Delay equations with engineering applications

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Contents

Delay equations arise in mechanical systems…

… by the information system (of control), and
by the contact of bodies.

- Linear stability & subcritical Hopf bifurcations
- **Force control** and balancing – human and robotic
- Contact problems

  Shimmying wheels (of trucks and motorcycles)

  Machine tool vibrations

Main references

Stepan, G., Haller, G., Quasiperiodic oscillations in robot dynamics, *Nonlinear Dynamics* **8** (1995) 513-528.

Stepan, G., Vibrations of machines subjected to digital force control, *International Journal of Solids and Structures* **38** (2001) 2149-2159.

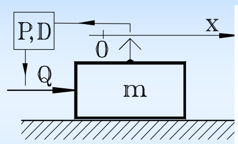
Kovecses J, Kovacs LL, Stepan G, Dynamics modeling and stability of robotic systems with discrete-time force control, *Archive of Applied Mechanics* **77** (2007) 293-299.

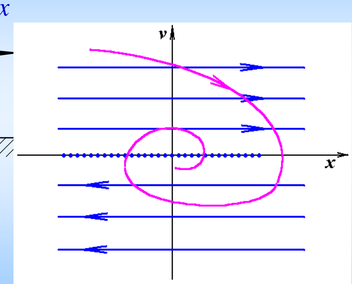
Kovacs LL, Kovecses J, Stepan G, Analysis of effects of differential gain on dynamic stability of digital force control, *International Journal of Non-Linear Mechanics* **43** (2008) 514-520.

Insperger T, Kovacs LL, Galambos P, Stepan G, Increasing the accuracy of digital force control process using the act-and-wait concept, *IEEE/ASME Transactions on Mechatronics*, **15** (2010) 291-298.

Position control

1 DoF models $\Rightarrow x$

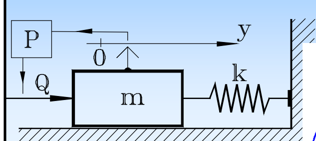


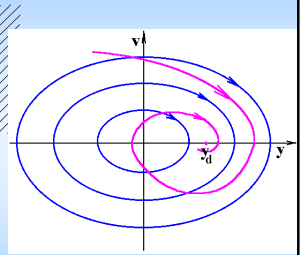


Blue trajectories:
$Q = 0$

Pink trajectories:
$Q = -Px - D\dot{x}$

Force control



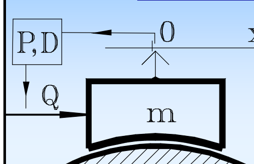


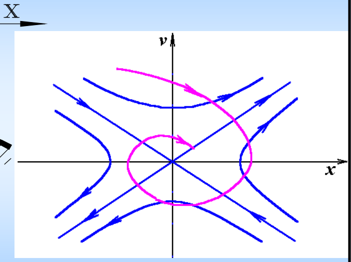
Desired contact force:
$F_d = ky_d$;

Sensed force:
$F_s = ky$

Control force: $Q = -P(F_d - F_s) - D\dot{F}_s + F_{s \text{ or } d}$

Stabilization (balancing)



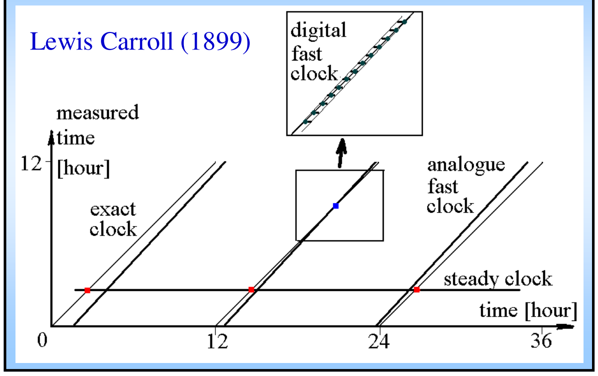


Control force:
$Q = -Px - D\dot{x}$

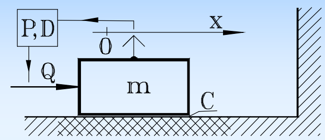
Special case of force control: with $k < 0$

Alice's Adventures in Wonderland

Lewis Carroll (1899)



Digital position control



Equation of motion

$$m\ddot{x}(t) + D\dot{x}(t) + Px(t) = -C \operatorname{sgn} \dot{x}(t)$$

Position error: $\Delta = C / P$

Stability $\Leftrightarrow$ $P > 0,\ D > 0$

Modeling sampling

Time delay τ and zero-order-holder

Dimensionless time

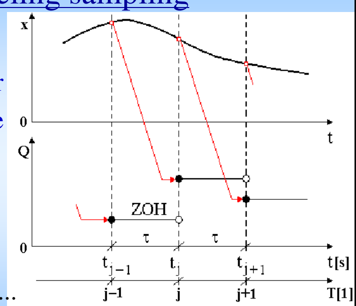
$$T = t / \tau$$

Equation of motion

$$\frac{m}{\tau^2} x''(T) = Q(T)$$

where for $j = 1,2,\ldots$

$$Q(T) \equiv -Px(j-1) - \frac{D}{\tau} x'(j-1), \quad T \in [j, j+1)$$



Stability of digital position control

$$x''(T) \equiv \underbrace{-px(j-1) - dx'(j-1)}_{=:\,a_j}, \quad T \in [j, j+1)$$

$$p = P\tau^2 / m,\ d = D\tau / m$$

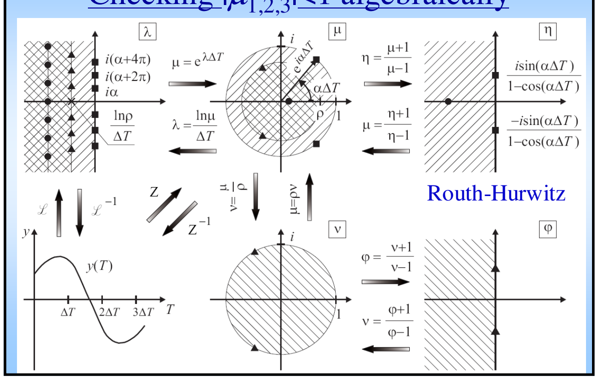
$$x(T) = x(j) + x'(j)(T-j) + \tfrac{1}{2} a_j (T-j)^2$$

$$x'(T) = x'(j) + a_j (T-j), \quad T \in [j, j+1)$$

$$\mathbf{z}^j := \begin{pmatrix} x(j) \\ x'(j) \\ a_j \end{pmatrix} \Rightarrow \mathbf{z}^{j+1} = \mathbf{A}\mathbf{z}^j, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ -p & -d & 0 \end{pmatrix}$$

$$\det(\mu\mathbf{I} - \mathbf{A}) = \mu^3 - 2\mu^2 + (1 + d + \tfrac{1}{2}p)\mu + (\tfrac{1}{2}p - d) = 0$$

Checking $|\mu_{1,2,3}| < 1$ algebraically

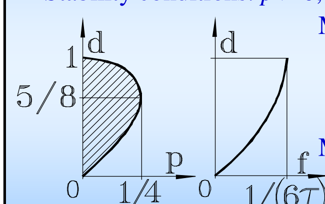


Stability chart

$$\mathrm{Re}\,\eta_{1,2,3} < 0$$

$$p\eta^3 + 2(d-p)\eta^2 + (4 - 4d + p)\eta + 2(2+d) = 0$$

Stability conditions: $p > 0$, $H_2 > 0$ ($= 0 \Rightarrow$ Hopf)



Maximum gain:

$$P_{\max} = \frac{1}{4} \frac{m}{\tau^2}$$

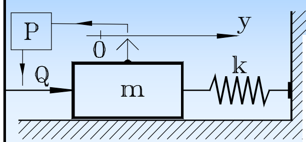
Minimum position error

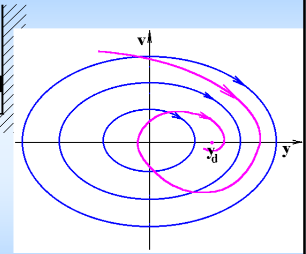
$$\Delta_{\min} \geq 4C\tau^2 / m$$

Self-excited vibration frequency: $0 < f < f_{\text{sampling}}/6$

The low-frequency vibrations

Force control





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Sensed force:
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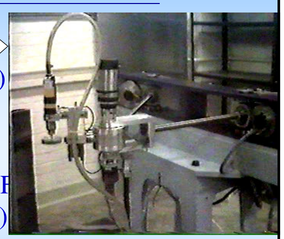
Control force: $Q = -P(F_d - F_s) - D\dot{F}_s + F_{s \text{ or } d}$

Force control – motivation

- Polishing turbine blade ⇨ (Newcastle/Parsons robot)
- Rehabilitation robotics (human/machine contact)
- Coupling force control (CF (between truck and trailer)
- Electronic brake force control (added to ABS systems)*

* ©Knorr-Bremse



Motivation

- Polishing turbine blade (Newcastle/Parsons robot)
- Rehabilitation robotics ⇨ (human/machine contact)
- Coupling force control (CF (between truck and trailer)
- Electronic brake force control (added to ABS systems)*

* ©Knorr-Bremse

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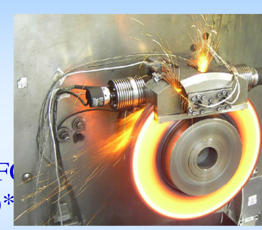
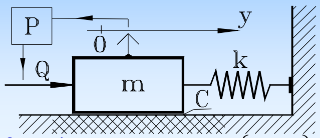


Digital force control



Equation of motion:

$$m\ddot{y}(t)+ky(t)=-P(ky(t)-F_d)+\left\{\begin{matrix} ky(t) \\ \text{or} \\ F_d \end{matrix}\right\}-C\,\mathrm{sgn}\,\dot{y}(t)$$

Equilibrium: $y_d = F_d / k$

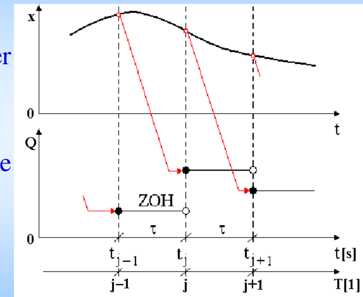
Force error: $\Delta_F = C/P$ or $C/(1+P)$ (Craig '86)

Stability for $y(t)=y_d+x(t)$, $m\ddot{x}+Pkx=0 \Rightarrow P>0$

Modeling sampling

Time delay τ and zero-order-holder (ZOH)

Dimensionless time $T=t/\tau$



Modeling sampling

Sampling time is τ, the j^{th} sampling instant is $t_j = j\tau$

$$Q(t)\equiv -P\big(ky(t_j-\tau)-F_d\big)+ky(t_j-\tau),\quad t\in[t_j, t_j+\tau)$$

Natural frequency: $f_n=\omega_n/(2\pi)=\sqrt{k/m}/(2\pi)$

Sampling frequency: $f_s = 1/\tau$ time: $T=t/\tau$

Dimensionless equations of motion: $T\in[j, j+1)$

$$x''(T)+(\omega_n\tau)^2x(T)=(\omega_n\tau)^2(1-P)x(j-1)$$

$$x(T)=x_h(T)+x_p(T)=$$ $x(j), x'(j)\Rightarrow B_1, B_2$

$$B_1\cos(\omega_n t)+B_2\sin(\omega_n t)+(1-P)x(j-1)$$

Stability of digital force control

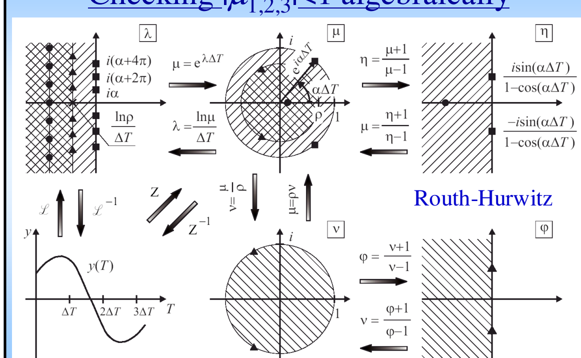
$$\mathbf{z}^j=\begin{pmatrix} x(j-1) \\ x(j) \\ x'(j)\end{pmatrix} \Rightarrow \mathbf{z}^{j+1}=\mathbf{A}\mathbf{z}^j \Rightarrow \det(\mu\mathbf{I}-\mathbf{A})=0$$

$|\mu_{1,2,3}|<1 \Leftrightarrow$ stability

$$\mathbf{A}=\begin{pmatrix} 0 & 1 & 0 \\ (1-P)(1-\cos(\omega_n\tau)) & \cos(\omega_n\tau) & \frac{1}{\omega_n\tau}\sin(\omega_n\tau) \\ (1-P)\omega_n\tau\sin(\omega_n\tau) & -\omega_n\tau\sin(\omega_n\tau) & \cos(\omega_n\tau)\end{pmatrix}$$

Parameters: $(\omega_n\tau)/(2\pi)=f_n/f_s$ and P

Checking $|\mu_{1,2,3}|<1$ algebraically

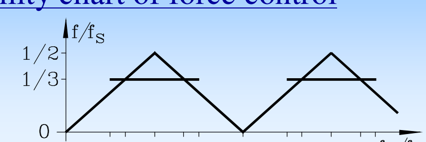


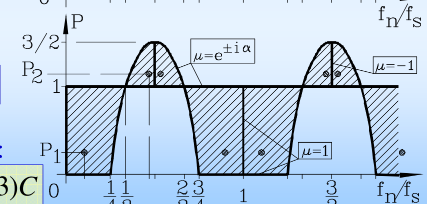
Stability chart of force control

Vibration frequency: $0<f<f_s/2$

Maximum gain: $P_{max}=1.5$

Minimum force error: $\Delta_{F,\min}\geq(2/3)C$



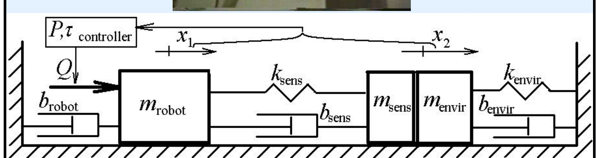


Conclusions on digital force control

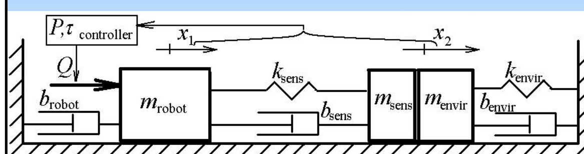
- All the 3 kinds of co-dimension 1 bifurcations arise in digital force control (Neimark-Sacker, flip, fold)
- Application of differential gain leads to loss of stable parameter regions
- Force derivative signal can be filtered with the help of sampling, but stability properties do not improve
- Do not use differential gain in force control

Stability problems along the blade

Turbine blade polishing



Mechanical model of polishing



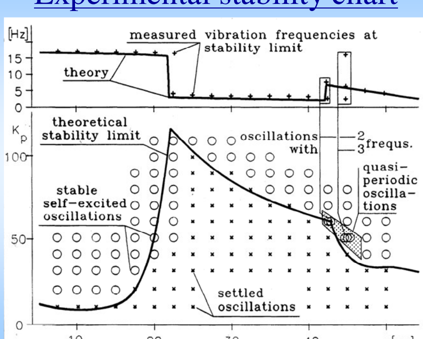
$m_r = 2500$ [kg] $b_r = 32$ [Ns/mm] $C = 150$ [N]

$m_s = 0.95$ [kg] $b_s = 2$ [Ns/m(!)] $k_s = 45$ [Ns/mm]

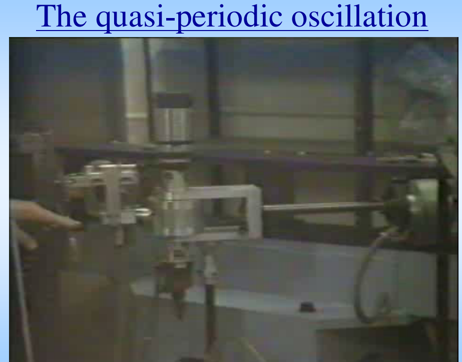
$m_e = 4.43$ [kg] $b_e = 3$ [Ns/m(!)] $k_s = 13$ [Ns/mm]

$F_d = 50$ [N]

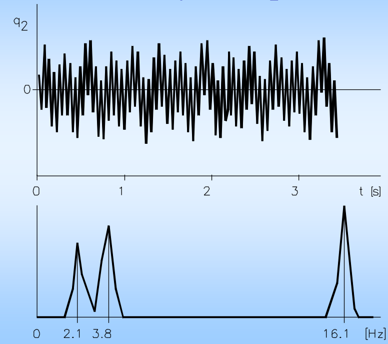
Experimental stability chart



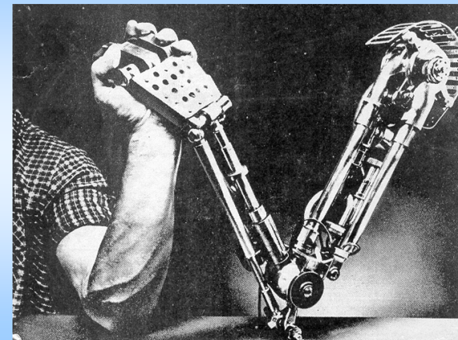
The quasi-periodic oscillation



Time-history and spectrum

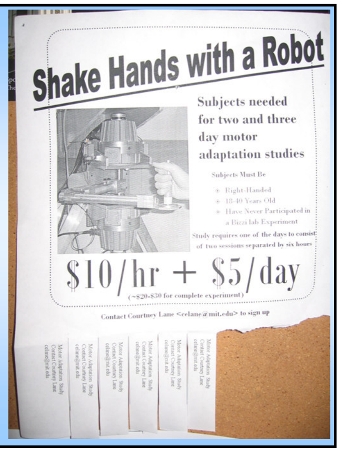


Human-robotic force control



Human-robotic force control

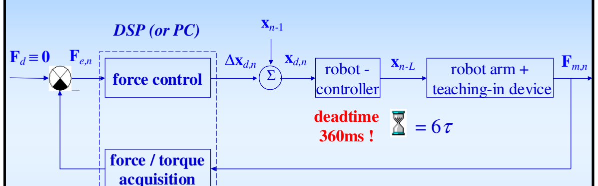
video



Delay and vibrations

Delay and vibrations

Outer-loop force control in RehaRob



Sampling time at outer loop with $\tau \approx 60$ [ms]

Sampling time at force sensor with $\Delta t \approx 4$ [ms]

Robotic rehabilitation

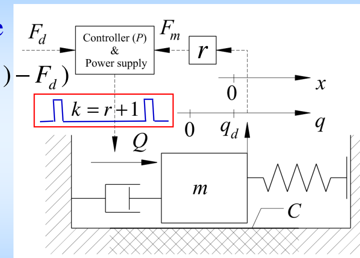
Force control model with large delay $r\Delta t$

Constant gain case

$$Q(t) = F_d - P(F_m(t_{j-r}) - F_d)$$

Steady force error

$$F_e = \frac{C}{1+P}$$

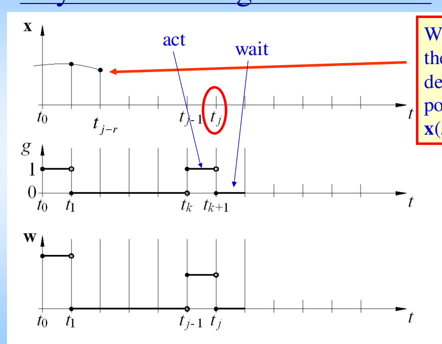


Act-and-wait

$$Q_{a\&w}(t) = \begin{cases} F_d - P(F_m(t_{j-r}) - F_d) & \text{if} \quad t \in [t_{hk}, t_{hk+1}), \quad h \in Z \\ F_d & \text{otherwise} \end{cases}$$

Physical meaning of act & wait

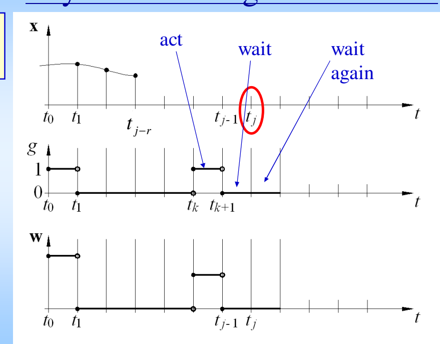
$r = 4$
$k = 5$



We know the delayed position $\mathbf{x}(t_{j-r})$!

Physical meaning of act & wait

$r = 4$
$k = 5$



Physical meaning of act & wait

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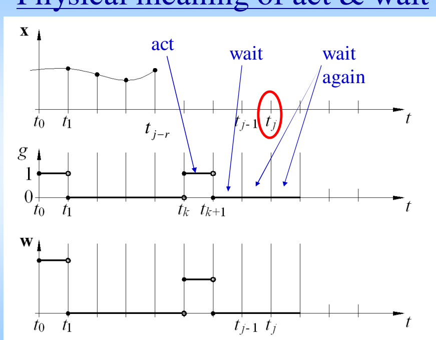

Physical meaning of act & wait

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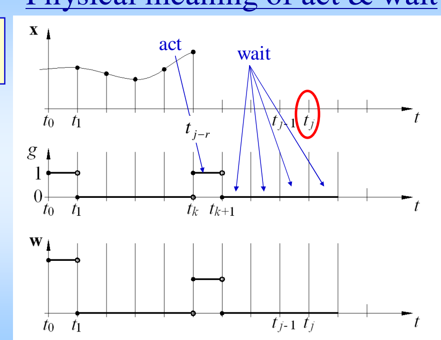

Physical meaning of act & wait

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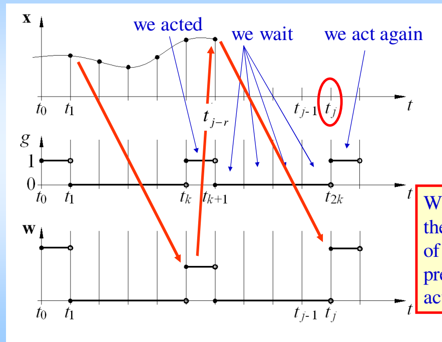
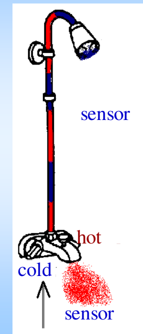


We know the result of our previous action!

Physical interpretation of act & wait



"Good memory causes trouble"

Regulating the shower temperature

large delay – small gains
small delay – large gains

Better solution for large delay – regulate taps with large gain then wait, and wait, and wait, etc...

Linear MIMO system with act & wait

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{w}(j-r), \quad t \in [j, j+1)$$
$$\mathbf{w}(j) = g_j \mathbf{H}\mathbf{x}(j), \quad \text{where} \quad \mathbf{H} = \mathbf{DC}$$
$$g_j = \begin{cases} 1 & \text{if} \quad j = hk, \quad h \in Z \\ 0 & \text{otherwise} \end{cases}$$

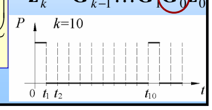
… and again, step-by-step solution:

$$\mathbf{x}(j+1) = \mathbf{Q}\mathbf{x}(j) + \mathbf{R}\mathbf{w}(j-r), \quad \text{with} \quad \mathbf{Q} = \mathrm{e}^{\mathbf{A}}, \quad \mathbf{R} = \mathrm{e}^{\mathbf{A}} \int_0^1 \mathrm{e}^{-\mathbf{A}s} \mathrm{d}\, s\, \mathbf{B}$$

$$\underbrace{\begin{pmatrix} \mathbf{x}(j+1) \\ \mathbf{w}(j) \\ \mathbf{w}(j-1) \\ \vdots \\ \mathbf{w}(j-r+1) \end{pmatrix}}_{\mathbf{z}_{j+1}} = \underbrace{\begin{pmatrix} \mathbf{Q} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R} \\ g_j\mathbf{H} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{pmatrix}}_{\mathbf{G}_j} \underbrace{\begin{pmatrix} \mathbf{x}(j) \\ \mathbf{w}(j-1) \\ \mathbf{w}(j-2) \\ \vdots \\ \mathbf{w}(j-r) \end{pmatrix}}_{\mathbf{z}_j}$$

Thus,

$$\mathbf{z}_{j+1} = \mathbf{G}_j \mathbf{z}_j,$$
$$\mathbf{z}_k = \mathbf{G}_{k-1} \ldots \mathbf{G}_1 \mathbf{G}_0 \mathbf{z}_0$$



Linear MIMO system with act & wait

The structure of the coefficient matrices:

$$\mathbf{G}_0 = \begin{pmatrix} \mathbf{Q} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R} \\ \mathbf{H} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{G}_1 = \ldots = \mathbf{G}_{k-1} = \begin{pmatrix} \mathbf{Q} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{pmatrix}$$

$$\mathbf{z}_{j+1} = \mathbf{G}_j \mathbf{z}_j, \quad \mathbf{z}_k = \mathbf{G}_{k-1} \ldots \mathbf{G}_1 \mathbf{G}_0 \mathbf{z}_0 = \underbrace{\mathbf{G}_1^{\,k-1} \mathbf{G}_0}_{\mathbf{\Phi}_{r,k}} \mathbf{z}_0$$

Algebraic structure of the map

$k = 1$

$$\mathbf{\Phi}_{r,1} = \mathbf{G}_0 = \begin{pmatrix} \mathbf{Q} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R} \\ \mathbf{H} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{pmatrix}$$

Algebraic structure of the map

$k = 2$

$$\mathbf{\Phi}_{r,2} = \mathbf{G}_1 \mathbf{G}_0 = \begin{pmatrix} \mathbf{Q} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{Q} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R} \\ \mathbf{H} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{pmatrix}$$

Algebraic structure of the map

$k=2$

$$\Phi_{r,2}=\begin{pmatrix} \mathbf{Q}^2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R} & \mathbf{QR} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{H} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

← zero row

Algebraic structure of the map

$k=r$

$$\Phi_{r,r}=\mathbf{G}_1^{r-1}\mathbf{G}_0=\begin{pmatrix} \mathbf{Q}^r & \mathbf{R} & \mathbf{QR} & \cdots & \mathbf{Q}^{r-1}\mathbf{R} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{H} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$$

(n-1) zero rows

Algebraic structure of the map

$k=r+1$

$$\Phi_{r,r+1}=\mathbf{G}_1^{\,r}\mathbf{G}_0=\begin{pmatrix} \mathbf{Q}^{r+1}+\mathbf{RH} & \mathbf{QR} & \mathbf{QR} & \cdots & \mathbf{Q}^{r-1}\mathbf{R} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$$

Feedback history is eliminated…
Stability is determined by the n-dimensional matrix: $\mathbf{Q}^{r+1}+\mathbf{RH}$

Algebraic structure of the map

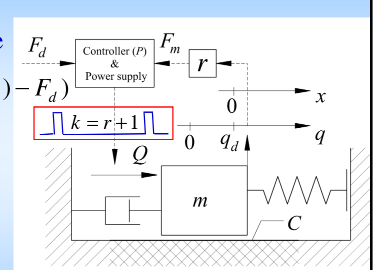
$k\geq r+1$

$$\Phi_{r,k}\big|_{k\geq r+1}=\begin{pmatrix} \mathbf{Q}^{k}+\mathbf{Q}^{k-r-1}\mathbf{RH} & \mathbf{Q}^{k-r}\mathbf{R} & \mathbf{Q}^{k-r+1}\mathbf{R} & \cdots & \mathbf{Q}^{k-1}\mathbf{R} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$$

Force control model with large delay $r\Delta t$

Constant gain case

$$Q(t)=F_d-P(F_m(t_{j-r})-F_d)$$



Steady force error

$$F_e=\frac{C}{1+P}$$

Act-and-wait

$$Q_{a\&w}(t)=\begin{cases} F_d-P(F_m(t_{j-r})-F_d) & \text{if} \quad t\in[t_{hk},t_{hk+1}), \quad h\in Z \\ F_d & \text{otherwise} \end{cases}$$

MIMO force control model – digital effects

Perturbation at desired force: $q(t)=q_d+x(t)$

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^{\,2}x(t)=-g_j\omega_n^{\,2}Px(t_{j-r})$$

$$t\in[t_j,t_{j+1}), \quad t_j=j\Delta t$$

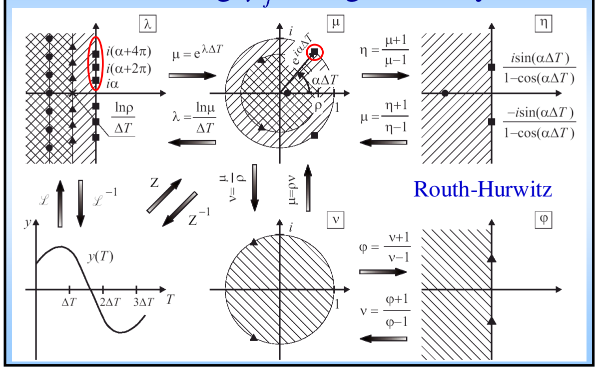
Discrete state space formulation

$$\mathbf{x}(j+1)=\mathbf{Qx}(j)+\mathbf{Rw}(j-r) \qquad \mathbf{Q}=\exp(\mathbf{A}\Delta t), \quad \mathbf{H}=[-\omega_n^2P \;\; 0]$$

$$\mathbf{w}(j)=g_j\mathbf{Hx}(j) \qquad \mathbf{R}=(\exp(\mathbf{A}\Delta t)-\mathbf{I})\mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{x}(t)=\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \; \mathbf{A}=\begin{bmatrix} 0 & 1 \\ -\omega_n^{\,2} & -2\zeta\omega_n \end{bmatrix}, \; \mathbf{B}=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \;\Rightarrow\; \mathbf{G}_j \;\Rightarrow\; \Phi_{r,k}$$

Checking $|\mu_j|<1$ algebraically

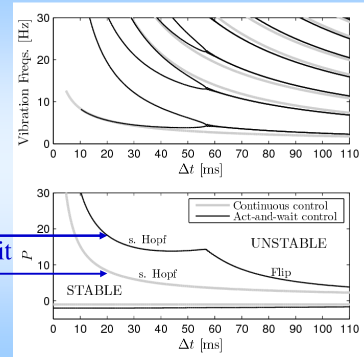


Continuous vs. act-and-wait control

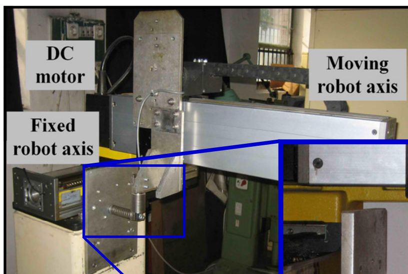
Theoretical predictions

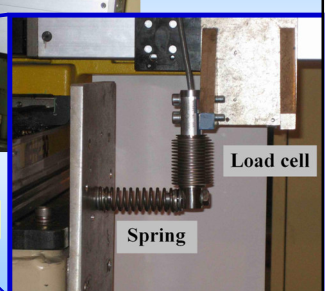
$r = 1, \; k = 2$

Increase of stability limit



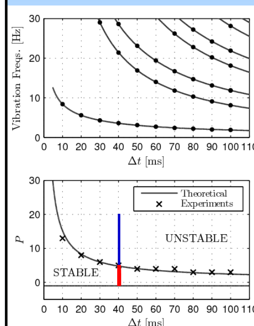
Experiments



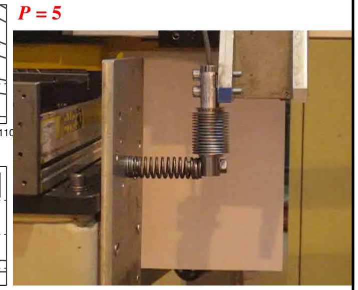


Hirata (MB-H180-500)

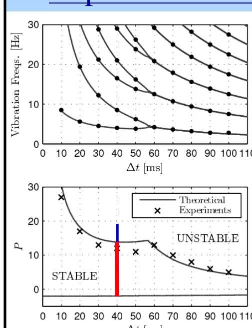
Experiments – constant gain controller



***P* = 5**

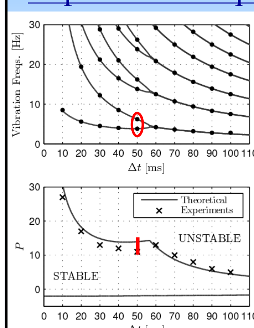


Experiments – act-and-wait controller

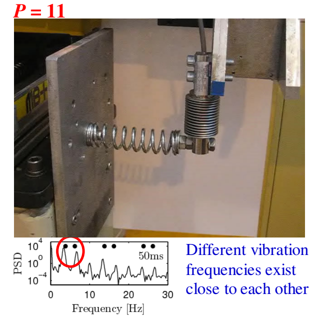


***P* = 12**

Experiments – quasi-periodic oscillations



***P* = 11**



Different vibration frequencies exist close to each other

I Robot