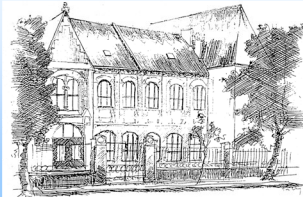


Delay equations with engineering applications

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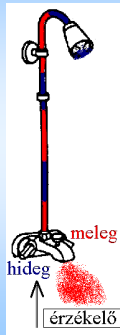
Contents

Delay equations arise in mechanical systems...

... by the information system (of control), and
by the contact of bodies.

- **Linear stability** & subcritical Hopf bifurcations
- Force control and balancing – human and robotic
- Contact problems
 - Shimmying wheels (of trucks and motorcycles)
 - Machine tool vibrations

Time delay – destabilizing effect

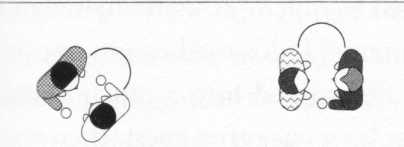


“Good memory causes trouble”

Shower

small delay – large control gains

large delay – small control gains
or periodic large control gains



Time delay models

Delay differential equations (DDE):

- simplest (populations) $\dot{x}(t) = x(t-1)$
- single delay $\dot{x}(t) = cx(t-\tau)$
 - (production based on past prices)
- average past values $\dot{x}(t) = \int_{-\tau}^0 x(t+\vartheta) d\vartheta$
 - (production based on statistics of past/averaged prices)
- weighted w.r.t. the past $\dot{x}(t) = \int_{-\tau}^0 w(\vartheta)x(t+\vartheta) d\vartheta$
 - (Roman law)

Delay Diff Equ (DDE) – Functional DE

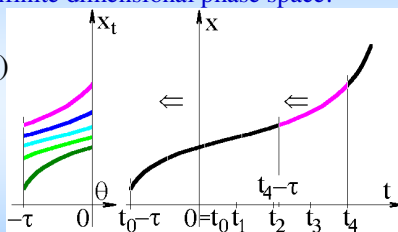
Time delay & infinite dimensional phase space:

$$x_t(\vartheta) = x(t + \vartheta)$$

$$\vartheta \in [-\tau, 0]$$

$$\dot{x}(t) = L x_t$$

$$\dot{x}(t) = \int_{-\tau}^0 x(t + \vartheta) d\eta(\vartheta) \quad \text{Riesz Representation Theorem}$$



Stieltjes integral representation

$$\eta(\vartheta) = \begin{cases} 0 & \vartheta \in [-r, -\tau) \\ -1 & \vartheta \in [-\tau, -h) \\ h - 1 + \vartheta + \frac{h}{\pi} \sin(\vartheta\pi/h) & \vartheta \in [-h, 0) \\ h & \vartheta = 0 \end{cases}$$

$$d\eta(\vartheta) = \eta(\vartheta + 0) - \eta(\vartheta - 0)$$

$$d\eta(\vartheta) = \eta'(\vartheta) d\vartheta$$

$$\dot{x}(t) = \int_{-\tau}^0 x(t + \vartheta) d\eta(\vartheta)$$

$$\int_{-\tau}^0 x(t + \vartheta) d\eta(\vartheta) = x(t) + \int_{-h}^0 (1 + \cos(\vartheta\pi/h)) x(t + \vartheta) d\vartheta - x(t - \tau)$$

Linear time-periodic time-delay models

Non-autonomous DDE:

- simplest periodic $\dot{x}(t) = p(t)x(t - \tau)$, $p(t) = p(t + \tau)$
- periodic delay $\dot{x}(t) = cx(t - \tau(t))$, $\tau(t) = \tau(t + T)$
- weighted past values
$$\dot{x}(t) = \int_{-\tau_{\max}}^0 w(t, \vartheta)x(t + \vartheta) d\vartheta$$

 $w(t, \vartheta) = w(t + T, \vartheta)$
- general form $\dot{x}(t) = L(t)x_t$, $L(t) = L(t + T)$
- Riesz Th
$$\dot{x}(t) = \int_{-\tau_{\max}}^0 x(t + \vartheta) d_{\vartheta} \eta(t, \vartheta)$$
, $\eta(t, \vartheta) = \eta(t + T, \vartheta)$

Semi-linear state-dependent time delay models

State-dependent DDE (SD-DDE):

- simplest $\dot{x}(t) = cx(t - \tau(x(t)))$, $\tau(x) = \tau_0 + \tau_1 \sin x$
- implicit $\dot{x}(t) = cx(t - \tau(x(t - \tau)))$, $\dot{x}(t) = cx(t - \tau(x_t))$
- generalized forms
Cooke, Huang (1996), Turi, Hartung (2000)
Hartung, Krisztin, Walther, Wu (2006)
 $\dot{x}(t) = L(t, x_t)x_t \Rightarrow \dot{x}(t) \neq L(t, 0)x_t$ for small x
with Riesz Theorem
$$\dot{x}(t) = \int_{-\tau_{\max}}^0 x(t + \vartheta) d_{\vartheta} \eta(t, \vartheta; x_t)$$
 "linearization" 0 No

Main references

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Stability of linear RFDEs of n DoF systems

Delayed mechanical systems include 2nd derivatives:

$$M\ddot{x}(t) + \int_{-h}^0 d_{\vartheta} B(t, \vartheta) \dot{x}(t + \vartheta) + \int_{-h}^0 d_{\vartheta} K(t, \vartheta) x(t + \vartheta) = 0$$

Autonomous systems: $B(t, \vartheta) \equiv B(\vartheta)$, $K(t, \vartheta) \equiv K(\vartheta)$

Trial solution: $x(t) = Ae^{\lambda t}$ $A \in \mathbb{R}^n$

Characteristic roots: $\text{Re } \lambda_j < 0, j=1, 2, \dots \Leftrightarrow$ stability

$$D(\lambda) = \det(M\lambda^2 + \int_{-h}^0 \lambda e^{\lambda \vartheta} d_{\vartheta} B(\vartheta) + \int_{-h}^0 e^{\lambda \vartheta} d_{\vartheta} K(\vartheta))$$

D-curves: $R(\omega) = \text{Re } D(i\omega)$, $S(\omega) = \text{Im } D(i\omega)$, $\omega \in [0, \infty)$

$$R(\rho_k) = 0, k=1, \dots, r: \quad S(\rho_k) \neq 0, k=1, \dots, r \quad \Leftrightarrow \text{stability}$$

$$\sum_{k=1}^r (-1)^k \text{sgn } S(\rho_k) = (-1)^n n$$

Examples with 1 DoF, $n = 1$

$$\ddot{x}(t) + c_0 x(t) = c_1 \int_{-1}^0 w(\vartheta) x(t + \vartheta) d\vartheta, \quad w(\vartheta) \equiv 1$$

$$D(\lambda) = \lambda^2 + c_0 - c_1 \int_{-1}^0 e^{\lambda \vartheta} d\vartheta = \lambda^2 + c_0 - c_1 \frac{1 - e^{-\lambda}}{\lambda}$$

$$R(\omega) = -\omega^2 + c_0 - c_1 \frac{\sin \omega}{\omega} \Rightarrow \lim_{\omega \rightarrow +\infty} R(\omega) = -\infty$$

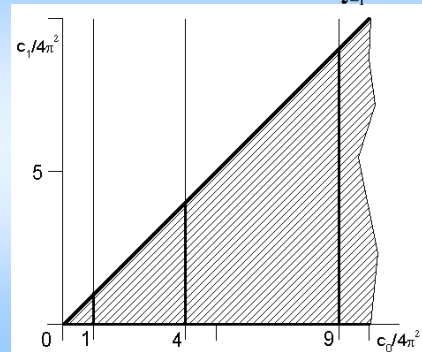
$$S(\omega) = c_1 \frac{1 - \cos \omega}{\omega} \Rightarrow S(\omega) > 0 \text{ for } c_1 > 0$$

$$\omega \neq 2k\pi, k=0, 1, \dots$$

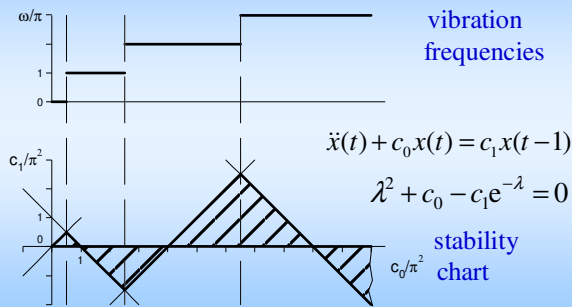
$$S(\rho_k) \neq 0, k=1, \dots, r \Rightarrow R(2k\pi) = -4k^2\pi^2 + c_0 \neq 0$$

$$\sum_{k=1}^r (-1)^k \underbrace{\text{sgn } S(\rho_k)}_{+1} = (-1)^n \underbrace{n}_{-1} \Rightarrow R(0) = c_0 - c_1 > 0$$

Stability chart $\ddot{x}(t) + c_0 x(t) = c_1 \int_{-1}^0 x(t + \vartheta) d\vartheta$



Delayed oscillators $w(\vartheta) = \delta(\vartheta + 1)$



Delayed oscillator with damping

