The classical Andronov-Hopf bifurcation

(Ref.: Y.A. Kuznetsov, Elements of Applied Bifurcation Theory, Springer Verlag)

Model equation for the supercritical case:

$$\begin{cases} \dot{x}_1 &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2). \end{cases}$$

α is a real parameter. For all α, $x_1 = x_2 = 0$ is an equilibrium point. The Jacobian at this point is

$$A_\alpha = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}.$$

The eigenvalues are $\lambda_{1,2} = \alpha \pm \mathrm{i}\,.$

Using polar coordinates, we obtain

$$\begin{cases} \dot{\rho} &= \rho(\alpha - \rho^2), \\ \dot{\varphi} &= 1. \end{cases}$$

The dynamics for the different values of α is quite clear:

- $\alpha < 0$: the origin is attracting.
- $\alpha = 0$: the origin is nonlinearly attracting.
- $\alpha > 0$: the origin is repelling, there is an attracting periodic orbit.

Model equations for the subcritical case:

$$\begin{cases} \dot{x}_1 &= \alpha x_1 - x_2 + x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= x_1 + \alpha x_2 + x_2(x_1^2 + x_2^2). \end{cases}$$

Taking polar coordinates, we get:

$$\begin{cases} \dot{\rho} &= \rho(\alpha + \rho^2), \\ \dot{\varphi} &= 1. \end{cases}$$

The dynamics for the different values of α is also clear:

- $\alpha < 0$: the origin is attracting, there is a repelling per. orbit.
- $\alpha = 0$: the origin is nonlinearly repelling.
- $\alpha > 0$: the origin is repelling.

The Hamiltonian-Hopf bifurcation

- À. Jorba and M. Ollé. Invariant curves near Hamiltonian-Hopf bifurcations of four-dimensional symplectic maps. *Nonlinearity*, 17:691–710, 2004.

Goal: to describe the dynamics in an extended neighbourhood of a fixed point of a 4-D symplectic map undergoing a Hamiltonian-Hopf bifurcation. We wand to compute the main invariant objects (invariant curves and their stable/unstable manifolds) and to use them to derive a qualitative description of the dynamics in an extended neighbourhood of the bifurcation point.

We will use numerical methods that only require to be able to evaluate the map and its Jacobian. In this way, these techniques can be applied to Poincaré sections of flows.

This bifurcation can be briefly described as follows.

On one side of the bifurcation the fixed point is linearly stable, this is, all the eigenvalues of the Jacobian matrix at the point have modulus one.

Due to the conditions imposed by the symplectic structure, these eigenvalues come in two couples of conjugate complex numbers, that move on the unit circle when we move along the family.

When we approach the bifurcation, the two pairs come close in such a way that, at the bifurcation point, they collapse in a single pair of complex eigenvalues (the collapse occurs on the unit circle but outside the real line).

After that, the four eigenvalues separate and move off the unit circle (two move outside and two move inside the unit disc) and the point becomes unstable. As none of the eigenvalues is real, the real stable and unstable manifolds are two-dimensional and, near the fixed point, the orbits inside them spiral in and out respectively.

We will focus on two maps, which can be seen as generalisations of the well-known standard map. They are defined by

$$T_s \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - \sin(x_1 + x_2) + L \sin(x_1 + x_2 + x_3 + x_4) \\ x_1 + x_2 \\ x_3 - L \sin(x_1 + x_2 + x_3 + x_4) \\ x_3 + x_4 \end{pmatrix}$$

$$T_t \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - \sin(x_1 + x_2) + L \tan(x_1 + x_2 + x_3 + x_4) \\ x_1 + x_2 \\ x_3 - L \tan(x_1 + x_2 + x_3 + x_4) \\ x_3 + x_4 \end{pmatrix}$$

The two maps have the same Jacobian at $x = 0$, but different nonlinear behaviour.

It is easy to check that we have a transition from (linear) stability to complex instability at $L = L_c = 0.25$

Although these maps have many extra properties (like symmetries), we will not take advantage of them.

Let us focus first on the map T_s.

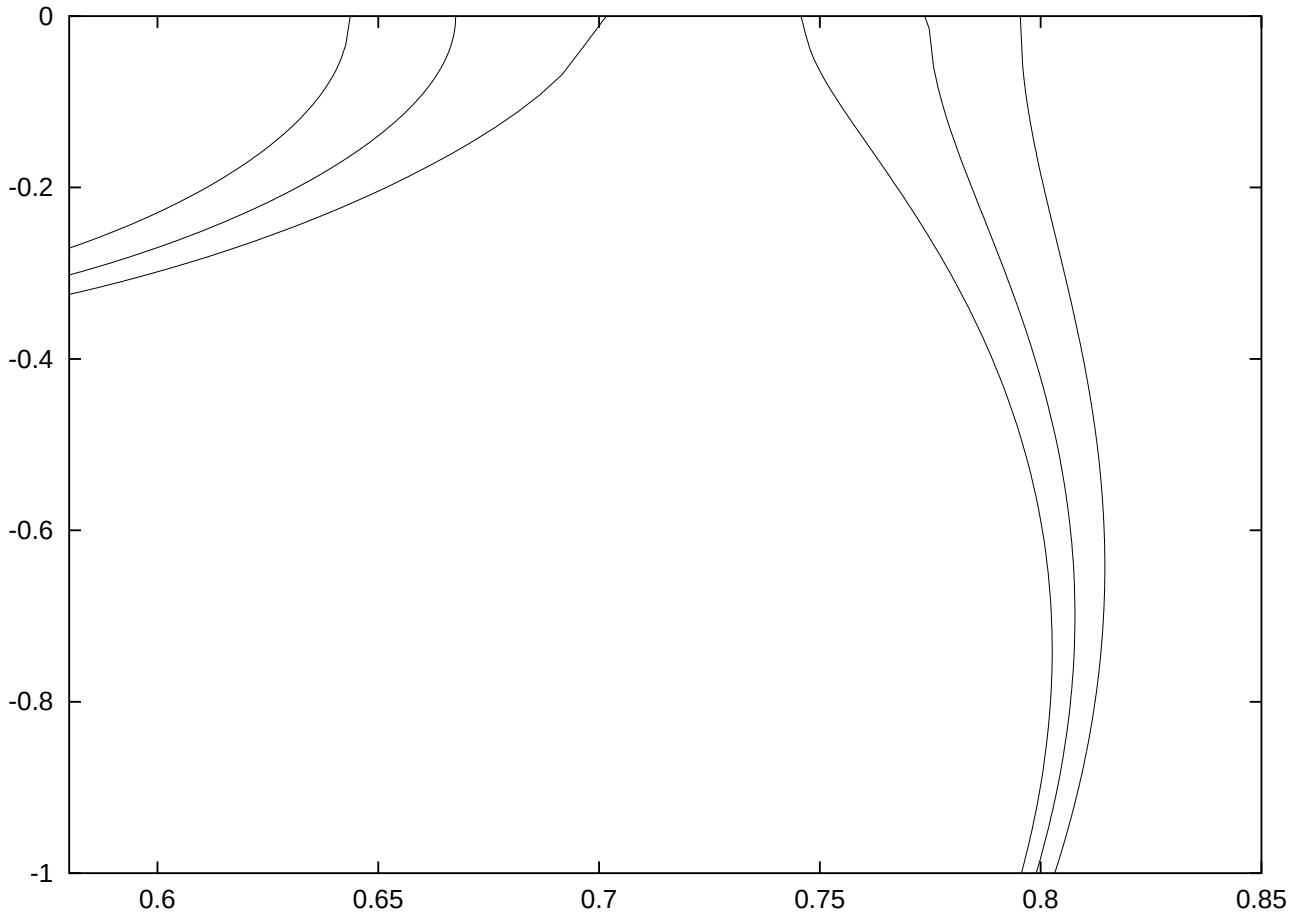
For $L < L_c = 0.25$, the eigenvalues of the Jacobian are given by $\exp(\pm \mathrm{i}\,\omega_1)$ and $\exp(\pm \mathrm{i}\,\omega_2)$. Therefore, the linear dynamics around the origin is described by the product of two harmonic oscillators.

Under generic conditions, it can be proved that each linear oscillation gives rise to a Cantorian 1-parametric family of invariant curves with a rotation number that tends to ω_i, $i = 1, 2$ when the invariant curves tend to the fixed point.

The parametrisation is only defined on a set of values of the parameter of positive Lebesgue measure and empty interior. If the frequencies of the linearization around the origin are Diophantine and the map is analytic, the size of the "holes" of the Cantor structure are exponentially small with the distance to the origin.

Due to the similarities with the periodic orbits of the well-known Lyapunov's centre theorem , we will refer to these families as Lyapunov families of invariant curves.

Note: The 2-D invariant tori (KAM tori) near the origin can be seen as the nonlinear continuation of the direct product of the two linear oscillations at $x = 0$.



Lyapunov families of invariant curves (rotation number ω versus $\tilde{x}_2$) of the mapping T_s. We plot both families (from the outer to the inner ones) for $L = 0.24$, 0.245 and 0.249.

For $L > L_c = 0.25$, the origin is complex unstable: the eigenvalues of the Jacobian are of the form $\lambda \exp(\pm \mathrm{i}\, \omega)$ and $\frac{1}{\lambda} \exp(\pm \mathrm{i}\, \omega)$.

They span a 2-D unstable manifold (and a 2-D stable manifold). The dynamics on them is, at least near $x = 0$, a combination of a rotation with an expansion (or a contraction).

Manifolds of a complex unstable fixed point

Assume that there is a couple of eigenvalues of the form $r_{1,2}\exp(\pm \mathrm{i}\,\omega)$, $0 < r_1 < 1$, $r_2 > 1$ and $\omega \notin \pi\mathbb{Z}$.

For the stable manifold, we select the eigenvalue $r_2 \exp(\mathrm{i}\,\omega)$ and we denote by $u + \mathrm{i}\,v$ (u and v are unitary vectors in $\mathbb{R}^4$) the corresponding eigenvector.

The vectors u and v span the linear approximation to the (2D) unstable manifold, that is a good approximation to the real manifold in a small neighbourhood of the point. To follow the manifold outside this small neighbourhood, we consider the closed curve on the linear approximation to the manifold defined by

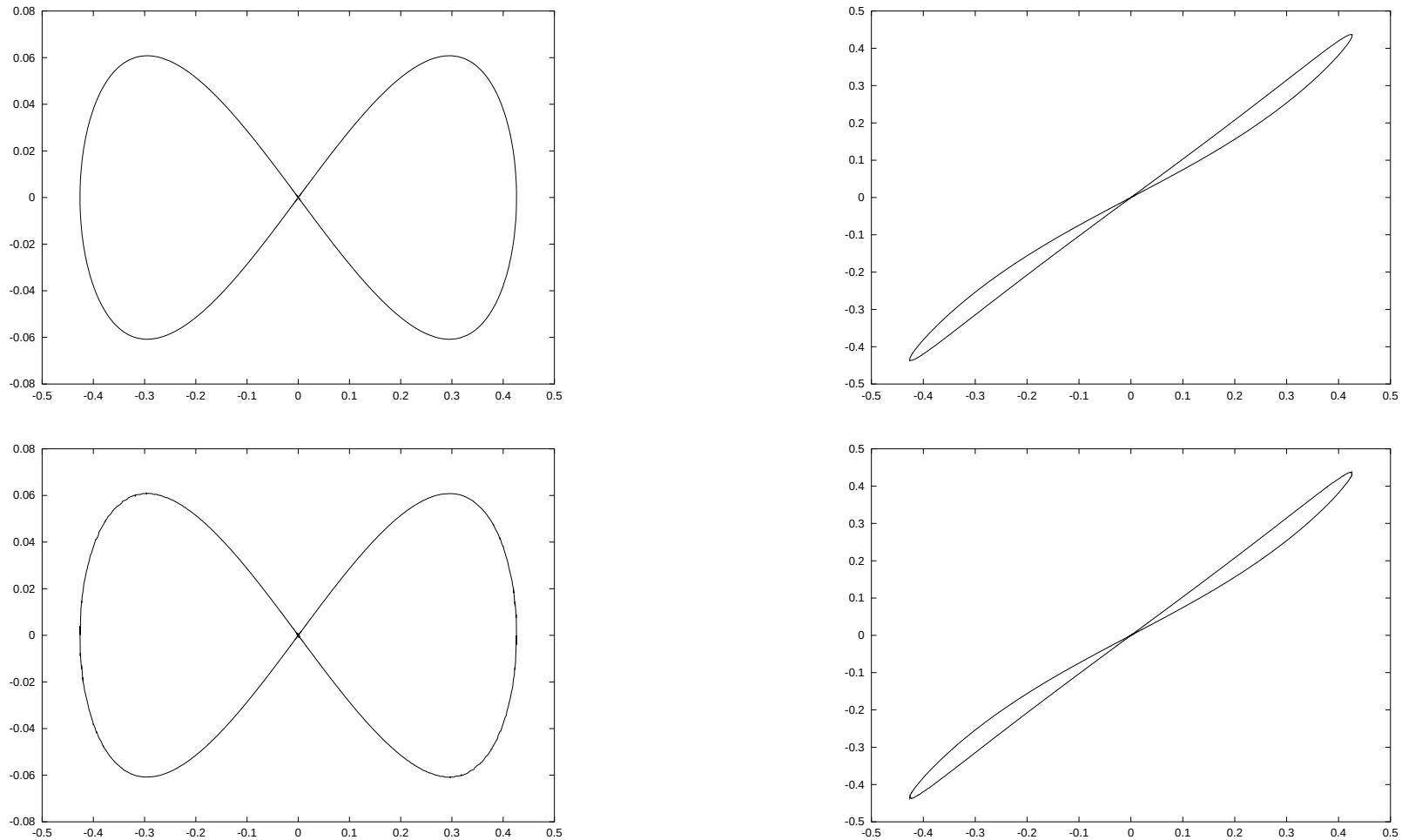
$$\sigma(s) = h\frac{\tilde{\sigma}(s)}{\|\tilde{\sigma}(s)\|}, \quad \text{where} \quad \tilde{\sigma}(s) = \cos(s)\cdot u + \sin(s)\cdot v, \quad s \in [0, 2\pi],$$

where h is a small quantity.

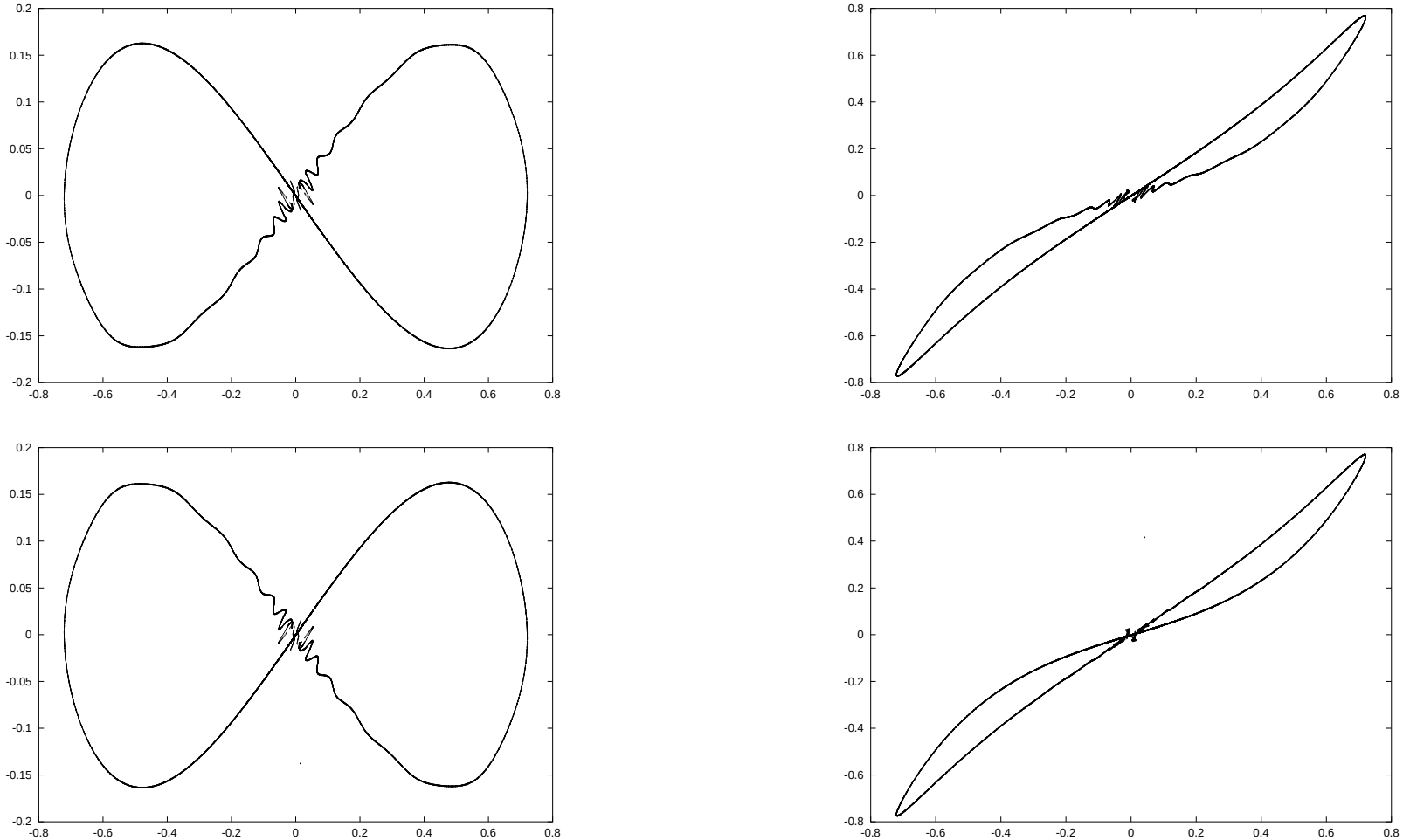
To check the smallness of h to guarantee that the tangent plane is a good enough approximation of the unstable invariant manifold, we carry out the following test: given $a = \alpha u + \beta v$, α, $\beta \in \mathbb{R}$, with $\|a\| = 1$, we have

$$T_i(ha) = h\left[(\alpha a + \beta b)u + (\beta a - \alpha b)v\right] + O(h^2), \qquad i \in \{s, t\},$$

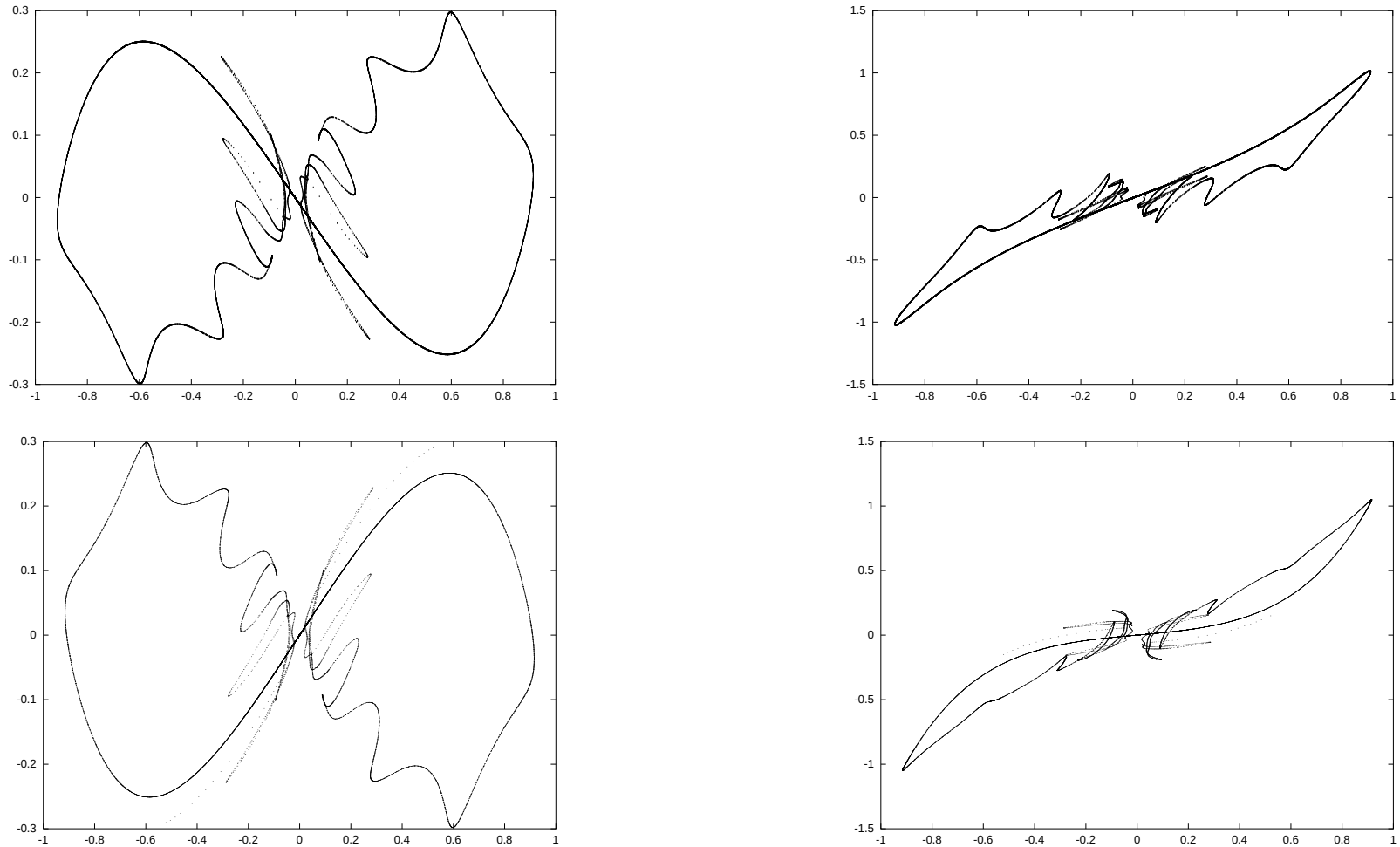
The difference (the term $O(h^2)$) can be computed explicitly and measures how far is the value $T_i(ha)$ from the plane generated by u and v. We have done such computation considering as a the points in the curve $\sigma(s)$, $s \in [0, 2\pi]$, and it turns out that, in our examples, if $h \leq 10^{-5}$ the term $O(h^2)$ is less than 10^{-15}. Then, we can use these values as starting points for trajectories on the unstable manifold.



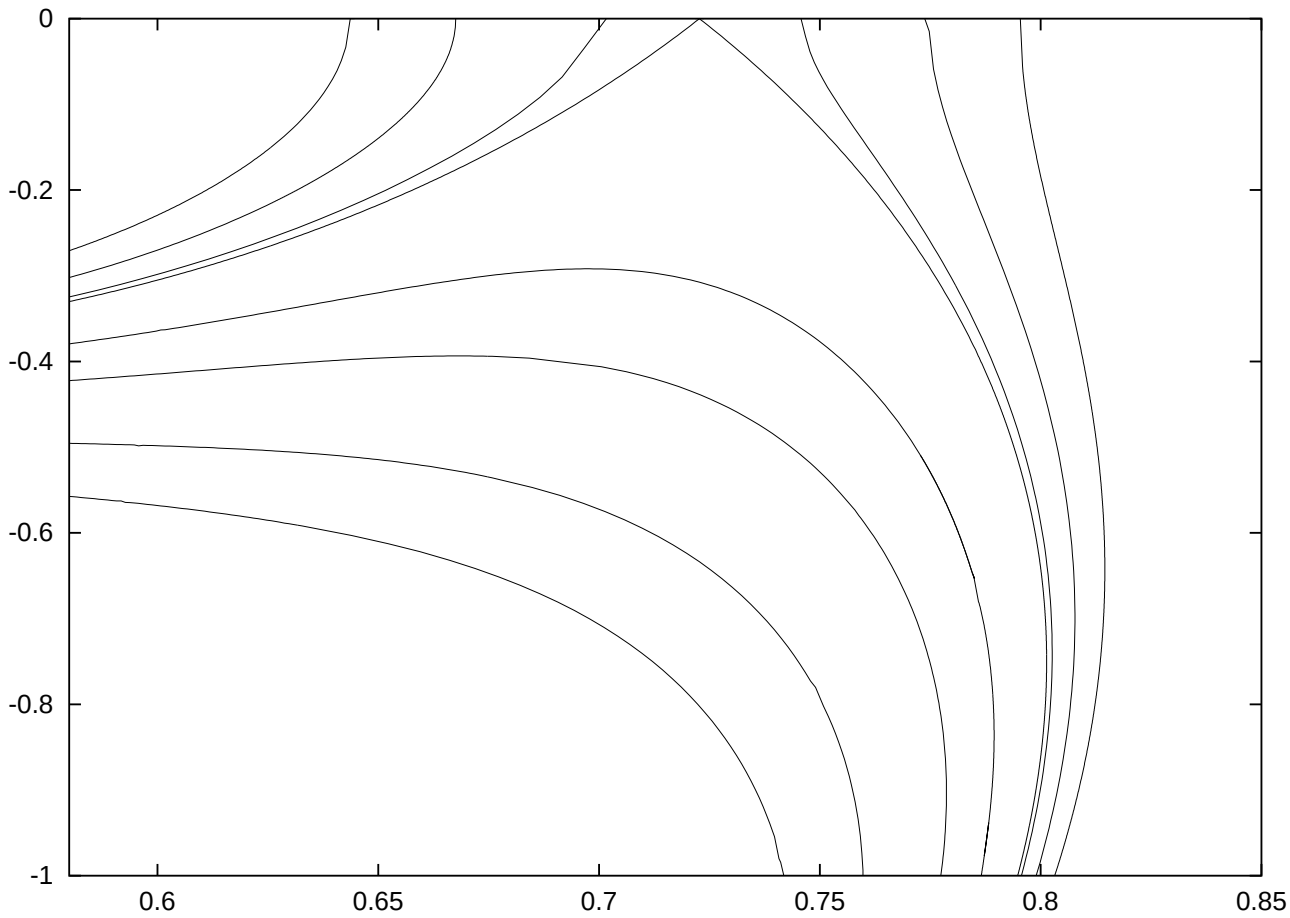
$L = 0.26$. Intersection between the invariant manifold and $x_1 = 0$. Top: W^u, (x_2, x_3) and (x_2, x_4) projections; bottom: W^s.



$L = 0.28$. Intersection between the invariant manifold and $x_1 = 0$. First row: W^u, (x_2, x_3) and (x_2, x_4) projections; second row: W^s.



As the previous one, but for $L = 0.30$.



Detachment of the Lyapunov families of invariant curves of the mapping T_s (rotation number ω versus $\tilde{x}_2$), when L crosses L_{crit}.

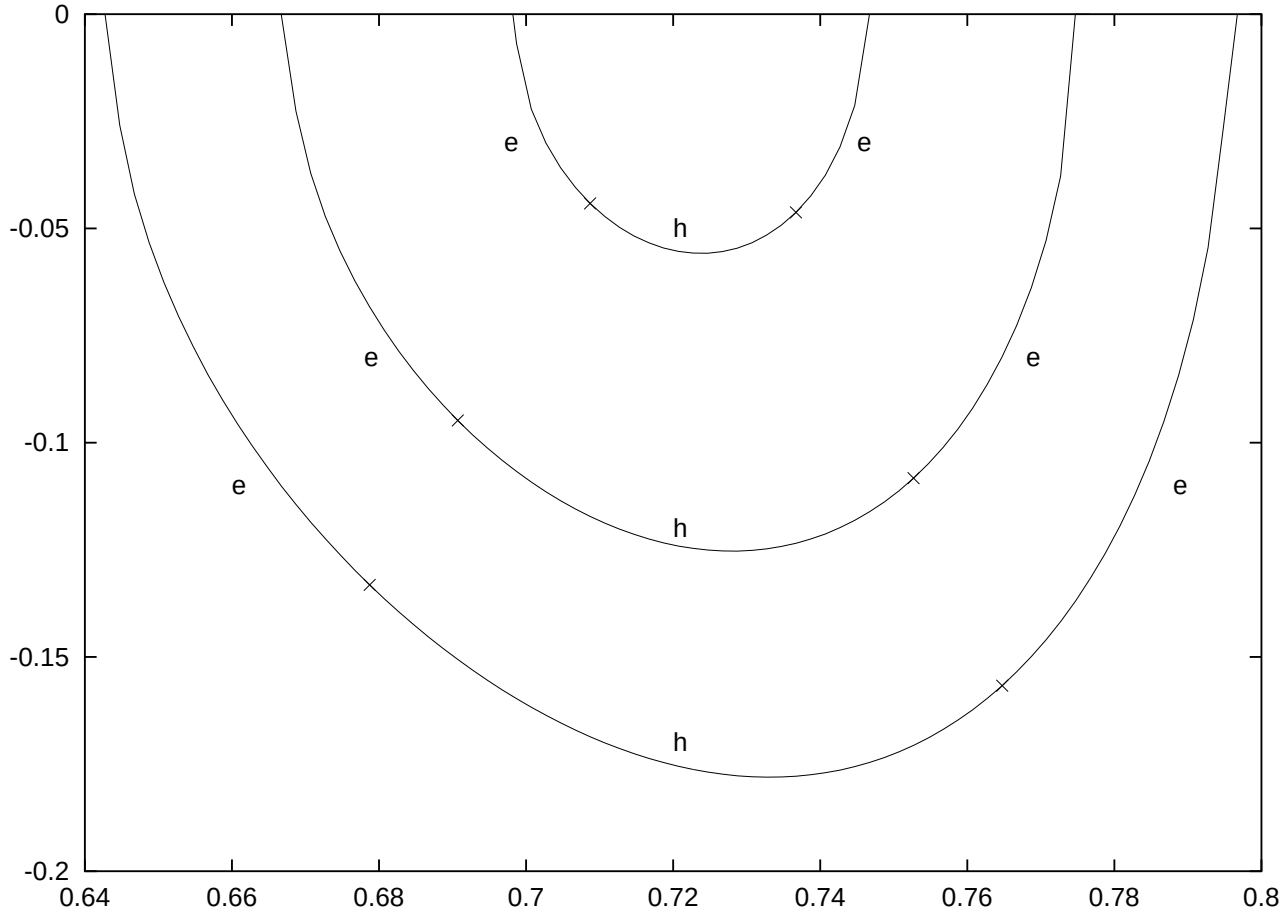
For $L < L_{crit}$, there are plenty of 2D invariant tori around the origin. When L crosses L_{crit}, the origin becomes hyperbolic and the (2D) unstable and stable manifolds are almost coincident so that they look like a small "loop", its size depending on the size of $L - L_{crit} > 0$.

Therefore, although the origin is unstable, the nearby trajectories are trapped* in a small neighbourhood.

When L gets bigger, the "loop" of the manifolds can grow up and, eventually, can allow the trajectories to escape. At the same time that the point becomes complex unstable, there is a family of invariant curves that detach from the origin.

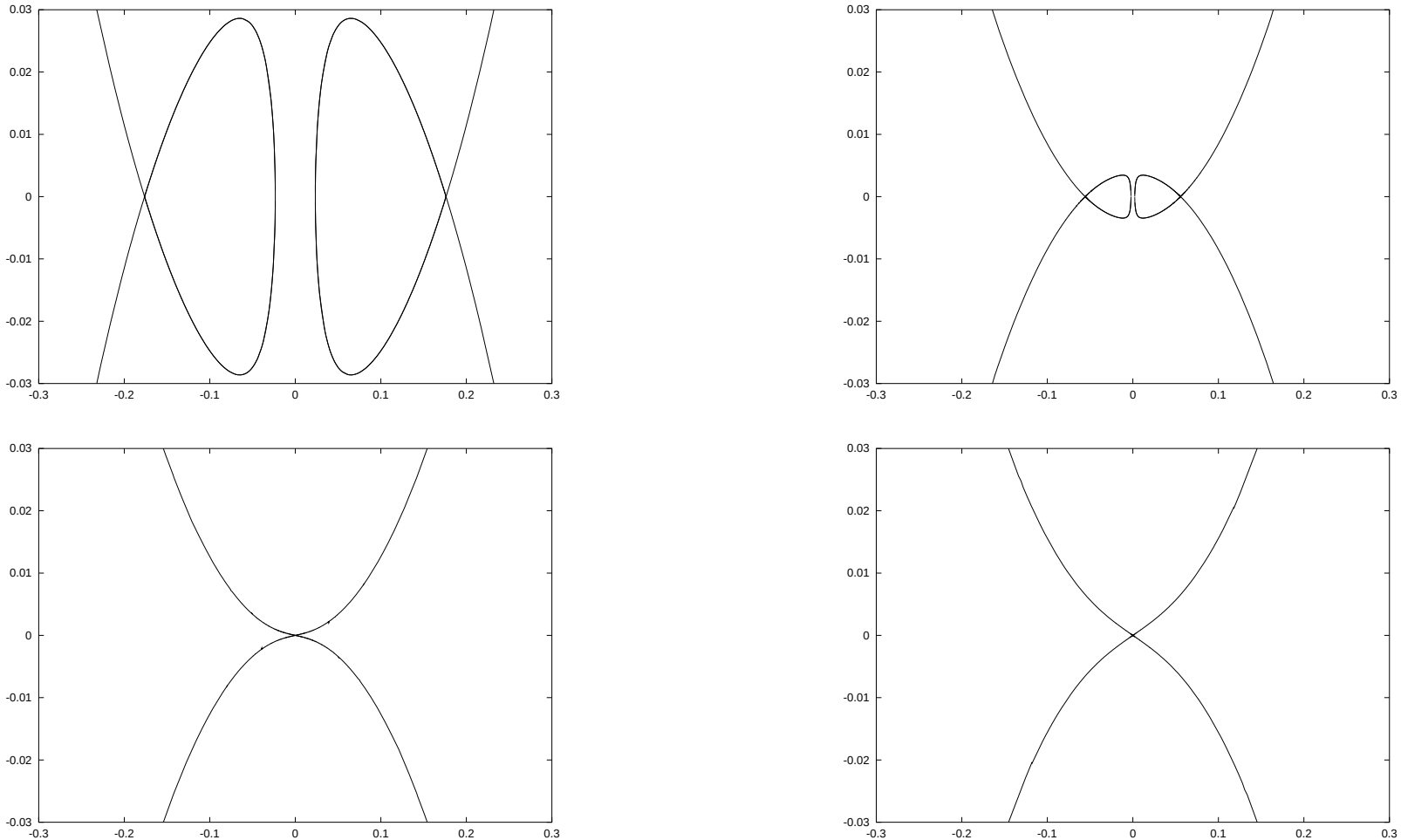
Let us now focus on the map T_t.

As before, for $L < L_{crit}$, we compute the two Lyapunov families of invariant curves that are born at the origin. The continuation process shows that these two families meet at some distance from the origin. In other words, for each $L < L_{crit}$ we have one global family which begins and ends at the origin. When L approaches L_{crit}, this "connecting loop" collapses to the origin.



The global family of invariant curves for $L = 0.24$, $L = 0.245$ and $L = 0.249$ (outer to inner ones); rotation number versus $\tilde{x}_2$. The marked points correspond to transition invariant curves.

Next plots show the (x_2, x_3) projections for the slice $x_1 = 0$ of the stable and unstable invariant manifolds of an invariant curve (top plots) and the origin (bottom plots), when the invariant curve collapses at $x = 0$.



Top left: $L = 0.24$; top right: $L = 0.249$; bottom left: $L = 0.2501$; bottom right: $L = 0.251$.

We can compare with the classical 2D (and dissipative) Hopf bifurcation.

This corresponds to the case in which there is an unstable periodic orbit that approaches to the (stable) origin. The orbit merges the fixed point that becomes unstable too.

This means that the dynamics after the bifurcation is “trivial”, in the sense that all the trajectories escape.