Delay differential equations with engineering applications

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Contents

Delay equations arise in mechanical systems…

… by the information system (of control), and by the contact of bodies.

- Linear stability & subcritical Hopf bifurcations
- Force control and balancing – human and robotic
- Contact problems
  - **Shimmying wheels (trucks and motorcycles)**
  - Machine tool vibrations

Main references

Stepan, G., Chaotic motion of wheels, *Vehicle System Dynamics* **20** (1991) 341-351.

Goodwine, B., Stepan, G., Controlling unstable rolling phenomena, *Journal of Vibration and Control* **6** (2000) 137-158.

Takacs D, Stepan G, Hogan SJ, Isolated large amplitude periodic motions of towed rigid wheels, *Nonlinear Dynamics* **52** (2008) 27-34.

Takacs D, Orosz G, Stepan G, Delay effects in shimmy dynamics of wheels with stretched string-like tyres, *European Journal of Mechanics A – Solids* **28** (2009) 516-525.

Takacs D, Stepan G, Experiments on quasiperiodic wheel shimmy, *ASME Journal of Computational and Nonlinear Dynamics* **4** (2009) Article number 031007.

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- **Stability and bifurcations in RFDEs**
- Shimmy (motorcycles, airplanes…)
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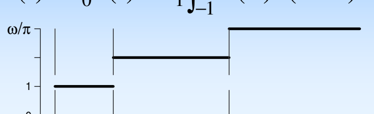
Delayed oscillators

$$w(\vartheta) = \delta(\vartheta + 1)$$

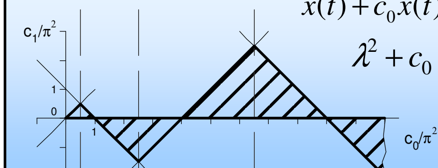
$$\ddot{x}(t) + c_0 x(t) = c_1 \int_{-1}^{0} w(\vartheta) x(t+\vartheta) \mathrm{d}\vartheta$$



vibration frequencies

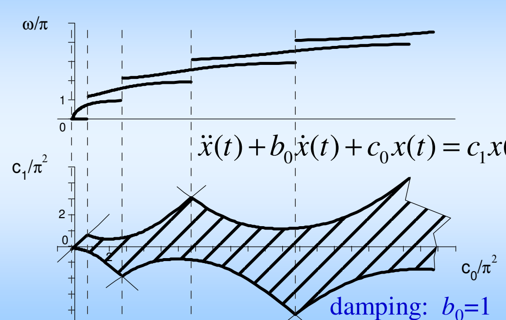
$$\ddot{x}(t) + c_0 x(t) = c_1 x(t-1)$$

$$\lambda^2 + c_0 - c_1 \mathrm{e}^{-\lambda} = 0$$



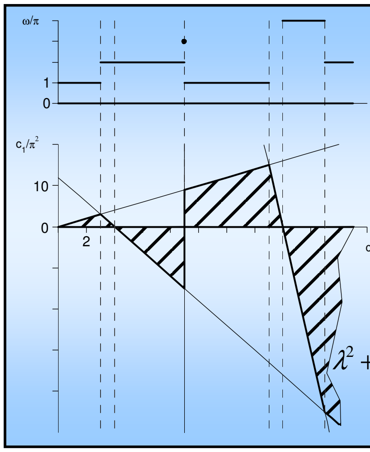
stability chart

Delayed oscillator with damping



$$\ddot{x}(t) + b_0 \dot{x}(t) + c_0 x(t) = c_1 x(t-1)$$

damping: b_0=1



$$\ddot{x}(t) + c_0 x(t) = c_1 \int_{-1}^{0} w(\vartheta) x(t+\vartheta) \mathrm{d}\vartheta$$

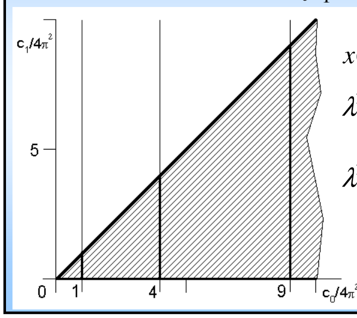
$$w(\vartheta) = -\frac{\pi}{2}\sin(\pi\vartheta)$$

$$\lambda^2 + c_0 - c_1 \frac{\pi^2}{2} \frac{1+\mathrm{e}^{-\lambda}}{\lambda^2+\pi^2} = 0$$

Stability chart !!!

$$w(\vartheta) \equiv 1$$

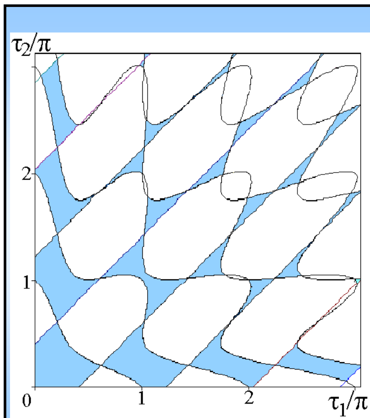
$$\ddot{x}(t) + c_0 x(t) = c_1 \int_{-1}^{0} x(t+\vartheta) \mathrm{d}\vartheta$$



$$x(t) = Ae^{\lambda t} \Rightarrow$$

$$\lambda^2 + c_0 = c_1 \int_{-1}^{0} e^{\lambda\vartheta} \mathrm{d}\vartheta \Rightarrow$$

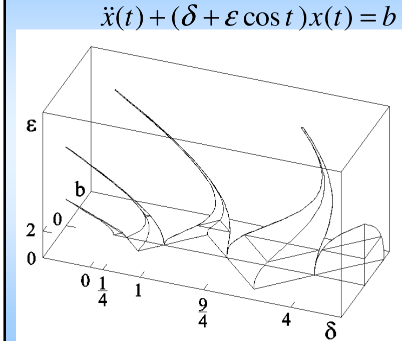
$$\lambda^2 + c_0 - c_1 \frac{1-e^{-\lambda}}{\lambda} = 0$$



$$\ddot{x}(t) + 6x(t) = x(t-\tau_1) + x(t-\tau_2)$$

Stability chart of delayed Mathieu

$$\ddot{x}(t) + (\delta + \varepsilon\cos t) x(t) = b\, x(t-2\pi)$$



Insperger, Stépán (2002)

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Shimmy – the dance (1920's…)

Shimmy – hospital trolley

Shimmy – Formula-1 trolley

Shimmy – motorcycle

Shimmy – motorcycle

Quasi-periodicity!

Mechanical degrees of freedom?

video1
video2
video3

Shimmy damper

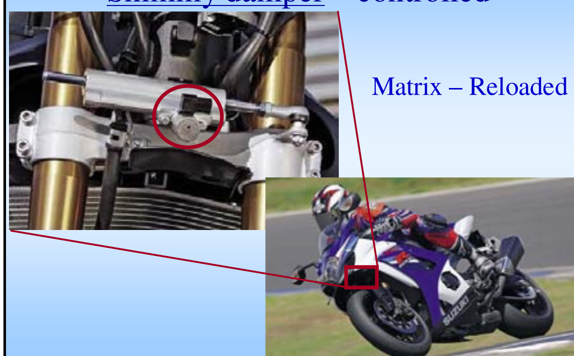
Shimmy damper on motorcycles

Shimmy damper on a Honda

Violent shimmy – ‘tank slapper’

Shimmy damper – manual tuning

Shimmy damper – controlled



Matrix – Reloaded

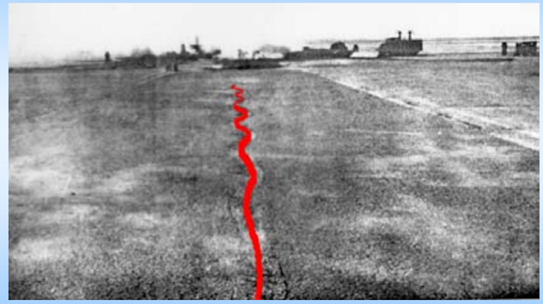
Shimmy on airplanes

Shimmy on airplanes

Hawker 125

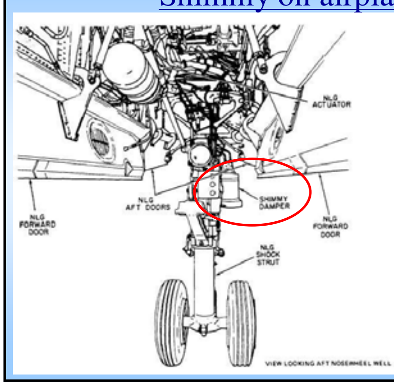
Shimmy on airplanes

Shimmy on airplanes



Shimmy on airplanes

Shimmy on airplanes



Concorde 2001

Articulated bus, trailer, RV, jeep

*video*1 *video2* *video3* *video4*

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Early publications on shimmy

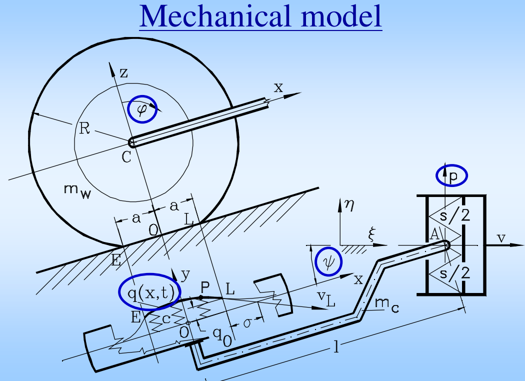
- Pacejka HB, *The Wheel Shimmy Phenomenon*. PhD thesis, TU Delft, 1966.
- Smiley RF, Correlation, evaluation, and extension of linearized theories for tyre motion and wheel shimmy. *Technical report* **1299** (1957) submitted to National Advisory Comm. for Aeronautics.
- von Schlippe B, Dietrich R, Shimmying of a pneumatic wheel. *Lilienthal-Gesellschaft für Luftfahrtforschung* **140** (1941) pp 125-160. (translated for the AAF in 1947)

Early publications on shimmy

- De Lavaud DS, Shimmy, pseudo-shimmy and tramp of an automobile, *C.R. Acad. Sci.* **185** (1927) pp. 254–257.
- Broulhiet G, The suspension of the automobile steering mechanism: Shimmy and tramp, *Bull Soc. Ing. Civ. Fr.* **78** (1925) pp. 540–554.
- Ramsesses II, The battle of Kadesh. *Transactions of the Luxor Society* **3** (1274 BC) ww 6-7.

Early publications on shimmy

Mechanical model



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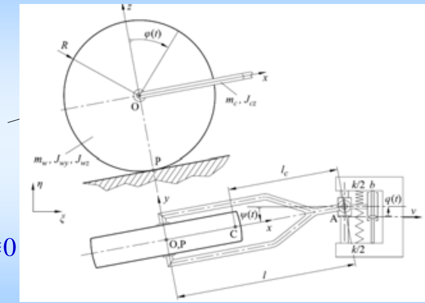
Single contact point model – rigid wheel

video



$q(t) := q(0,t) \equiv 0$

p ≠ 0

Viscous damping at king pin

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Nonlinear vibrations – without delay

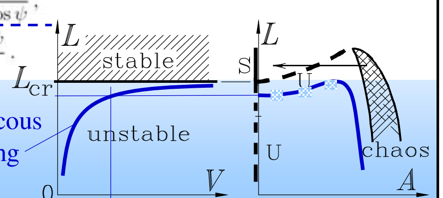
$$\dot\psi = \Omega\,,$$

$$\dot\Omega = -\frac{\frac{v}{l}\left(\frac{1}{\cos^2\psi} - \frac{1}{2} + \frac{3m_w}{2m_c}\tan^2\psi\right)\Omega + \frac{s}{lm_c}p + \left(1 + \frac{3m_w}{2m_c}\right)\frac{\tan\psi}{\cos\psi}\Omega^2}{\left(\frac{1}{3} + \tan^2\psi\right)\cos\psi + \frac{m_w}{4m_c}\left(\frac{R^2}{l^2}\cos\psi + 6\tan^2\psi\cos\psi\right)}\,,$$

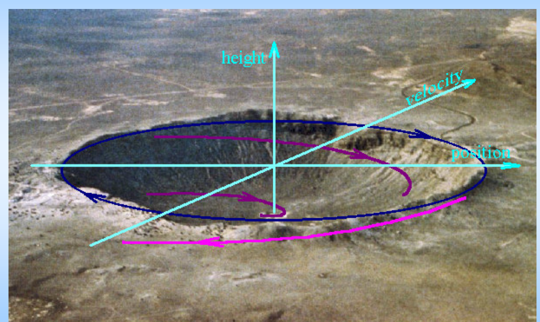
$$\dot p = v\tan\psi + \frac{\Omega l}{\cos\psi}\,,$$

$$\dot\varphi = \frac{v + \Omega l\sin\psi}{R\cos\psi}\,.$$

With viscous damping

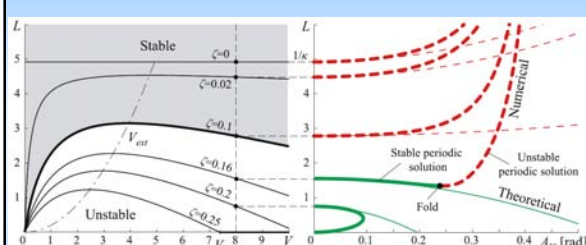


Unstable limit cycle – "ghost" vibration



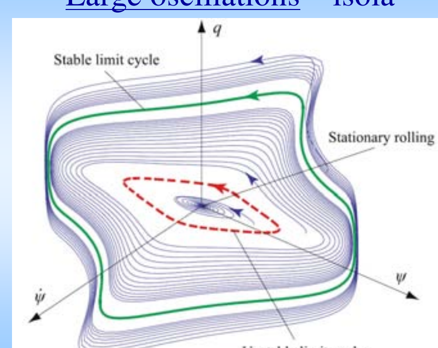
video

Large damping, high speed

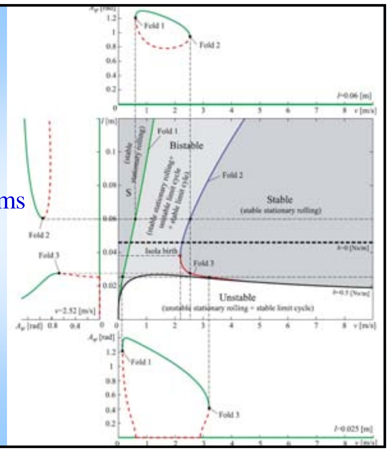


Takacs,Stepan,Hogan (*J Nonl Dyn*, 2007)

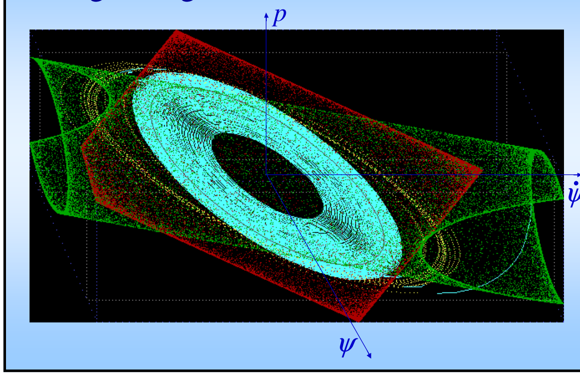
Large oscillations – isola



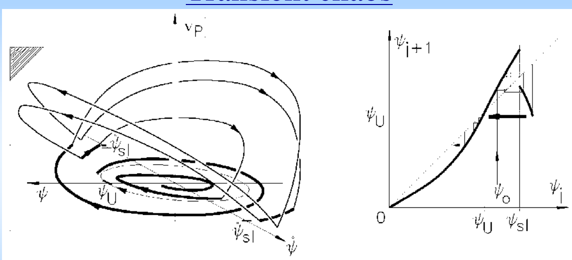
Stability chart & bifurcation diagrams for realistic parameters



Sliding at large oscillation – transient chaos



Transient chaos



Non-conservative system –
even without viscous damping!

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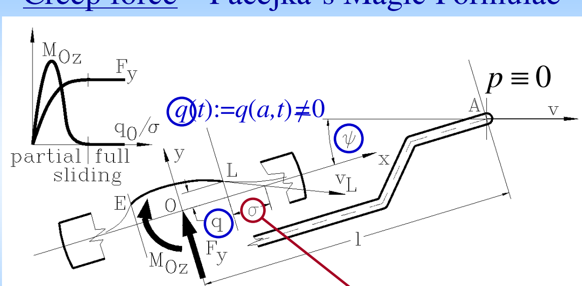
Creep force – Pacejka's Magic Formulae

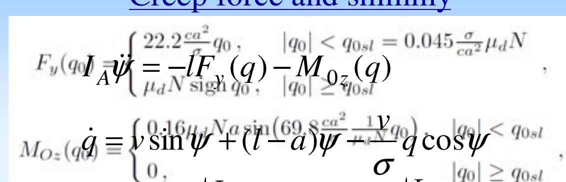


Stationary idea inserted in dynamic model
Stretched string model of tyre – relaxation length

Creep force and shimmy

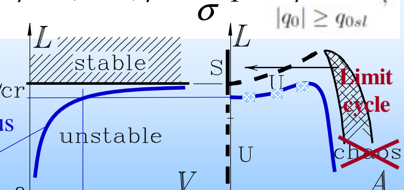
$$I_A\ddot\psi = -lF_y(q) - M_{0z}(q)$$

$$\dot q \equiv v\sin\psi + (l-a)\dot\psi - \frac{v}{\sigma}q\cos\psi$$



$$1+\frac{\sigma}{a} = L_{cr}$$

With viscous damping

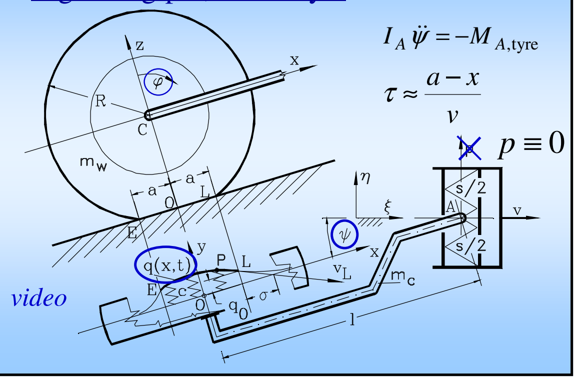


Rigid king-pin, elastic tyre – brush model

$$I_A \ddot{\psi} = -M_{A,\mathrm{tyre}}$$

$$\tau \approx \frac{a-x}{v}$$

$$p \equiv 0$$



video

Governing equations & memory effect

$$I_A \ddot{\psi}(t) = -c\int_{-a}^{a}(l-x)q(x,t)\mathrm{d}x \qquad p \equiv 0$$

$$\dot{q}(x,t) = v\psi(t) + (l-x)\dot{\psi}(t) + q'(x,t)v + \text{h.o.t.}$$

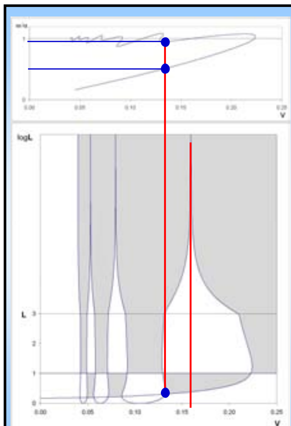
$$x \in [-a,a], \quad t \in [t_0,\infty), \quad \text{and} \quad q(a,t) = 0$$

Traveling wave solution of the PDE:

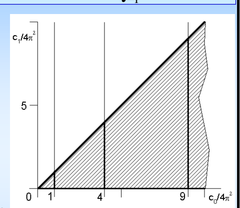
$$q(x,t) = (a-x)\psi(t) + (l-a)(\psi(t) - \psi(t - \underbrace{\tfrac{a-x}{v}}_{=\tau})) + \ldots$$

$$V^2\ddot{\psi}(t) + \psi(t) = \frac{L-1}{L^2+1/3}\int_{-1}^{0}(L-1-2\vartheta)\psi(t+\vartheta)\mathrm{d}\vartheta + \ldots$$

$$V = \frac{v}{2a\omega_n}, \quad L = \frac{l}{a}, \quad \omega_n = \frac{2ac(l^2+a^2/3)}{I_A}$$



$$\ddot{x}(t) + c_0 x(t) = c_1\int_{-1}^{0} x(t+\vartheta)\mathrm{d}\vartheta$$

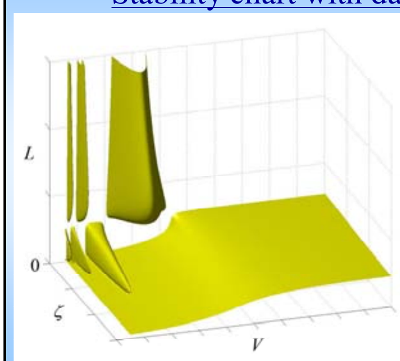


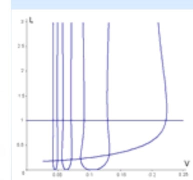
$$L \to \infty$$

$$V^2\ddot{\psi}(t) + \psi(t) = \int_{-1}^{0}\psi(t+\vartheta)\mathrm{d}\vartheta$$

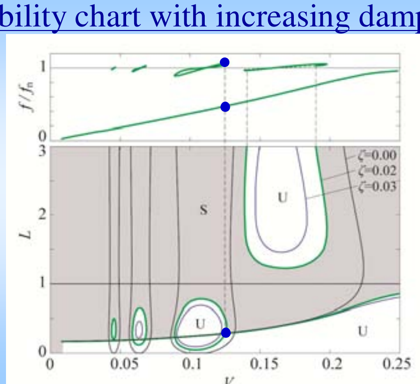
$$V \neq \frac{1}{2\pi} \approx 0.159, \frac{1}{4\pi}, \frac{1}{6\pi}, \ldots$$

Stability chart with damping



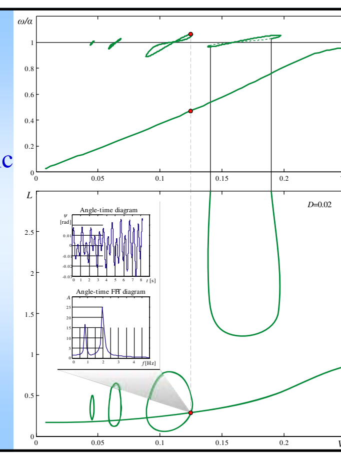


Stability chart with increasing damping

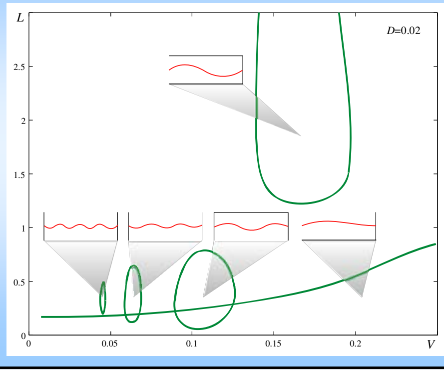


Simulations

to find quasi-periodic oscillations



Simulation



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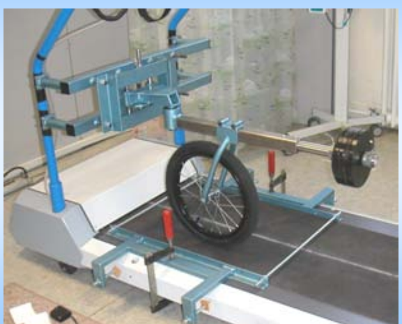
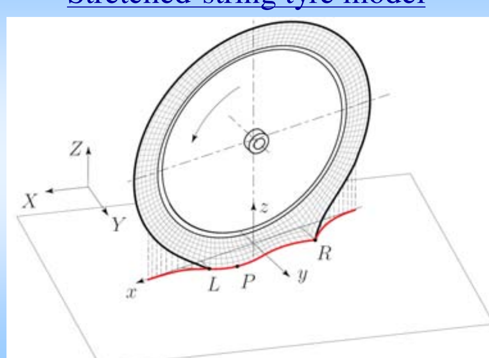
Experiments on conveyor belt



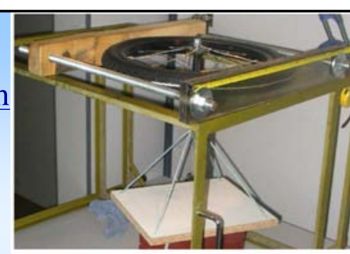
Experiments on lateral stiffness

video

Stretched-string tyre model

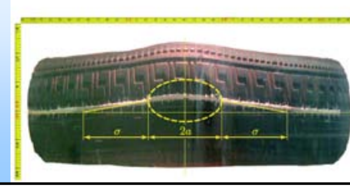


Relaxation length measurement

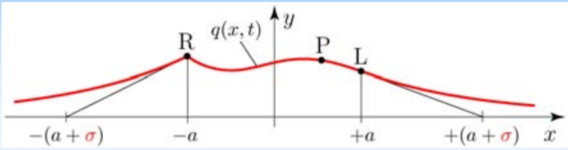


$\sigma = 1.8a$

$\Sigma := \dfrac{\sigma}{a} = 1.8$



Stretched-string tyre model and delay

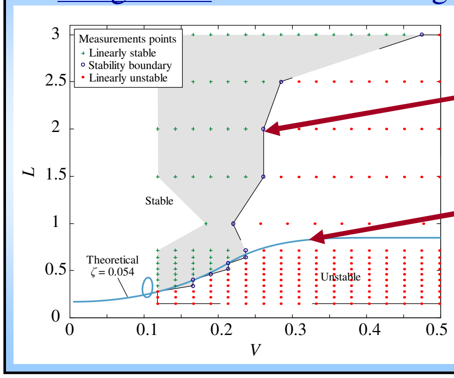


$$q(x,t) = \begin{cases} q(a,t)\,\mathrm{e}^{-(x-a)/\sigma}, & \text{if} \quad x \in [a,\infty), \\ q(-a,t)\,\mathrm{e}^{(x+a)/\sigma}, & \text{if} \quad x \in (-\infty,-a] \end{cases}$$

$$\dot{q}(x,t) = v\sin\psi(t) + (l-x)\dot{\psi}(t) + q'(x,t)\Big(v\cos\psi(t) - q(x,t)\dot{\psi}(t)\Big)$$

'No kink' at leading edge ⇒ $q'(a,t) = -\dfrac{q(a,t)}{\sigma}$

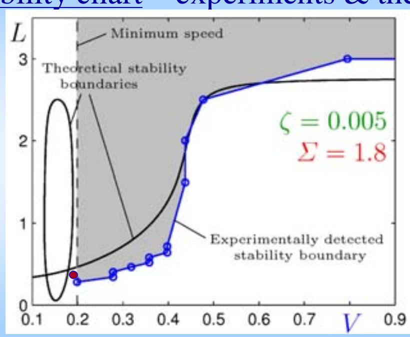
Long caster – stretched-string model



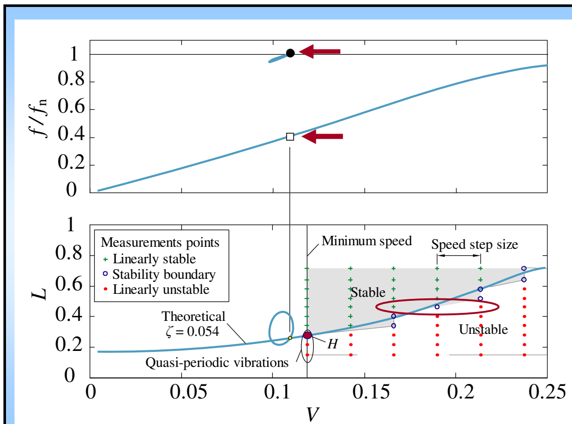
experiments

Brush model (theory)

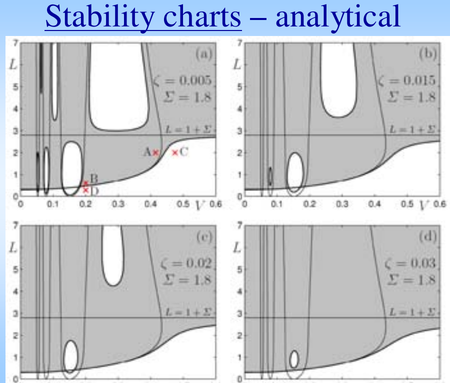
Stability chart – experiments & theory

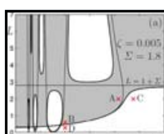


Takacs,Orosz,Stepan (*Eur J Mech A*, 2009)

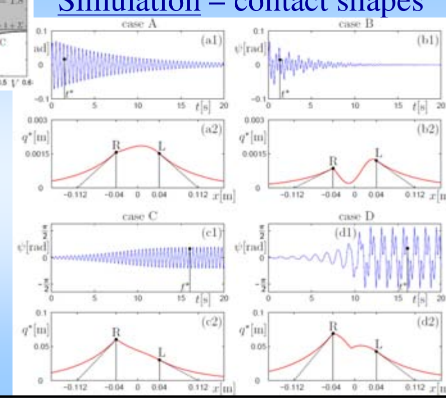


Stability charts – analytical

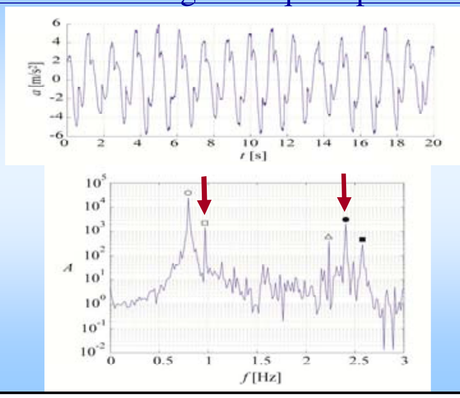




Simulation – contact shapes

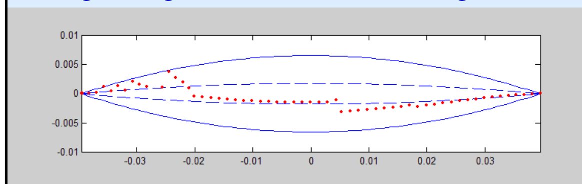


Acceleration signal at quasi-periodicity



Sticking-sliding in the contact-patch

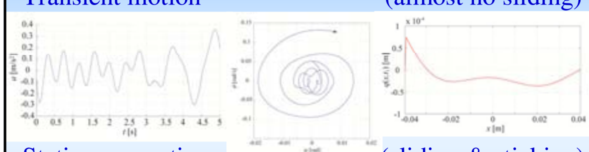
Coefficient of static friction 0.7 does not provide large enough friction forces for sticking



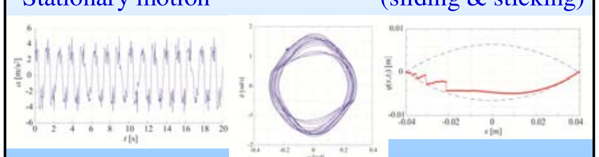
When tread points sliding, friction force is limited by coefficient of dynamic friction 0.2

Simulation of sliding as nonlinearity

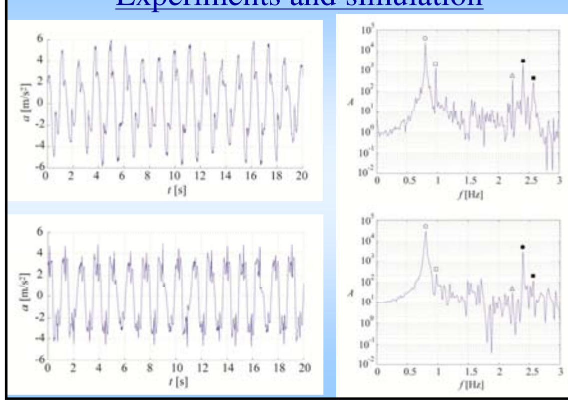
Transient motion (almost no sliding)



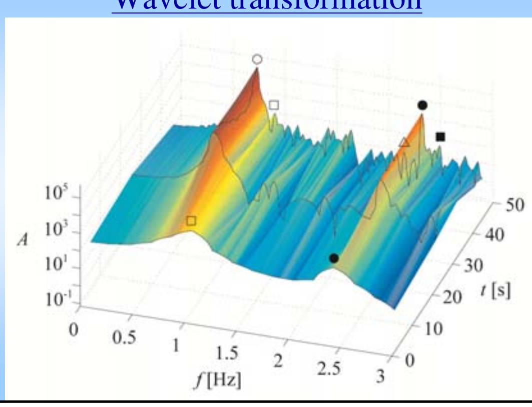
Stationary motion (sliding & sticking)

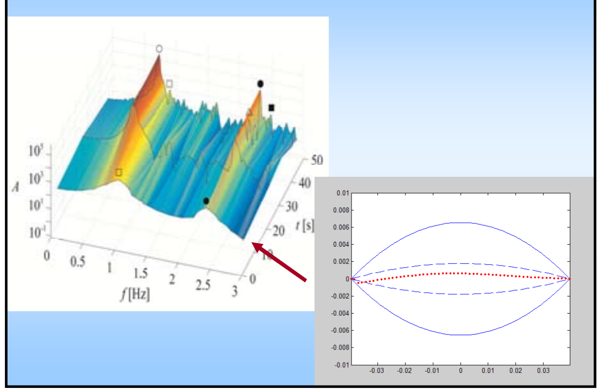


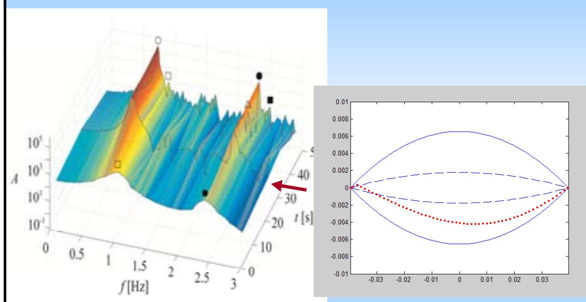
Experiments and simulation

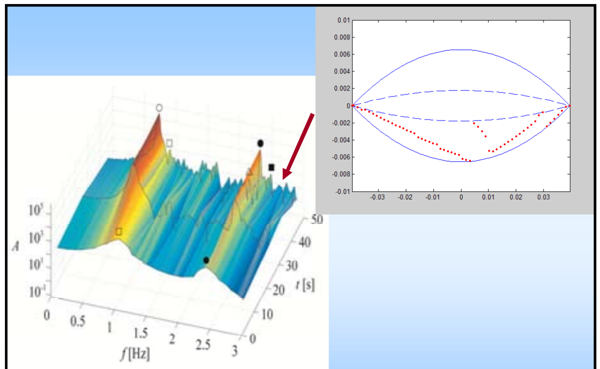


Wavelet transformation









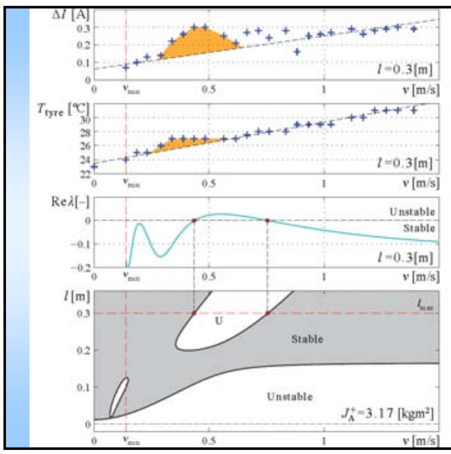
Takacs,Stepan (*ASME J Comp & Nonlin Dynamics*, 2009)

Nonlinear RFDE for shimmy

$$V^2\ddot{\psi}(t)+\psi(t)-\frac{7}{6}\psi^3(t)=$$
$$\frac{L-1}{L^2+1/3}\Big(\int_{-1}^{0}(L-1-2\vartheta)\psi(t+\vartheta)\mathrm{d}\vartheta+f(\psi_t)+g(\psi_t,\dot{\psi}_t)\Big)$$

$$\begin{aligned}
f(\psi_t) &= -\tfrac{5}{2}\psi^2(t)\textstyle\int_{-1}^{0}(L-1-2\vartheta)\psi(t+\vartheta)\mathrm{d}\vartheta\\
&\quad+2\psi(t)\textstyle\int_{-1}^{0}(L-1-\vartheta)\psi^2(t+\vartheta)\mathrm{d}\vartheta\\
&\quad-\tfrac{2}{3}\textstyle\int_{-1}^{0}(L-1-\tfrac{1}{2}\vartheta)\psi^3(t+\vartheta)\mathrm{d}\vartheta\,,\\
g(\psi_t,\dot{\psi}_t) &= \tfrac{1}{2}\psi^2(t)\textstyle\int_{-1}^{0}(L-1-2\vartheta)^2\dot{\psi}(t+\vartheta)\mathrm{d}\vartheta\\
&\quad-\psi(t)\textstyle\int_{-1}^{0}(L-1-\vartheta)(L-1-2\vartheta)\psi(t+\vartheta)\dot{\psi}(t+\vartheta)\mathrm{d}\vartheta\\
&\quad+\tfrac{1}{2}(L-1)\textstyle\int_{-1}^{0}(L-1-2\vartheta)\psi^2(t+\vartheta)\dot{\psi}(t+\vartheta)\mathrm{d}\vartheta\,.
\end{aligned}$$

It is likely to be subcritical, again – video

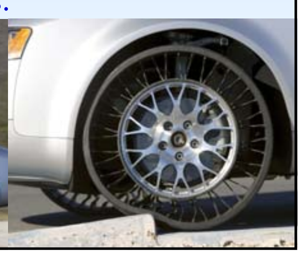


*video*1

*video*2

Conclusion

Quasi-periodic oscillations and tyre micro-slips/warming is explained with nonlinear, delayed, low (mechanical) DoF wheel models.



Parking

Thank you for your attention!