Delay equations with engineering applications

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Contents

Delay equations arise in mechanical systems…
… by the information system (of control), and
by the contact of bodies.

- **Linear stability & subcritical Hopf bifurcations**
- Force control and balancing – human and robotic
- Contact problems
  - Shimmying wheels (of trucks and motorcycles)
  - Machine tool vibrations

Main references

Insperger T, Stepan G, Stability chart for the delayed Mathieu equation, *Proceedings of the Royal Society London A* **458** (2002) 1989-1998.

Insperger T, Stepan G, *Semi-discretization for time-delay systems – Engineering applications*, to appear, Springer, New York, 2011.

Insperger T, Stepan G, Updated semi-discretization method for periodic delay-differentilal equations with discrete delay, *International Journal for Numerical Methods in Engineering* **61** (2004) 117-141.

Stepan G, Insperger T, Szalai R, Delay, parametric excitation, and the nonlinear dynamics of cutting processes, *International Journal of Bifurcation and Chaos* **15** (2005) 2783-2798.

Orosz G, Stepan G, Subcritical Hopf bifurcations in a car-following model with reaction-time delay, *Proceedings of the Royal Society London A* **462** (2006) 2643-2670.

Non-autonomous linear RFDEs

$$M\ddot{x}(t)+\int_{-h}^{0}\mathrm{d}_{\vartheta}B(t,\vartheta)\dot{x}(t+\vartheta)+\int_{-h}^{0}\mathrm{d}_{\vartheta}K(t,\vartheta)x(t+\vartheta)=0$$

Time-periodic systems: $B(t+T,\vartheta)=B(t,\vartheta)$

Trial solution: $\boxed{x(t)=p(t)\mathrm{e}^{\lambda t}}$ $K(t+T,\vartheta)=K(t,\vartheta)$

$$p(t+T)=p(t)=\sum_{k=0}^{+\infty}(A_k\cos(k\tfrac{2\pi}{T}t)+B_k\sin(k\tfrac{2\pi}{T}t))$$

Hill's infinite dimensional determinant $\Rightarrow$

characteristic function $\Rightarrow$ characteristic roots λ

Re $\lambda_j<0$, j=1,2,… $\Leftrightarrow$ stability $\Leftrightarrow$ $|\mu_j|<1$, j=1,2,…

for characteristic multipliers $\mu=\mathrm{e}^{\lambda T}$ of fund. op. at T

The delayed Mathieu equation

Analytically constructed stability chart for testing numerical methods and algorithms

$$\ddot{x}(t)+(\delta+\varepsilon\cos t)x(t)=b\,x(t-2\pi)$$

Time delay and time periodicity are equal:

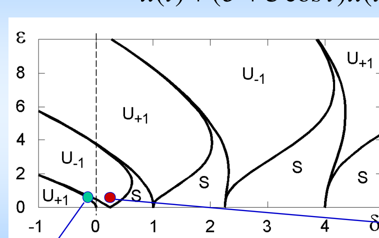
$$T=\tau=2\pi$$

$b=0$ Mathieu equation (1868)

$\varepsilon=0$ Delayed oscillator (1941)

Stability chart – Mathieu equation

$$\ddot{x}(t)+(\delta+\varepsilon\cos t)x(t)=0$$



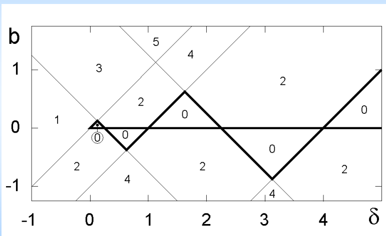
Floquet (1883)
Hill (1886)
Rayleigh(1887)
van der Pol & Strutt (1928)
Swing (2000BC)

Strutt – Ince diagram (1956)

Stephenson (1908), Swinney (2004), Zelei (2005)

Stability chart – delayed oscillator

$$\ddot{x}(t)+\delta\, x(t)=b\, x(t-2\pi)$$



Pontryagin (1942)
Nyquist (1949)
Bellman & Cooke (1963)
in engienering:
Schlippe (1941)
Minorsky (1942)

Hsu & Bhatt (1966)

The delayed Mathieu equation

$$\ddot{x}(t)+(\delta+\varepsilon\cos t\,)x(t)=b\, x(t-2\pi)$$

$$x(t)=\sum_{k=0}^{\infty}\left(A_k\,\mathrm{e}^{\mathrm{i}kt}+B_k\,\mathrm{e}^{-\mathrm{i}kt}\right)\mathrm{e}^{\lambda t}+\sum_{k=0}^{\infty}\left(\bar{A}_k\,\mathrm{e}^{-\mathrm{i}kt}+\bar{B}_k\,\mathrm{e}^{\mathrm{i}kt}\right)\mathrm{e}^{\bar{\lambda} t}$$

$$x(t)=\sum_{k=-\infty}^{\infty}C_k\,\mathrm{e}^{(\lambda+\mathrm{i}k)t}+\bar{C}_k\,\mathrm{e}^{(\bar{\lambda}-\mathrm{i}k)t}$$ Harmonic balance $\Rightarrow$ Hill's determinant

$$\det\begin{pmatrix} \ddots & & \ddots & & \ddots & 0 & 0\ldots \\ \ldots 0 & \frac{\varepsilon}{2} & & \delta+(\lambda+\mathrm{i}k)^2-b\,\mathrm{e}^{-2\pi(\lambda+\mathrm{i}k)} & & \frac{\varepsilon}{2} & 0\ldots \\ \ldots 0 & 0 & \ddots & & \ddots & & \ddots \end{pmatrix}=0$$

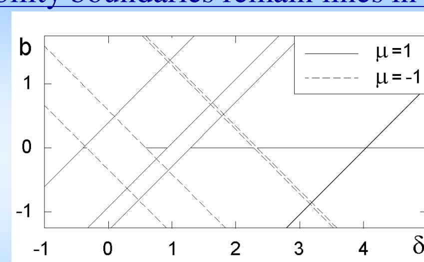
Stability boundaries

$$\lambda=\pm\mathrm{i}\,\omega \;\Rightarrow\; \delta-(\omega+k)^2-b\cos(2\pi\omega)\pm\mathrm{i}\,b\sin(2\pi\omega)$$

In the main diagonal, there are 4 cases to check if 0:

1) $\omega\neq j/2,\; b\neq 0$ then the main diagonal is complex, the Hill's determinant is non-zero
2) $b=0$ then there are possible stability boundaries
3) $b\neq 0,\; \omega=j/2=h \;\Rightarrow\; \delta-b-(k+j/2)^2$ then $D(\mathrm{i}\omega,\delta,b,\varepsilon)=f(\delta-b,\varepsilon)$ and $\mu=\mathrm{e}^{\mathrm{i}h\pi}=1$ (SN)
4) $b\neq 0,\; \omega=j/2=(2h+1)/2 \;\Rightarrow\; \delta+b-(k+j/2)^2$ $D(\mathrm{i}\omega,\delta,b,\varepsilon)=f(\delta+b,\varepsilon)$ and $\mu=\mathrm{e}^{\mathrm{i}(2h+1)\pi}=-1$(flip)

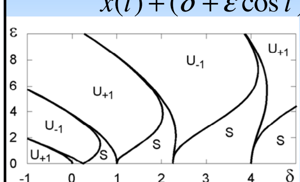
Stability boundaries remain lines in (δ,b)



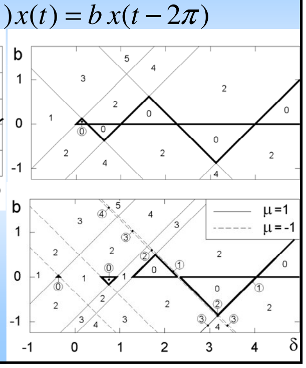
To select stable ones: $$\frac{\partial\,\mathrm{Re}\,\lambda}{\partial b}=\frac{2\pi\Gamma_N^{\;2}}{\left(2\pi b(-1)^j\Gamma_N\right)^2+\Omega_N^{\;2}}\,b$$

The delayed Mathieu – stability charts

$$\ddot{x}(t)+(\delta+\varepsilon\cos t\,)x(t)=b\, x(t-2\pi)$$



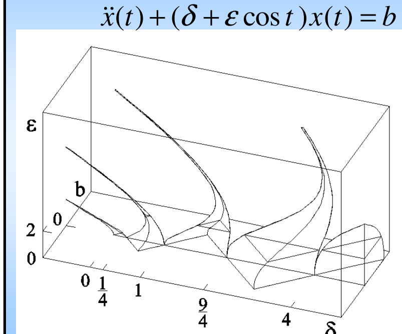
b=0 (Strutt-Incze, 1928)



ε=1

Stability chart of delayed Mathieu

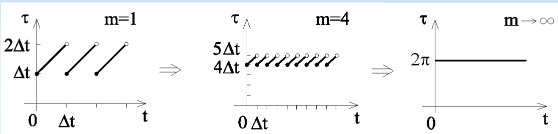
$$\ddot{x}(t)+(\delta+\varepsilon\cos t\,)x(t)=b\, x(t-2\pi)$$



Insperger, Stépán (2002)

Semi-discretization method – introduction

$$\ddot{x}(t)+c_0 x(t)=c_1 x(t-\tau) \qquad \tau=2\pi$$



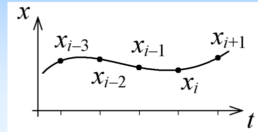
The approximating DDE is *non-autonomous*

$$\ddot{x}(t)+c_0 x(t)=c_1 x(t-\tau(t)), \quad \tau(t)=t+(m-\mathrm{int}(t/\Delta t))\Delta t$$

$$t\in[t_i,t_{i+1})=[i\Delta t,(i+1)\Delta t) \qquad \Delta t=2\pi/(m+1/2)$$

$$\Rightarrow\ x(t-\tau(t))\equiv x((i-m)\Delta t)=x_{i-m}$$

Introduction to SDM – delayed oscillator



$$\ddot{x}(t)+c_0 x(t)=c_1 x_{i-m}$$
$$x(t_i)=x_i$$
$$\dot{x}(t_i)=\dot{x}_i$$

$$x(t)=K_{1i}\cos(\sqrt{c_0}t)+K_{2i}\sin(\sqrt{c_0}t)+c_1 x_{i-m}/c_0$$

$$\dot{x}(t)=-K_{1i}\sqrt{c_0}\sin(\sqrt{c_0}t)+K_{2i}\sqrt{c_0}\cos(\sqrt{c_0}t)$$

$$x_{i+1}=a_{00}x_i+a_{01}\dot{x}_i+a_{0m}x_{i-m}$$
$$\dot{x}_{i+1}=a_{10}x_i+a_{11}\dot{x}_i+a_{1m}x_{i-m}$$

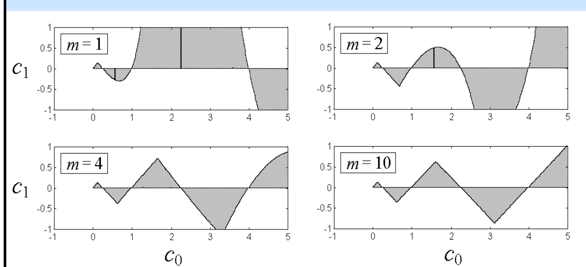
$$\mathbf{y}_i=\mathrm{col}(\dot{x}_i\ x_i\ x_{i-1}\ \ldots\ x_{i-m})$$

$$\boxed{\mathbf{y}_{i+1}=\mathbf{A}\,\mathbf{y}_i}$$

$$\det(\mu\mathbf{I}-\mathbf{A})=0\ \Rightarrow\ |\mu_{1,2,\ldots,m+2}|<1 \Leftrightarrow \text{stability}$$

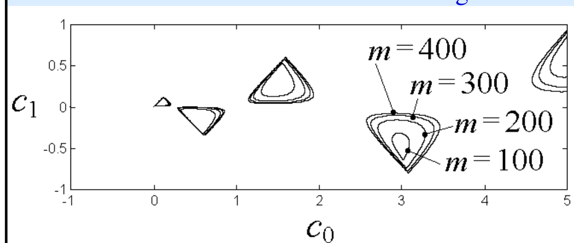
Delayed oscillator – stability chart by SDM

$$\ddot{x}(t)+c_0 x(t)=c_1 x(t-\tau(t)), \quad \tau(t)=t+(m-\mathrm{int}(t/\Delta t))\Delta t$$



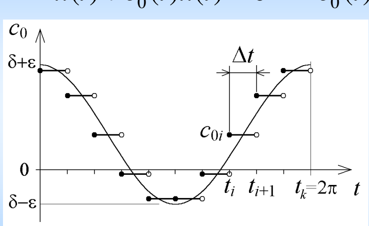
Full discretization - comparison

Discretization also w.r.t. time derivatives – slow convergence



Introduction to SDM – Mathieu equation

$$\ddot{x}(t)+c_0(t)x(t)=0 \qquad c_0(t)=\delta+\varepsilon\cos t$$



$$t\in[t_i,t_{i+1})$$
$$\ddot{x}(t)+c_{0i}x(t)=0$$
$$x(t_i)=x_i$$
$$\dot{x}(t_i)=\dot{x}_i$$
$$i=0,1,\ldots,k-1$$

$$x(t)=x_i\cos\left(\sqrt{c_{0i}}(t-t_i)\right)+\frac{\dot{x}_i}{\sqrt{c_{0i}}}\sin\left(\sqrt{c_{0i}}(t-t_i)\right)$$

SDM for Mathieu equation

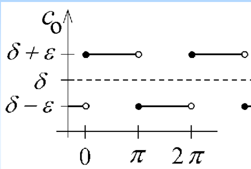
$$\mathbf{y}_i=\begin{pmatrix}x(t_i)\\ \dot{x}(t_i)\end{pmatrix} \qquad \mathbf{A}_i=\begin{pmatrix}\cos\left(\Delta t\sqrt{c_{0i}}\right) & \dfrac{\sin\left(\Delta t\sqrt{c_{0i}}\right)}{\sqrt{c_{0i}}}\\ -\sqrt{c_{0i}}\sin\left(\Delta t\sqrt{c_{0i}}\right) & \cos\left(\Delta t\sqrt{c_{0i}}\right)\end{pmatrix}$$

$$\begin{pmatrix}x(2\pi)\\ \dot{x}(2\pi)\end{pmatrix}=\mathbf{y}_k=\mathbf{A}_{k-1}\mathbf{A}_{k-2}\ldots\mathbf{A}_0\mathbf{y}_0=\mathbf{\Phi}(2\pi)\begin{pmatrix}x(0)\\ \dot{x}(0)\end{pmatrix}$$

$$\det(\mu\mathbf{I}-\mathbf{\Phi}(2\pi))=0\ \Rightarrow\ |\mu_{1,2}|\le 1 \Leftrightarrow \text{stability}$$

for k=2 intervals ⇒ Meissner (1918)
for $k\to\infty$ ⇒ van der Pol, Strutt, Ince (1928)

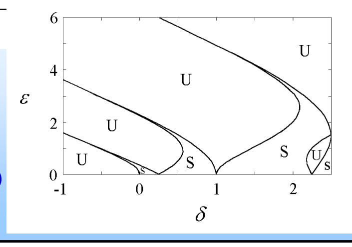
SDM for Mathieu equation



$$c_0(t)=\begin{cases}\delta+\varepsilon & \text{if} \quad 0\le t<\pi \\ \delta-\varepsilon & \text{if} \quad \pi\le t<2\pi\end{cases}$$

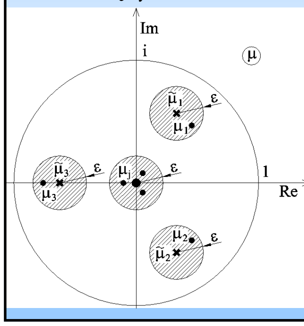
$$\ddot{x}(t)+c_0(t)x(t)=0$$

Meissner (1918)



Semi-discretization – general case

$$\dot{x}(t)=\int_{-\tau}^{0}\mathrm{d}_{\vartheta}\eta(t,\vartheta)x(t+\vartheta),\quad \eta(t+T,\vartheta)=\eta(t,\vartheta)$$



$\forall\varepsilon>0,\ \exists M(\varepsilon),\ \forall m>M(\varepsilon)$

$\Rightarrow$

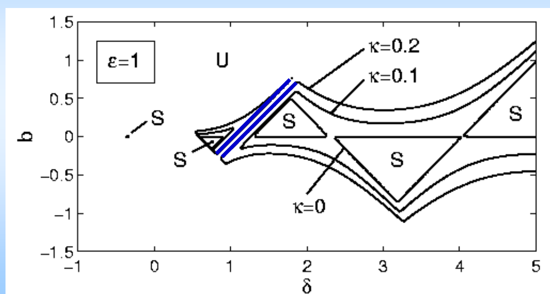
$\mu_j\in\bigcup_{j=1}^{mn}S_{\tilde{\mu}_j,\varepsilon},\ j=1,\dots,mn$

$|\mu_j|<\varepsilon,\qquad j=mn+1,\dots$

Insperger, Stepan: *Int. J. of Numerical Methods in Engineering* (2002)

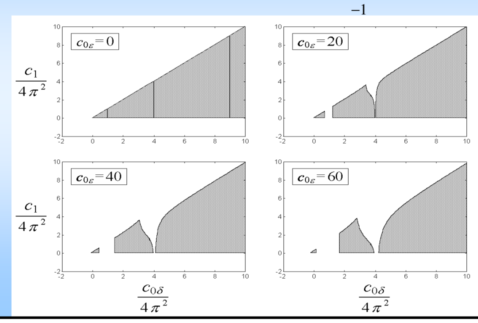
Test on (damped) delayed Mathieu

$$\ddot{x}(t)+\kappa\dot{x}(t)+(\delta+\varepsilon\cos t)x(t)=b\,x(t-2\pi)$$

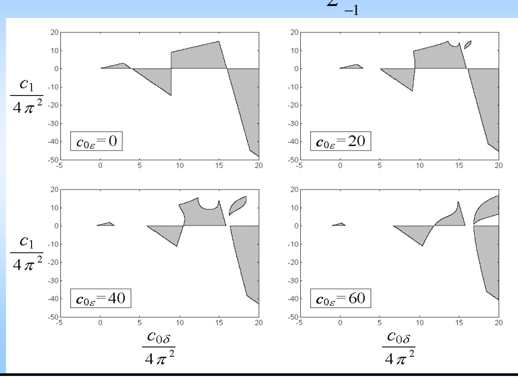


Examples – test on delayed Mathieu

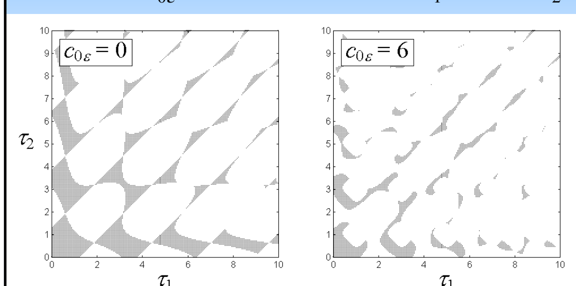
$$\ddot{x}(t)+(c_{0\delta}+c_{0\varepsilon}\cos(4\pi t))x(t)=c_1\int_{-1}^{0}x(t+\vartheta)\mathrm{d}\vartheta$$



$$\ddot{x}(t)+(c_{0\delta}+c_{0\varepsilon}\cos(4\pi t))x(t)=c_1\frac{\pi}{2}\int_{-1}^{0}\sin(\pi\vartheta)x(t+\vartheta)\mathrm{d}\vartheta$$



$$\ddot{x}(t)+(6+c_{0\varepsilon}\cos(2\pi t))x(t)=x(t-\tau_1)+x(t-\tau_2)$$



Nonlinear RFDEs in Engineering

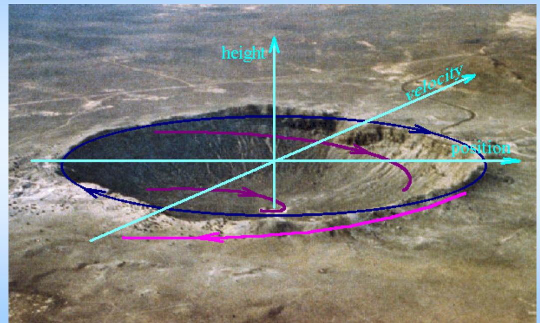
Stability analysis of steady-states is followed by Center Manifold reduction & bifurcation analysis

Hopf bifurcation – self-excited vibrations

Supercritical case: easy to avoid vibrations by knowing the linear stability behavior

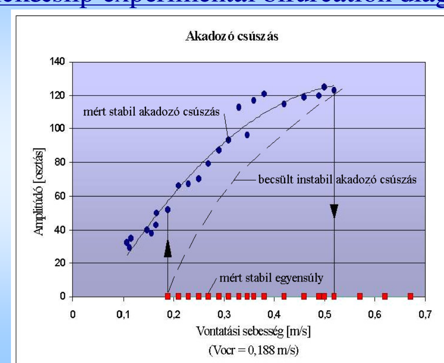
Subcritical case: the unstable periodic solutions mean a limited domain of attraction for the desired steady-state behavior – *cannot* be predicted by linear stability analysis.

Unstable limit cycle – "ghost" vibration

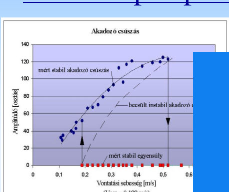


Stick&slip

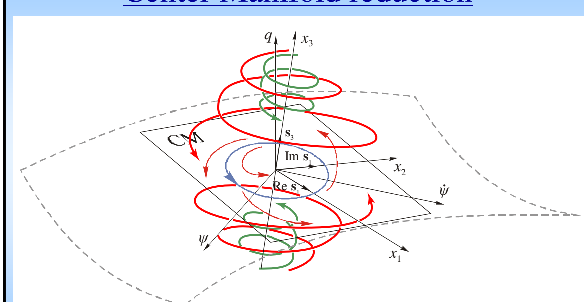
Stick&slip experimental bifurcation diagram



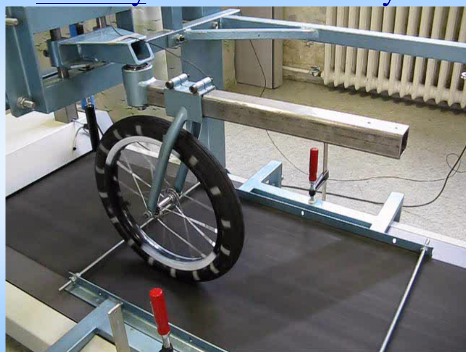
Stick&slip experimental bifurcation diagram



Center Manifold reduction



Shimmy – unstable limit cycle

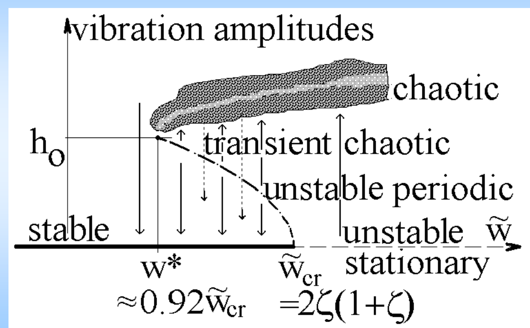


Shimmy – quasi-periodic oscillations

Machine tool vibrations

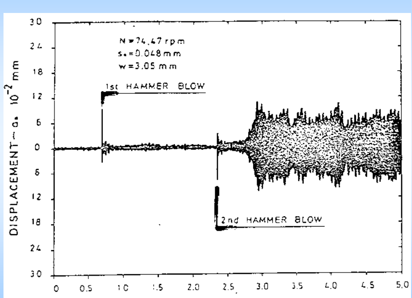
video

Machine tool vibration



The unstable periodic motion

Shi, Tobias (1984) – impact experiment



Balancing