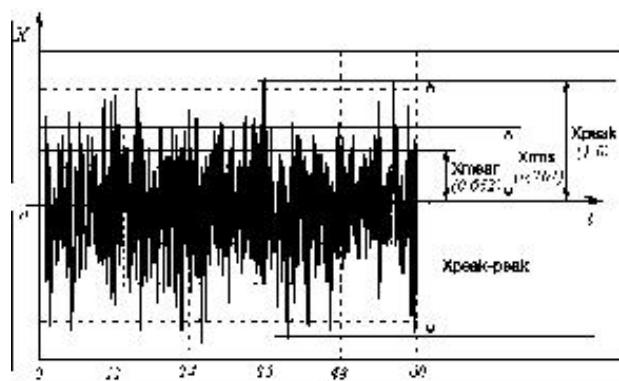


The Restricted Three Body Problem



$$H = \frac{1}{2} (p_x^2 + p_y^2 + p_z^2) + (yp_x - xp_y) - \frac{1-\mu}{\sqrt{(x-\mu)^2 + y^2 + z^2}} - \frac{\mu}{\sqrt{(x-\mu+1)^2 + y^2 + z^2}}$$

FREQUENCY ANALYSIS



Frequency Analysis: Close-to-integrable systems and KAM tori (I)

$$H(I, \theta) = H_0(I) + \varepsilon H_1(I, \theta), \quad (I, \theta) \in B^n \times \mathbb{T}^n, \quad B^n \subset \mathbb{R}^n$$

- For $\varepsilon = 0$:

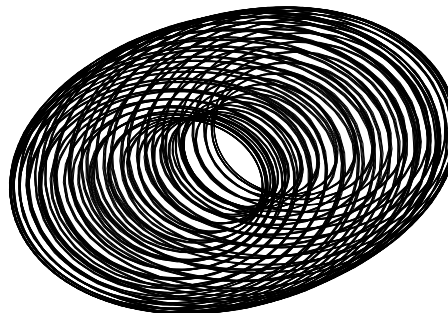
$$\dot{I}_j = 0, \quad \dot{\theta}_j = \frac{\partial H_0}{\partial I_j} \equiv \omega_j(I); \quad j = 1, \dots, n$$

If $\det \left(\frac{\partial \omega(I)}{\partial I} \right) \neq 0$ (non-degenerate system), the *frequency map*

$$\begin{aligned} F : B^n &\longrightarrow \mathbb{R}^n \\ I &\longmapsto \omega(I) \end{aligned}$$

is a diffeomorphism on its image.

Motion in phase space takes place on *invariant tori*.



Frequency Analysis: Close-to-integrable systems and KAM tori (II)

$$H(I, \theta) = H_0(I) + \varepsilon H_1(I, \theta), \quad (I, \theta) \in B^n \times \mathbb{T}^n, \quad B^n \subset \mathbb{R}^n$$

- For $\varepsilon \neq 0$: The *KAM theorem* asserts that, for sufficiently small ε , there exists a Cantor set $\Omega_\varepsilon \ni \omega$, for which the invariant tori of the unperturbed system persist: *KAM tori*.

For $\omega \in \Omega_\varepsilon$, the solution lies on a torus and can be parametrized, for instance, by a Fourier series:

$$z^j(t) = z_0^j e^{i\omega_j t} + \sum_k a_k^j(\omega) e^{i\langle k, \omega \rangle t}, \quad j = 1, \dots, n$$

If we fix $\theta \in \mathbb{T}^n$ to some value $\theta = \theta_0$, we obtain the frequency map on B^n :

$$F_{\theta_0} : B^n \longrightarrow \mathbb{R}^n$$

Frequency Analysis: Quasi-periodic approximation (I)

Let

$$f(t) = \sum_{k \in \mathbb{Z}^m} a_k e^{i \langle k, \omega \rangle t}, \quad a_k \in \mathbb{C},$$

be a quasi-periodic function for which we know a table of equidistant values in the time span $[0, T]$: $f(t_k)$, $t_k \in [0, T]$, $k = 1, \dots, N$.

The frequency analysis algorithm provides (numerically) the values of the:

- Frequencies: $\tilde{\omega}_k$
- Amplitudes: $\tilde{a}_k$

of a function

$$\tilde{f}(t) = \sum \tilde{a}_k e^{i \tilde{\omega}_k t}$$

that approximates $f(t)$ in $[0, T]$.

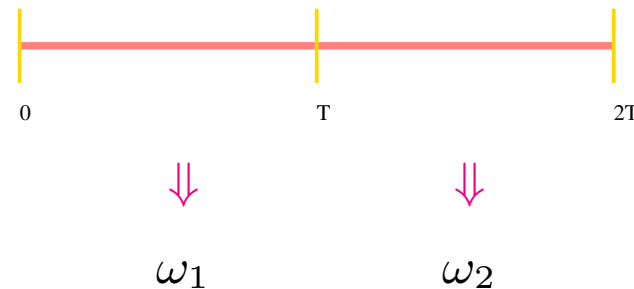
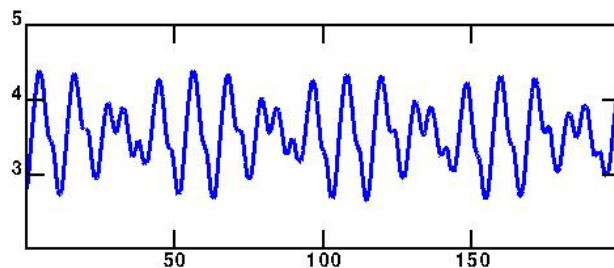
Frequency Analysis: Quasi-periodic approximation (II)

Phase space of a close-to-integrable system:



Quasi-Periodic Motions

Frequency decomposition of particular trajectories



Diffusion index:

$$\delta = \left| 1 - \frac{\omega_2}{\omega_1} \right|$$

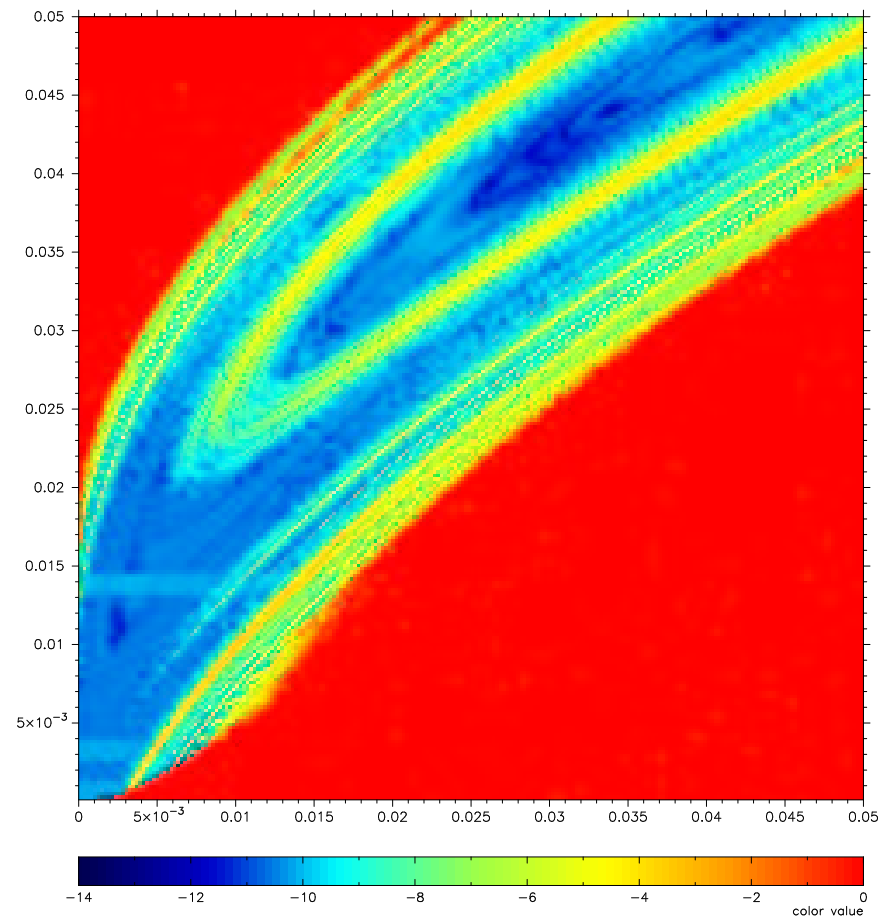
- δ gives an estimation of the “chaoticity” of the particular orbit.
- If the trajectory associated to an initial condition is quasi-periodic, δ is zero.

Frequency Analysis: Example I

RTBP near L_5 : $H(I, \varphi) = \nu_1 I_1 + \nu_2 I_2 + \sum_{l=3}^{+\infty} h_l(I, \varphi)$

Diffusion index:

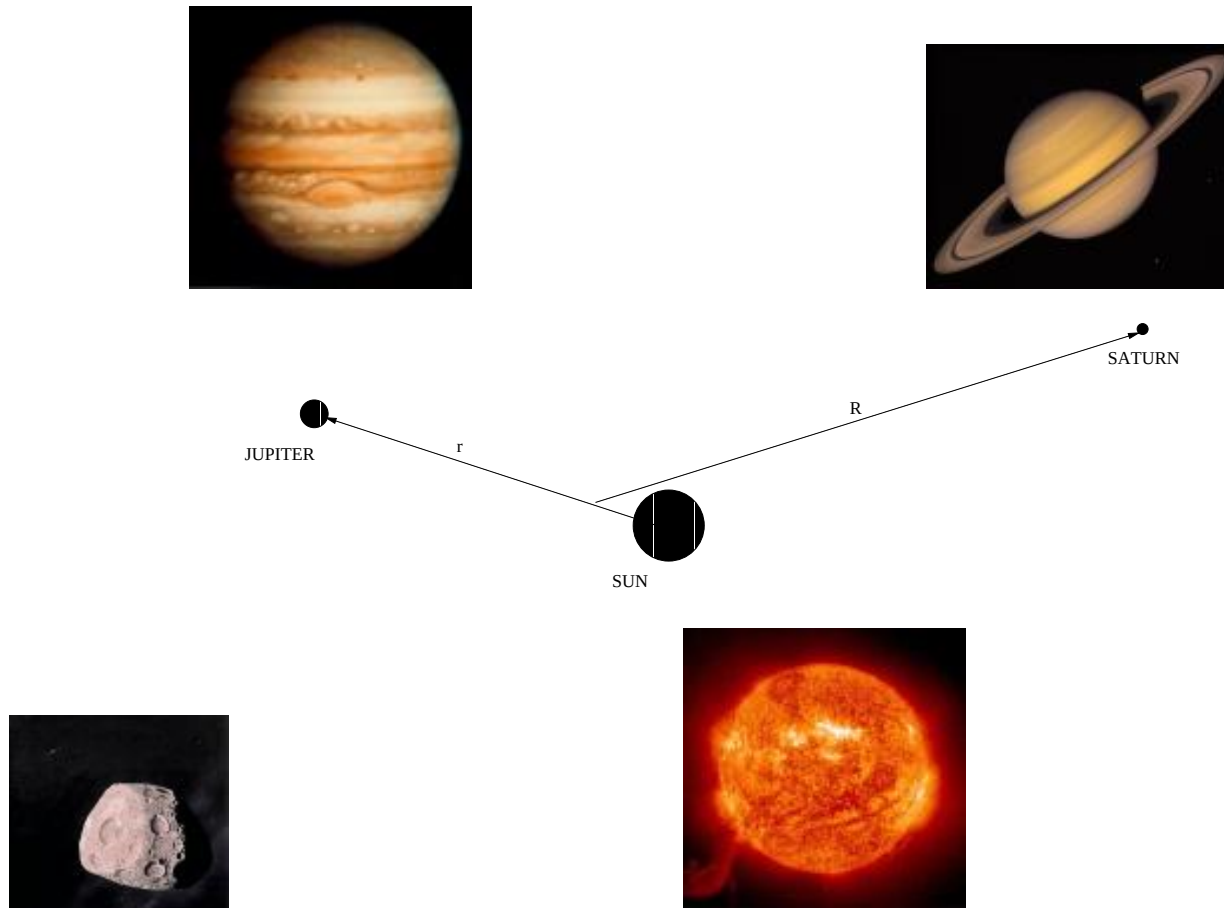
$$\sigma = \log \left| 1 - \frac{\omega_2}{\omega_1} \right|$$



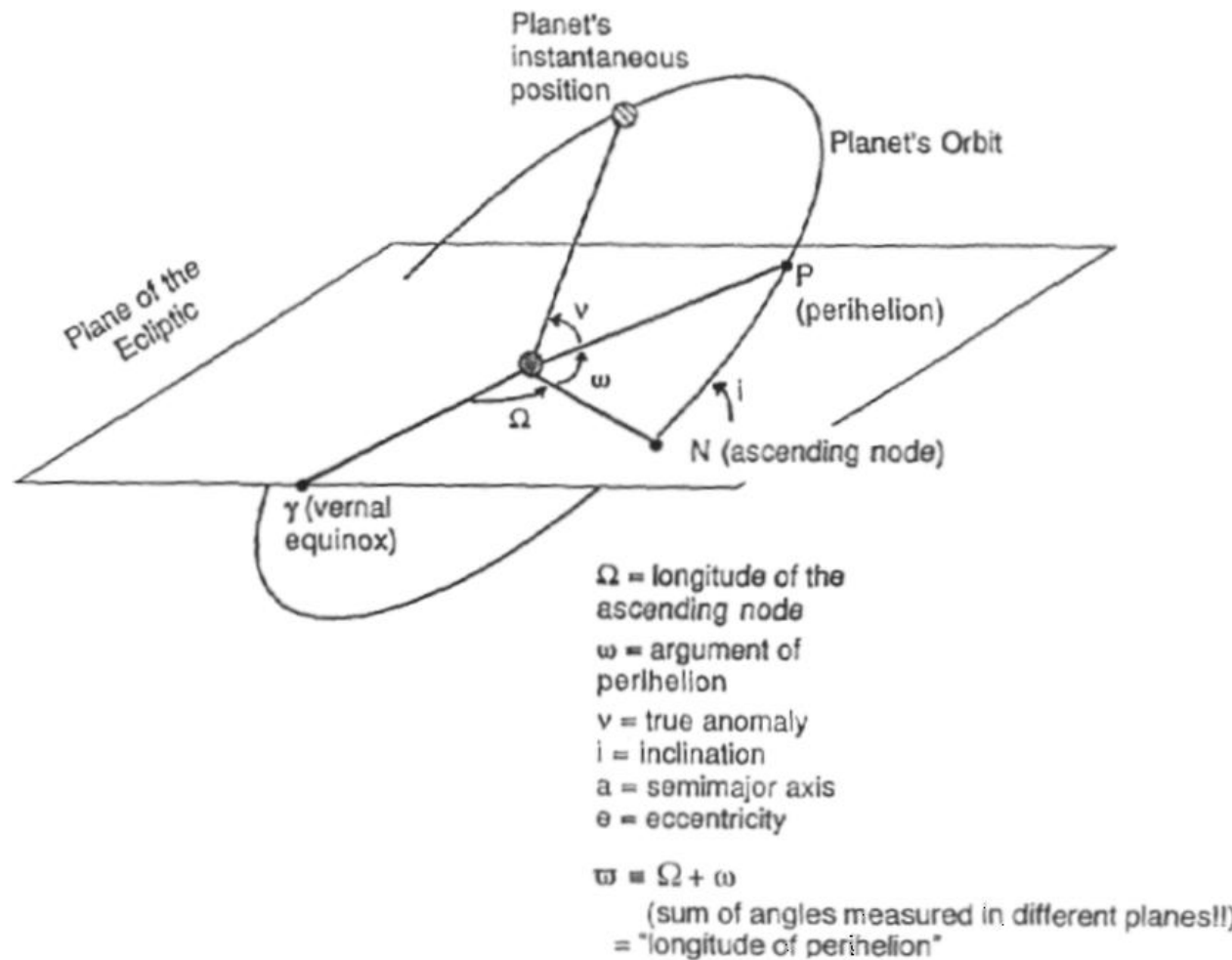
GLOBAL DYNAMICS OF THE TROJANS PHASE SPACE



Trojan Global dynamics: Sun–Jupiter–Saturn Model



Trojan Global dynamics: Orbital Elements



Trojan Global dynamics: Basic Frequencies of the planets (I)

Time window: 10 Million years.

Motion of the planets can be considered quasi-periodic.

Apply frequency analysis to:

$$\begin{aligned}\alpha_p(t) &= a_p(t) \exp(i\lambda_p(t)), \\ \beta_p(t) &= e_p(t) \exp(i\varpi_p(t)), \\ \gamma_p(t) &= \sin\left(\frac{I_p(t)}{2}\right) \exp(i\Omega_p(t)),\end{aligned}$$

where

- $p = 5 \longrightarrow$ Jupiter $p = 6 \longrightarrow$ Saturn.
- $\lambda = \varpi + M$: mean longitude (M : mean anomaly).
- $\varpi = \Omega + \omega$: longitude of the perihelion.

Trojan Global dynamics: Basic Frequencies of the planets (II)

	frequency (″/yr)	Period (yr)
n_5	109254.63165	11.8622
n_6	43995.34975	29.4577
g_5	4.02760	321780
g_6	28.00657	46274.9
s_6	−26.03912	49771.3

- $\{n_5, n_6\}$: Proper Mean Motion of the planets.
- $\{g_5, g_6, s_6\}$: Secular frequencies.

Trojan Global dynamics: Frequencies of the asteroids (I)

Time window: 5 Million years.

Apply frequency analysis to:

$$\begin{aligned}\alpha_j(t) &= a_j(t) \exp(i(\lambda_j(t) - \lambda_5(t))), \\ \beta_j(t) &= e_j(t) \exp(i\varpi_j(t)), \\ \gamma_j(t) &= \sin\left(\frac{I_j(t)}{2}\right) \exp(i\Omega_j(t)).\end{aligned}$$

FREQUENCY MAP:

$$\begin{aligned}F_{\theta_0} &: (a, e, I) \mapsto (\nu, g, s), \\ \theta_0 &= (\lambda_0, \varpi_0, \Omega_0)\end{aligned}$$

Trojan Global dynamics: Frequencies of the asteroids (II)

- Fixed phase:

$$\lambda_0 = \lambda_5 \mp \frac{\pi}{3}, (L_4, L_5)$$

$$\varpi_0 = \varpi_5 \mp \frac{\pi}{3}, (L_4, L_5)$$

$$\Omega_0 = \Omega_5.$$

- Fixed inclination, $I = I_5$:

FREQUENCY MAP:

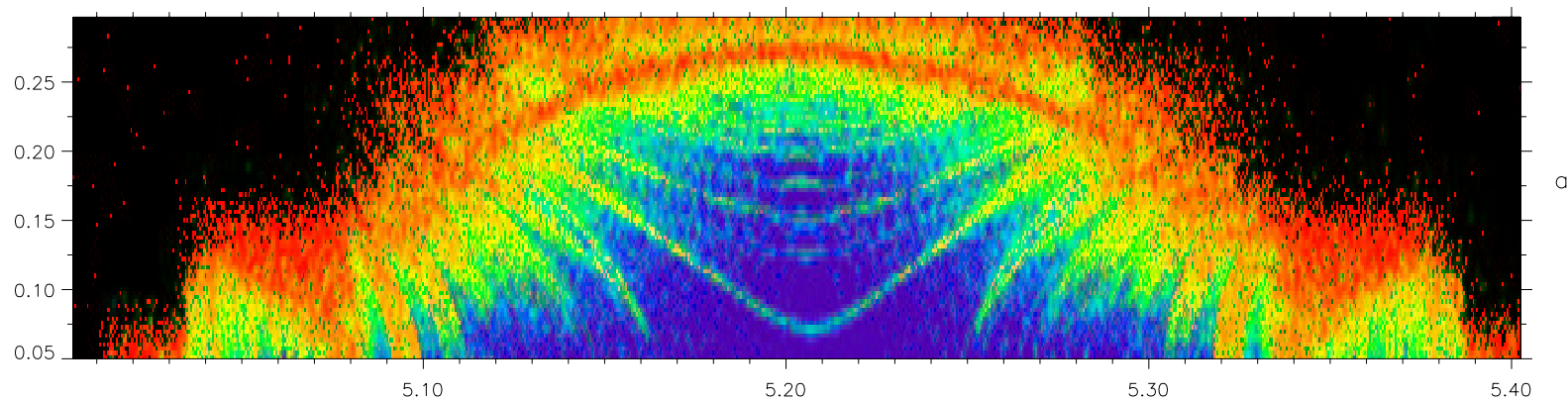
$$F_{(I_5, \theta_0)} : (a, e) \mapsto (\nu, g, s).$$

Trojan Global dynamics: Diffusion

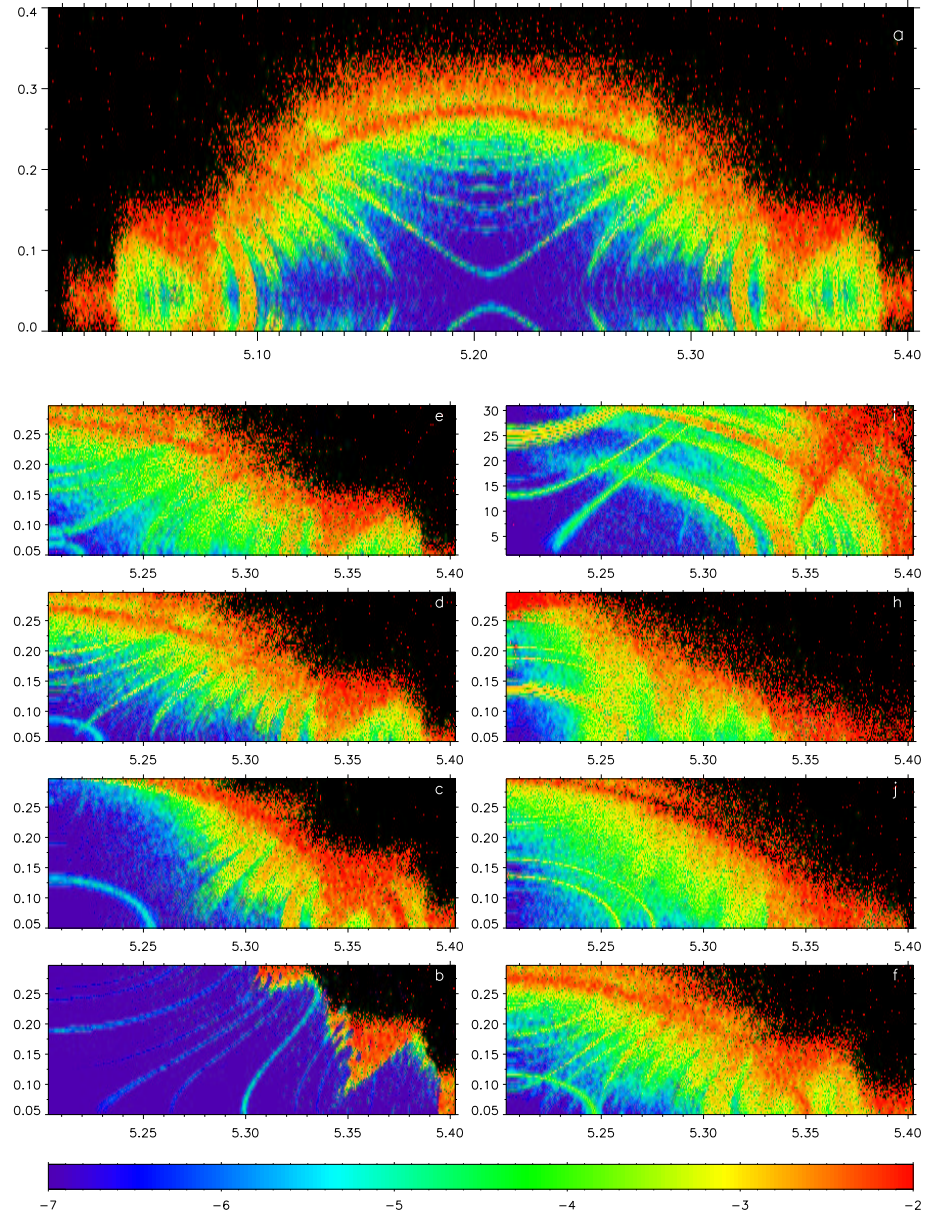
Frequency of libration:	Diffusion index:
$\nu \rightsquigarrow [0, T]$	$\delta\nu = 1 - \frac{\nu'}{\nu}$
$\nu' \rightsquigarrow [T, 2T]$	

Color code:

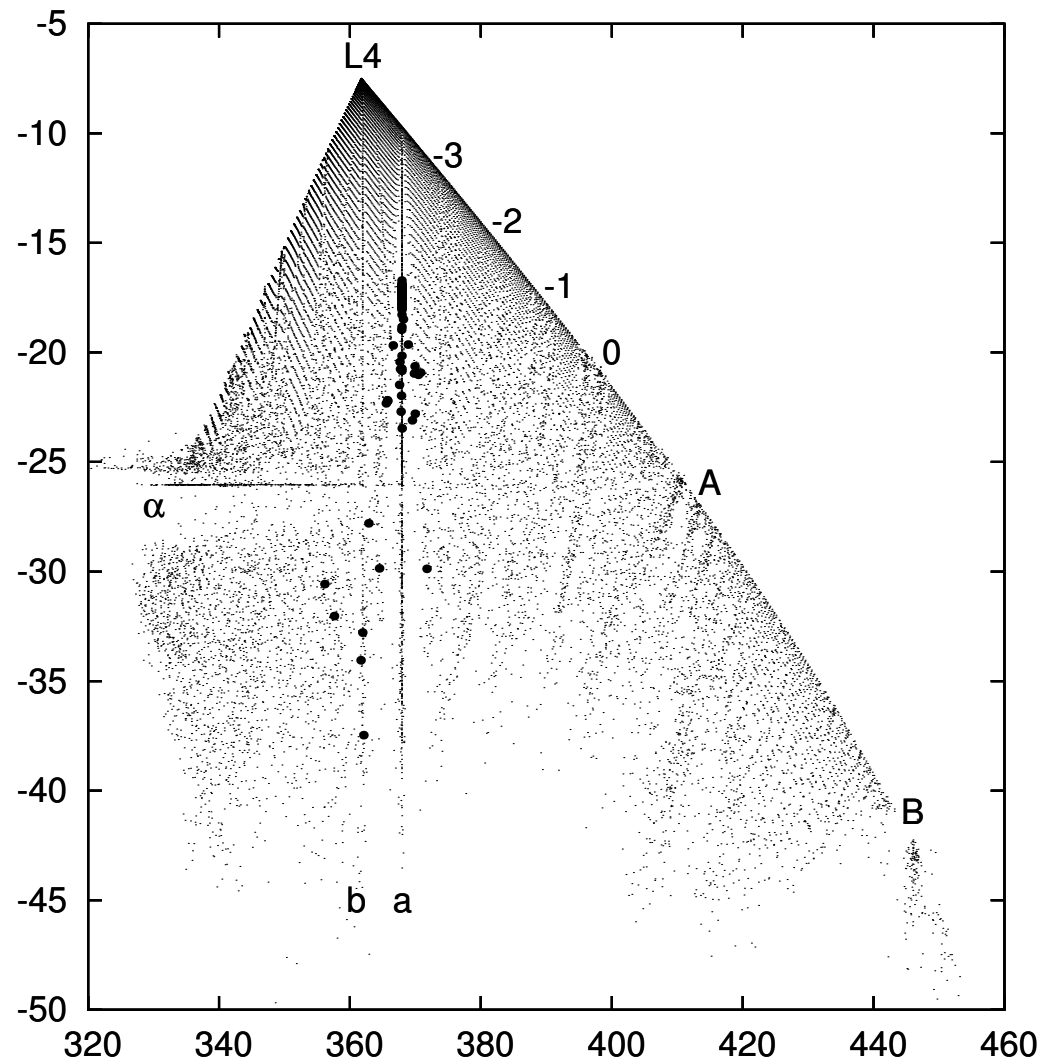
- $\delta\nu > 10^{-2}$: Strongly irregular motion. [RED]
- $\delta\nu < 10^{-7}$: Close-to quasi-periodic motion. [BLUE]
- [BLACK] : Escaping orbits.



GLOBAL DYNAMICS



Trojan Global dynamics: Frequency Space (g, s)



Global dynamics: Observed Trojans (I)

How to locate the observed Trojans in the global dynamical pictures?

- FREQUENCY MAP: $F_{\theta_0} : (a, e, I) \mapsto (\nu, g, s)$.
- PHASE PROBLEM: For every observed Trojan $\theta_j \neq \theta_0$.
- IDEA: Look at the *Frequency Space*.

For the Trojan asteroid j :

$$(a_j, e_j, I_j, \lambda_j, \varpi_j, \Omega_j) \implies F_{\theta_j}(a_j, e_j, I_j) = (\nu_j, g_j, s_j)$$

Let $\mathcal{D}$ be the domain of initial conditions: $(a, e, I) \in \mathcal{D}$ and $\mathcal{F} = F_{\theta_0}(\mathcal{D})$.

OBS: $F_{\theta_j}(a_j, e_j, I_j) \in \mathcal{F}$. Then, we define (well defined if H integrable)

$$(\tilde{a}_j, \tilde{e}_j, \tilde{I}_j) = F_{\theta_0}^{-1} \circ F_{\theta_j}(a_j, e_j, I_j).$$

$2 \neq$ trajectories lie on the SAME TORUS $\implies$ DYNAMICALLY EQUIVALENT

$\hookrightarrow (\tilde{a}_j, \tilde{e}_j, \tilde{I}_j, \theta_0)$ vs. $(a_j, e_j, I_j, \theta_j)$

FREQUENCY VECTOR $\iff$ INVARIANT TORUS

Global dynamics: Observed Trojans (II)

In practice, we find two main difficulties:

1. H is not integrable in an open subset of the phase space

$\Downarrow$

$F_{\theta_0}^{-1}$ is not invertible.

2. The domain of initial conditions $\mathcal{D}$ is a discrete set: $\tilde{\mathcal{D}}$

$\Downarrow$

Discrete Frequency Space: $\tilde{\mathcal{F}} = F_{\theta_0}(\tilde{\mathcal{D}})$

For a given Trojan, we approximate $(\tilde{a}_j, \tilde{e}_j, \tilde{I}_j) \approx (a_j^*, e_j^*, I_j^*)$ such that

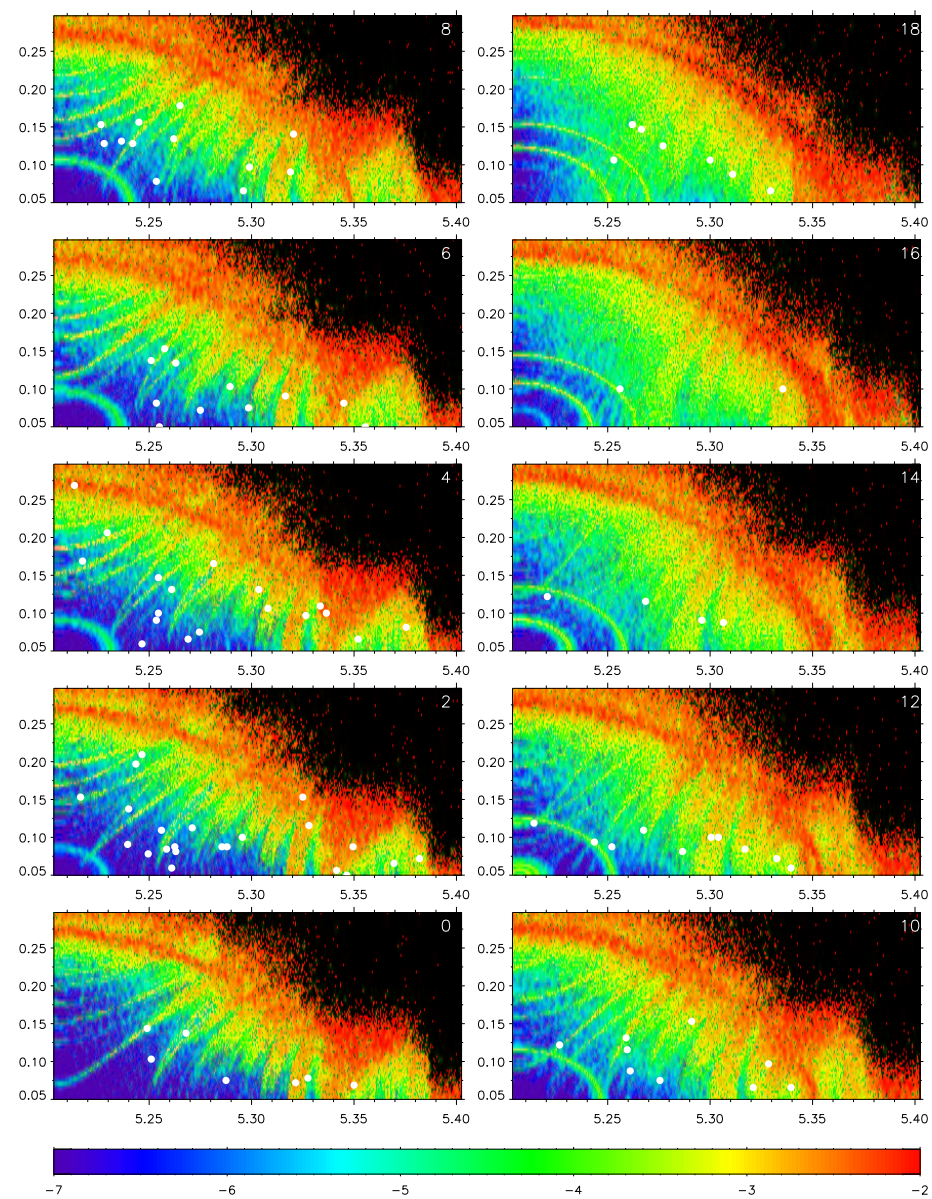
$$d_{j,0} = \|F_{\theta_j}(a_j, e_j, I_j) - F_{\theta_0}(a^*, e^*, I^*)\|_2$$

is minimal in the grid $\tilde{\mathcal{D}}$.

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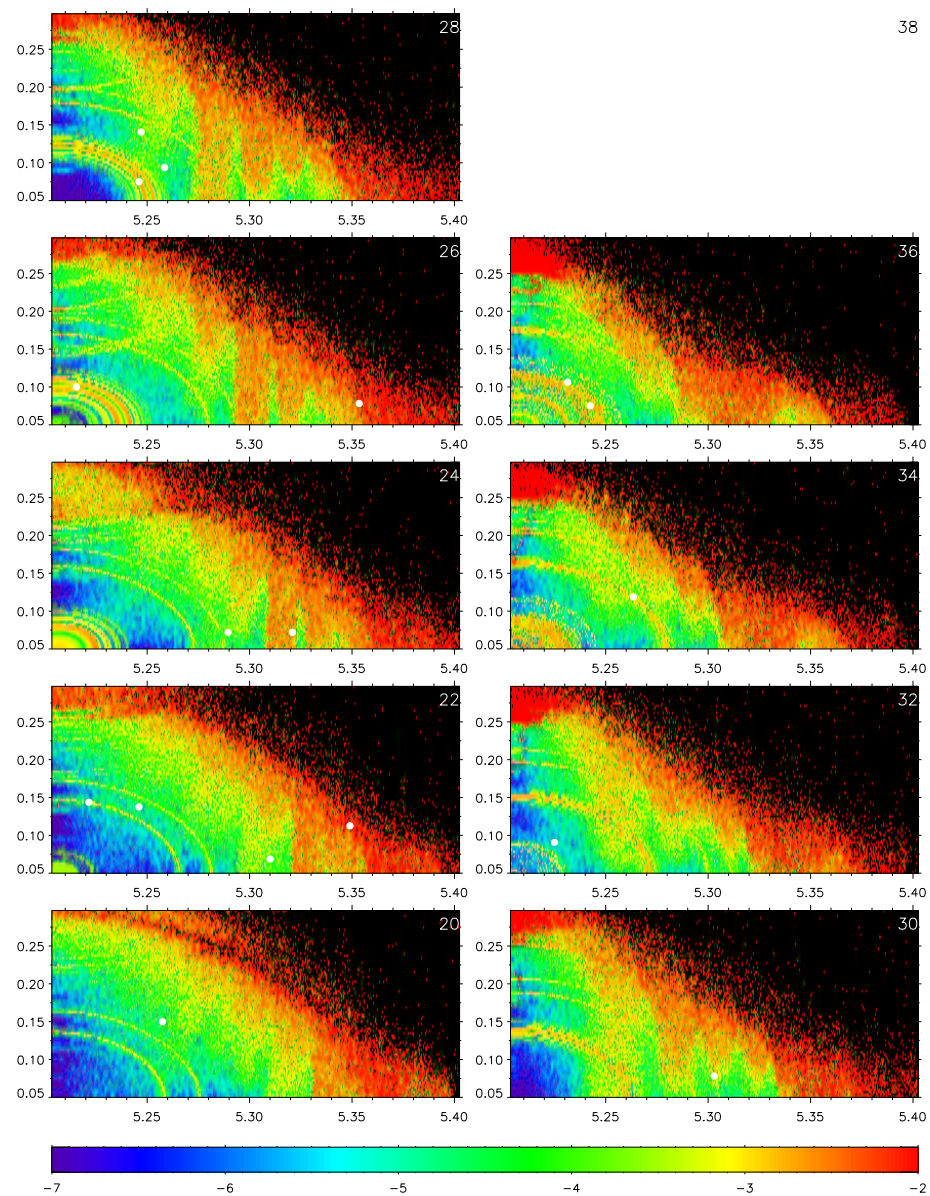
(I)



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(II)

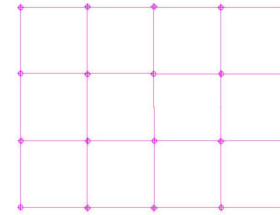


Global dynamics: Technical details (I)

1. Initial conditions (orbital elements) of Jupiter and Saturn.

Take a grid of points in the (a, e) plane:

2. $(a, e) \in \mathcal{A} = [5.20, 5.40] \times [0.05, 0.30]$ and
 $I = I_5, \lambda = \lambda_5 - \frac{\pi}{3}, \varpi = \varpi - \frac{\pi}{3}, \Omega = \Omega_5.$



3. For every initial conditions (a, e) , perform a 5 Myr. integration with the *symplectic integrator* of the family $SABA_n$: J. Laskar and P. Robutel, “High order symplectic integrators for perturbed Hamiltonian systems”, *Celestial Mech.*, 80:39–62, 2001.
 - (a) Take a sample of N equidistant points and do frequency analysis of this sample of the trajectory.
 - (b) Keep the first 10 frequencies (with larger amplitude).
 - (c) Choose a base for (ν, g, s) . In our case,

$$\nu \in [7000, 9500] \quad g \in [250, 450] \quad s \in [-50, 10] \quad ("/\text{yr})$$

$$\text{Periods: } \nu \rightsquigarrow 150 \text{ yr} \quad g \rightsquigarrow 4000 \text{ yr} \quad s \rightsquigarrow 65000 \text{ yr}$$

- (d) Identify (ν, g, s) of the particular trajectory.

Global dynamics: Technical details (II)

4. Repeat point (3.) for the interval 5 Myr. to 10 Myr.
Keep the frequencies (ν', g', s') .

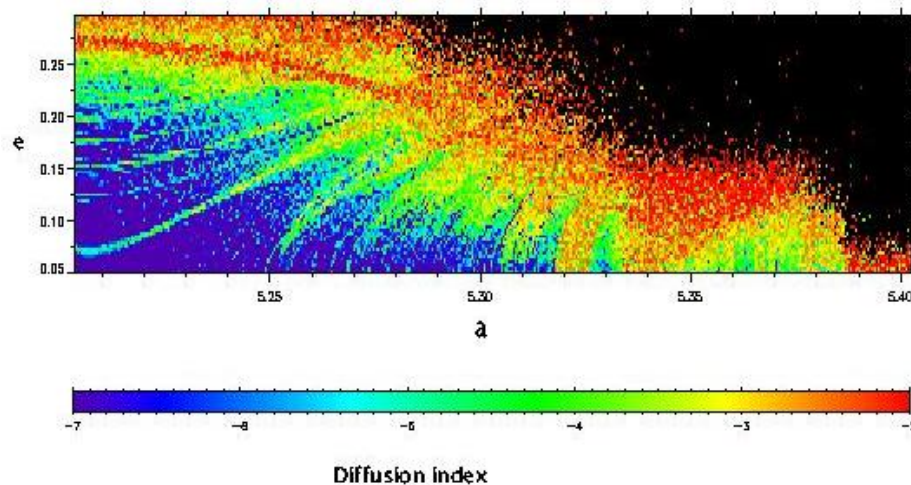
5. Compute the diffusion index $\delta\nu = 1 - \frac{\nu'}{\nu}$

6. Save a file where every entry (line) is

$a \quad e \quad \nu \quad g \quad s \quad \delta\nu \quad \delta g \quad \delta s$

7. Perform a contour plot assigning a color to $\delta\nu$ for every point (a, e) :

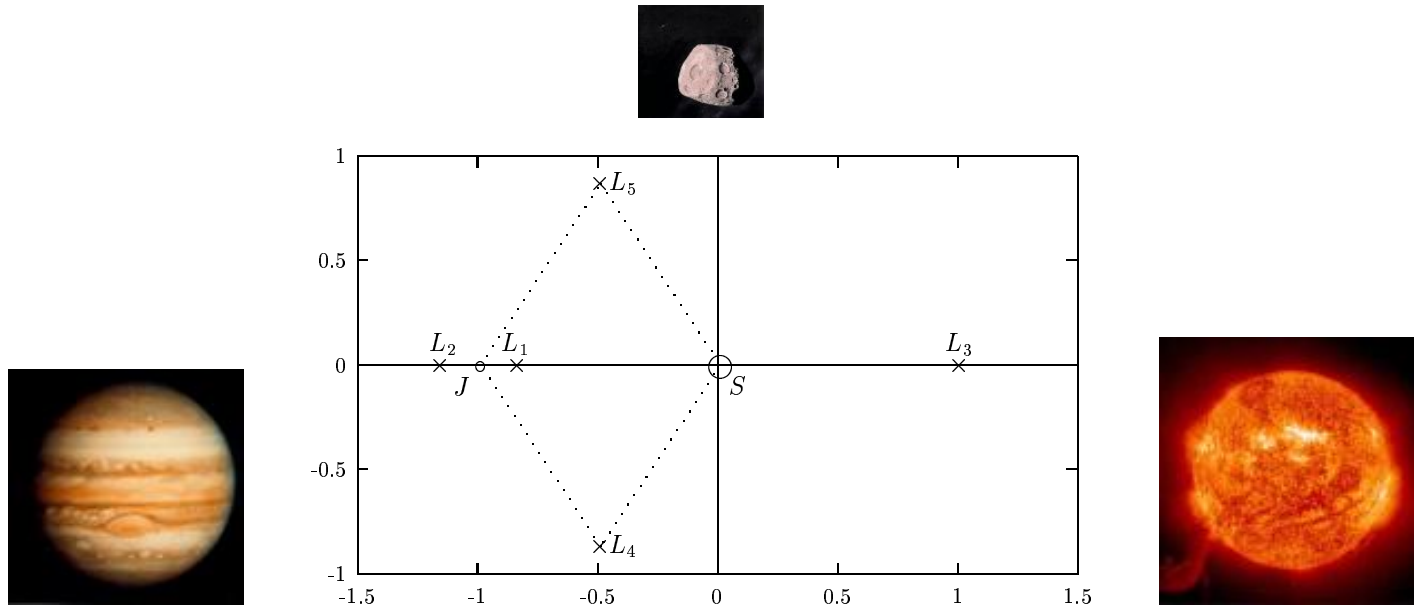
- PGPLOT
- Gnuplot
- Matlab



COMPUTER LAB



Computer Lab: 2 dof model, planar RTBP



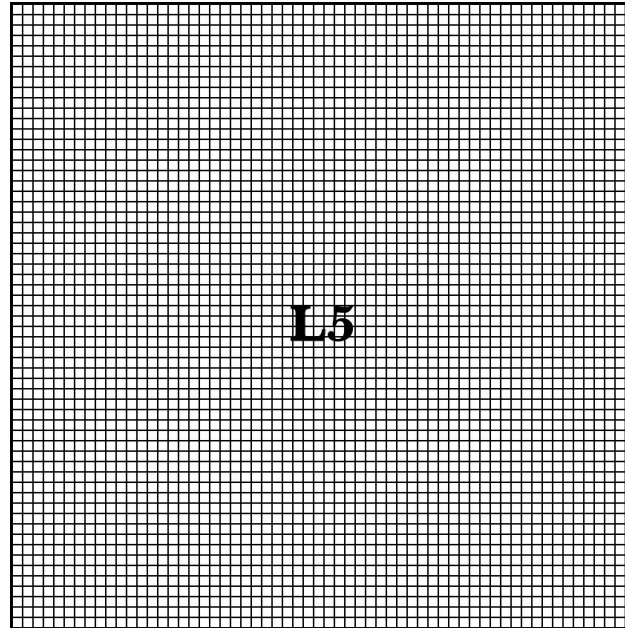
$$H = \frac{1}{2} (p_x^2 + p_y^2) + (yp_x - xp_y) - \frac{1-\mu}{\sqrt{(x-\mu)^2 + y^2}} - \frac{\mu}{\sqrt{(x-\mu+1)^2 + y^2}}$$

Computer Lab: Initial conditions

$$H(x, y, p_x, p_y) : \quad (x, y, p_x, p_y) \in \mathbb{R}^4$$

$$y(L_5) + 0.05$$

$$y(L_5) - 0.05$$



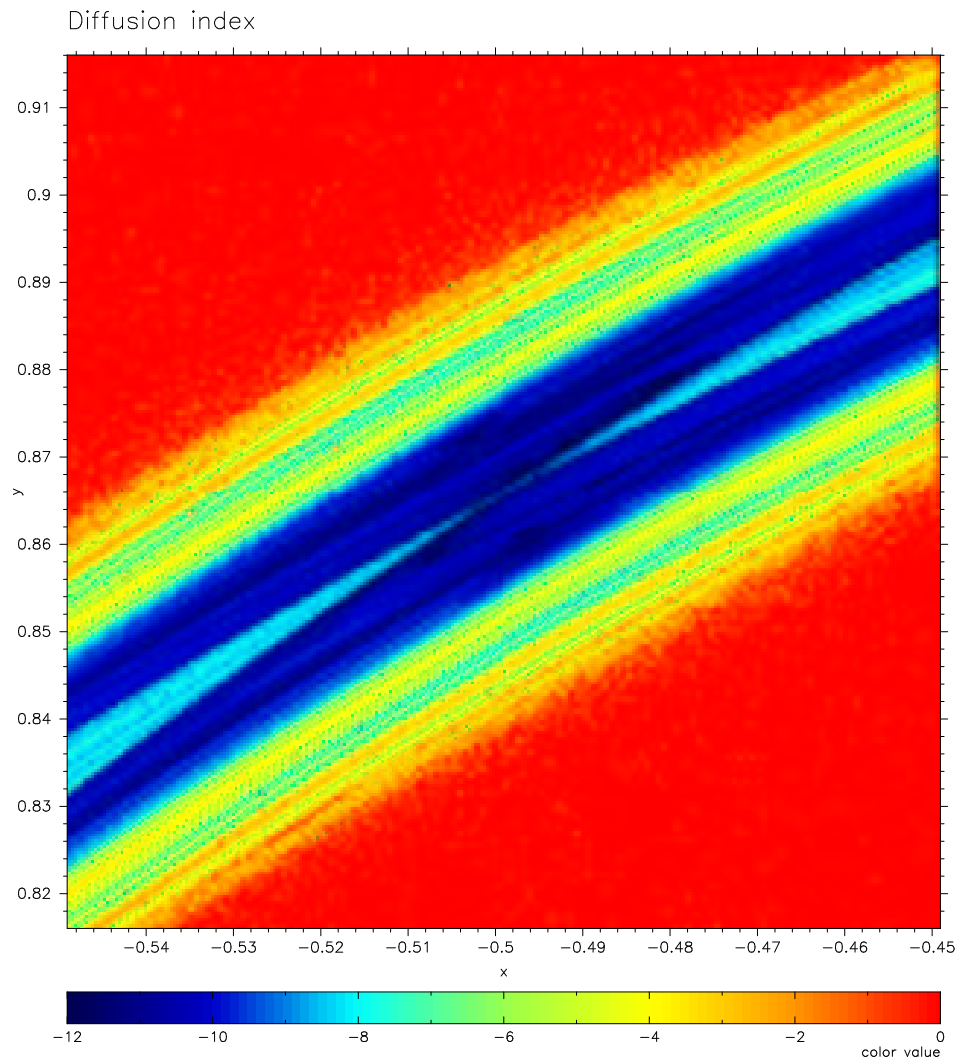
$$x(L_5) - 0.05$$

$$x(L_5) + 0.05$$

$$L_5 : \quad x(L_5) = \mu - 0.5 \quad y(L_5) = \frac{\sqrt{3}}{2} \quad p_x(L_5) = -y(L_5) \quad p_y(L_5) = x(L_5)$$

$$\text{Fix: } p_x = p_x(L_5) \text{ and } p_y = p_y(L_5).$$

Computer Lab: Final Goal



Computer Lab: Tools

- Numerical Integration of ODEs:
 - Taylor method.
 - Your favorite integrator.
- Furiane: Refined Fourier Analysis
- PGPLOT.

See you tomorrow at the LAB !!

