

MULTI-FREQUENCY OSCILLATIONS IN DYNAMICAL SYSTEMS

Session 1. Multi-frequency Phenomena

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Course Outline

Session 1. Multi-frequency Phenomena

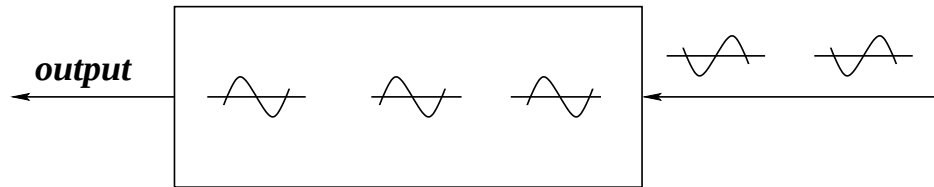
Session 2. Monotone Systems

Session 3. Damped Oscillatory Systems

Session 4. Conservative Systems

Session 5. Bio-chemical Oscillations

1. Physical Motivations



- Harmonic response to the multi-frequency in physical systems (biological, electrical, chemical, mechanical oscillators)
- Seasonal variations (population, climate models)
- Complexity caused by interaction of multi-frequency (resonance, chaos, turbulence)

2. Fundamental dynamics issues

- Regular dynamics (periodic, quasi-periodic, almost periodic)
 - Existence of quasi-periodic or almost periodic motions: Favard, Bochner, Von Neumann, Sobolev, Kolmogorov, Arnold, Moser, Levitan, Amerio, Sacker, Sell, Johnson ...
- Periodicity v.s. non-periodicity (similarity, difference)
- Transient dynamics (bifurcation, irregularity, order to chaos)
- Complex dynamics driven by multi-frequencies (mixing, transitivity, chaos)

3. Irregularity due to Multi-frequency

I. Monotone systems

- No internal frequency.
- Arise in population genetics, climate models, etc.
- Model equations: Scalar ODE

$$x' = f(x, t), \quad x \in R^1.$$

- a) For periodic time dependence, bounded solution $\implies$ periodic solution;
- b) For almost periodic time dependence,
 - bounded solution $\not\implies$ almost periodic solution;
 - Denjoy type of solutions can exist.

- Example (Denjoy 1932):

$$\dot{u} = f(u, \rho t, t), \quad u \in R,$$

where $\rho \notin Q$, $f : R \times T^2 \rightarrow R$.

- a) All solutions are bounded but none is almost periodic;
- b) Typical solution has the form $u(t) = U(\rho t, t)$, where $U : T^2 \rightarrow R^1$ is not continuous;
- c) $cl\{(U(\rho t, t) + \rho t, t) \bmod Z^2\}$ is a Cantorus (Cross sections are Denjoy cantor sets).

II. Damped oscillatory systems

- Both internal and external frequencies but oscillations lie in a compact region.
- Arise in classical mechanics, electrical network, etc.
- Model equations: Quasi-periodically forced nonlinear oscillators (van der Pol, Josephson junctions, Duffing equations, etc)

$$\ddot{u} + f(u, \dot{u}, \omega t) = 0,$$

where $\omega = (\omega_1, \dots, \omega_k)$, $f : R^2 \times T^k \rightarrow R$.

- a) Without time dependence, periodic solutions are generally expected;
- b) Even with periodic time dependence,
 - quasi-periodic solutions are generally not expected;
 - non-chaotic strange attractors can exist (Grebogi, Ott, Pelikan, Yorke, etc).

- Examples:

a) Quasi-periodically forced van der Pol equation:

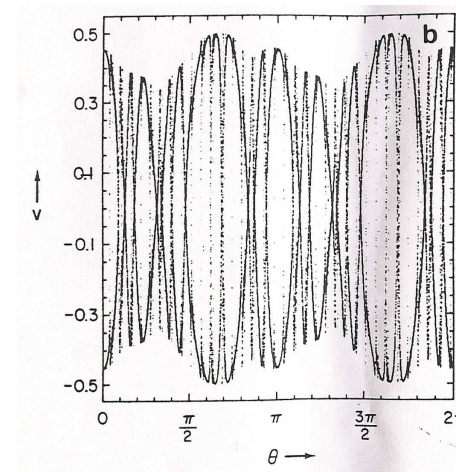
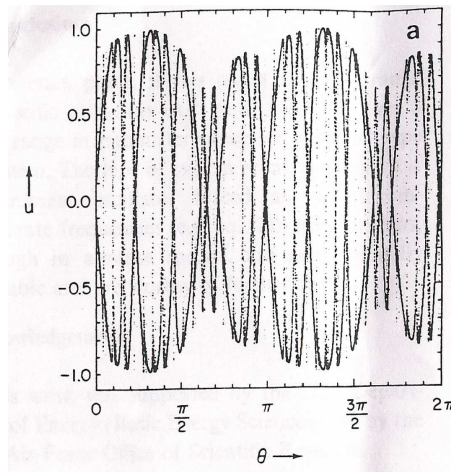
$$\ddot{u} - \alpha(1 - u^2)\dot{u} + u = \varepsilon b(u, \dot{u}, \omega t),$$

where $\alpha > 0$, $|\varepsilon| \ll 1$, $b : R^2 \times T^k \rightarrow R$.

b) Quasi-periodically forced Josephson junction equation:

$$\ddot{u} + \beta\dot{u} + \sin u = F(\omega t),$$

where $\beta \geq 2$, $F : T^k \rightarrow R$.



III. Conservative systems

- Both internal and external frequencies but oscillations can be everywhere.
- Arise in celestial mechanics, vertex dynamics and quantum physics, etc.
- Model equations: Hamiltonian systems (coupled pendulums, N-body problems, etc)

$$\begin{cases} \dot{x} = -\frac{\partial H}{\partial y}(x, y, t) \\ \dot{y} = \frac{\partial H}{\partial x}(x, y, t), \quad x, y \in R^n. \end{cases}$$

- a) In 1-degree of freedom, periodic solutions typically exist;
- b) If the degree of freedom is bigger than 1, then
 - quasi-periodic solutions need not exist especially when frequencies are close to resonance or the system is less integrable;

- Aubry-Mather sets (Cantori) may exist.
- Example (Periodically forced pendulum):

$$\ddot{u} + V_u(u, t) = 0,$$

$$V(u, t) = V(u + 1, t) = V(u, t + 1).$$

a) For $\forall \rho \notin Q$, $\exists U : T^2 \rightarrow R$ discontinuous in general, s.t.

$$u(t) = \rho t + U(\rho t, t)$$

is a solution (Aubry-Daeron 1983, Mather 1982).

b) $\text{cl}\{(u(t), \dot{u}(t), t)\} \subset T^1 \times R^1 \times T^1$ is an Aubry-Mather set (Denjoy type of Cantorus).

4. Dynamical systems

Let

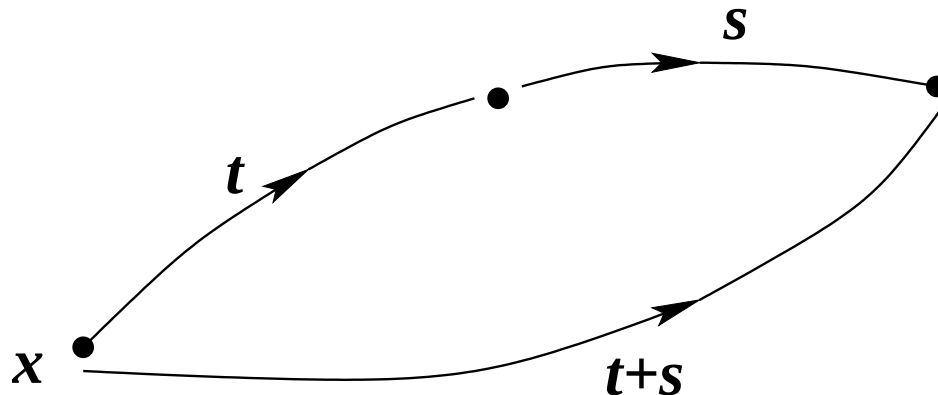
X – *phase space*, a metric space

$\mathbb{T}$ – *phase group*, an Abelian topological group

• A *dynamical system* (or *flow*) $(X, \mathbb{T})$ is an action of $\mathbb{T}$ on X which is jointly continuous and satisfies

i) $x \cdot 0 = x, \forall x \in X;$

ii) $x \cdot (t + s) = (x \cdot t) \cdot s, \forall x \in X, t, s \in \mathbb{T}.$



• **Mappings as dynamical systems:** Let $f : X \rightarrow X$ be a homeomorphism.

Then f induces a (discrete) dynamical system $(X, \mathbb{Z})$:

$$x \cdot n = f^n(x).$$

• **ODEs as dynamical systems:** Consider

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f \in C^1.$$

The equation generates a (continuous) dynamical system $(\mathbb{R}^n, \mathbb{R})$:

$$x_0 \cdot t = x(x_0, t),$$

where $x(x_0, t)$ is the solution with initial value x_0 .

- **Non-autonomous ODEs as dynamical systems:** Consider

$$\dot{x} = f(x, t), \quad x \in R^n.$$

— *Solution space:* R^n

— *Coefficient space:* $H(f) = cl\{f_s | s \in R\}$ — the *hull* of f , where $f_s(x, t) = f(x, t + s)$

— *Dynamics on coefficient space:* $(H(f), \mathbb{R})$: $g \cdot t = g_t$

— *Skew-product flow* $(R^n \times H(f), \mathbb{R})$:

$$(x_0, g) \cdot t = (x(x_0, g, t), g \cdot t)$$

where $x(x_0, g, t)$ is the solution of

$$\dot{x} = g(x, t)$$

with the initial value x_0 (Miller 1965, Sell 1971).

• **PDEs as dynamical systems:**

$$\begin{cases} u_t = \Delta u + f(u, \nabla u, x, t), & x \in \Omega, \quad t > 0, \\ u|_{\partial\Omega} = 0 \quad \text{or} \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, & t > 0. \end{cases}$$

— *Solution space:* $X \hookrightarrow C^1(\bar{\Omega})$ (e.g. a fractional power space)

— *Coefficient space:* $H(f)$

— *Skew-product semiflow:* $(X \times H(f), \mathbb{R}^+)$:

$$(U, g) \cdot t = (u(U, g, \cdot, t), g_t), \quad t > 0$$

where $u(U, g, x, t)$ solves the same equation with $g \in H(f)$ in place of f , and $u(U, g, x, 0) = U(x)$.

— If $u(U, g, \cdot, t)$ is bounded, then it exists for all $t > 0$, and $\omega(U, g)$ is compact invariant (for all $t \in \mathbb{R}$).

5. Multi-frequency Functions

Let $f : \mathbb{T} \rightarrow X$ be a continuous function.

a) Almost periodic functions

- ε -period: $\forall \varepsilon > 0$, t_ε is an ε -period if

$$d(f(t + t_\varepsilon), f(t)) < \varepsilon, \quad \forall t \in \mathbb{T}.$$

- *Almost periodic functions*: f is almost periodic (a. p.) if for $\forall \varepsilon$ $\{t_\varepsilon\}$ is relatively dense in $\mathbb{T}$ (Bohl 1893, Bohr 1925).
- f is a. p. iff for any sequence $\{t_n\} \subset \mathbb{T}$ there exists a subsequence $\{t_{n_k}\}$ such that $f(t + t_{n_k})$ converges uniformly on $\mathbb{T}$.

- *Harmonic properties:*

- *Fourier series:*

$$f(t) \sim \sum a_\lambda \mathcal{X}_\lambda(t)$$

well defined, unique, uniformly convergent.

- *Frequency module:*

$$\mathcal{M}(f) = \text{span}\{\lambda | a_\lambda \neq 0\}.$$

- *Quasi-periodic function:* If $F : T^k \rightarrow X$ is continuous and $\omega \in R^k$ is non-resonant, then $f(t) = F(\theta_0 + \omega_1 t, \dots, \theta_k + \omega_k t)$ is quasi-periodic for $\forall \theta = (\theta_0, \dots, \theta_k) \in T^k$.

- e.g. $f(t) = \sin t + \sin \sqrt{2}t$.

b) Almost automorphic functions

- ε, N -period: $\forall \varepsilon > 0$ and $\forall N \subset \mathbb{T}$ compact, $t_{\varepsilon, N}$ is an ε, N -period if

$$d(f(t + t_{\varepsilon}), f(t)) < \varepsilon, \quad \forall t \in N.$$

- *Almost automorphic function*: f is almost automorphic (a. a.) if for $\forall \varepsilon, N$ $\{t_{\varepsilon, N}\}$ is relatively dense in $\mathbb{T}$ (Levitan 1953, Bochner 1955).
- Theorem (Veech 65). The followings are equivalent:
 - f is a. a.;
 - f is a pointwise limit of a jointly a. a. sequence of a. p. functions;
 - Whenever $f(t + t_k) \rightarrow g(t)$ uniformly on compact sets, then also $g(t - t_k) \rightarrow f(t)$ uniformly on compact sets.

- *Harmonic property:*

- Fourier series well defined, non-unique, pointwise convergent (Veech 1967).

- Frequency module for an a. a. function is uniquely defined (Y. 1998).

- *Weak quasi-periodic functions:* Let $\omega \in R^k$ be non-resonant and $F : T^k \rightarrow X$ be measurable and continuous on the set $\{\omega t\} \subset T^k$. Then

$$f(t) = F(\omega_1 t, \dots, \omega_k t)$$

is a. a. if it is uniformly continuous.

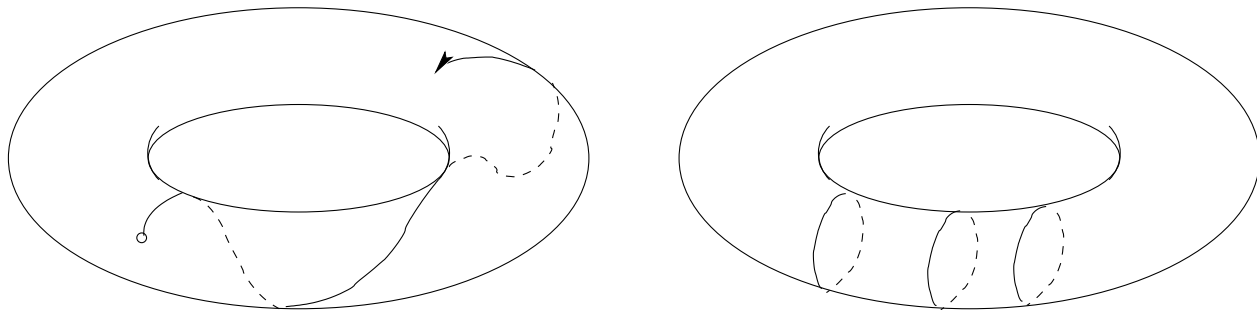
- e.g. $f(t) = \frac{2 + e^{it} + e^{i\sqrt{2}t}}{|2 + e^{it} + e^{i\sqrt{2}t}|}$.

6. Multi-frequency Flows

a) a. p. flows

- *Almost periodic minimal set*: $Y \subset X$ is an a. p. minimal set if it is a closure of an a. p. orbit.

— e.g. Quasi-periodic minimal set $(T^n, \mathbb{R})$: $\theta \cdot t = \theta + \omega t \pmod{Z^n}$, where $\omega \in \mathbb{R}^n$.



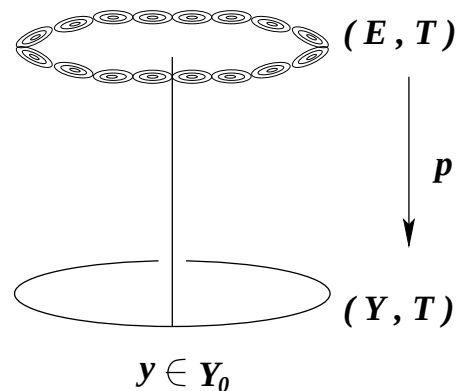
- *Structural property:* Y is a. p. minimal iff the flow induces an Abelian topological group structure on Y (Ellis 1957).
- *Harmonic property:*
 - $\mathcal{M}(Y) =: \mathcal{M}(y_0 \cdot t), y_0 \in Y;$
 - $\mathcal{M}(Y) \simeq Y'.$
- *Dynamical complexity:* An a. p. minimal set is uniquely ergodic with zero topological entropy.

b) a. a. minimal flows

- *Almost automorphic minimal set*: M is an a. a. minimal set if it is a closure of an a. a. orbit.

— e.g.: Denjoy sets and Aubry-Mather sets are a. a. minimal.

- *Structural property*: E is a. a. minimal iff it is an almost 1-cover of an a. p. minimal set Y (Veech 1965).



– If all points of E are a. a., then E is a. p.

– An almost N -cover ($N > 1$) of an a. p. minimal set need not be a. a.

- *Harmonic property:*

- $\mathcal{M}(E) =: \mathcal{M}(x_0 \cdot t)$ for an a. a. point $x_0 \in E$;

- $\mathcal{M}(E) \simeq Y'$ (Y. 1998).

- *Dynamical complexity:* a. a. flow can be non-uniquely ergodic and can admit positive topological entropy ($\mu(Y_0) = 0, 1$).

- Almost periodicity and almost automorphy are generalizations to periodicity in the strongest and weakest sense respectively.

- An a. a. motion may be interpreted as an a. p. motion covered with noise.

7. a. a. flows with zero topological entropy

- (X, π^t) – compact flow
- $\mathcal{A} = \{t_n\} \subset \mathbb{R}_+$ an increasing sequence $t_n \rightarrow +\infty$.

For any open cover $\mathcal{U}$ of X , define

$$h_{\mathcal{A}}(\mathcal{U}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log N\left(\bigvee_{i=1}^n \pi^{-t_i} \mathcal{U}\right).$$

- *Sequential entropy*:

$$h_{\mathcal{A}}(X) = \sup_{\mathcal{U}} h_{\mathcal{A}}(\mathcal{U}).$$

- *Null flow*: $(X, \mathbb{R})$ is said to be null if $h_{\mathcal{A}}(X) = 0$, $\forall \mathcal{A}$.
- Any a. p. minimal flow is null;
- A minimal null flow must be a. a. and uniquely ergodic (Dou-Huang-Ye 06).

8. a. a. flows with positive topological entropy

$S = \{s_1, \dots, s_m\}$ – symbols

$\Sigma = S^{\mathbb{Z}^d}$ – the space of symbolic arrays

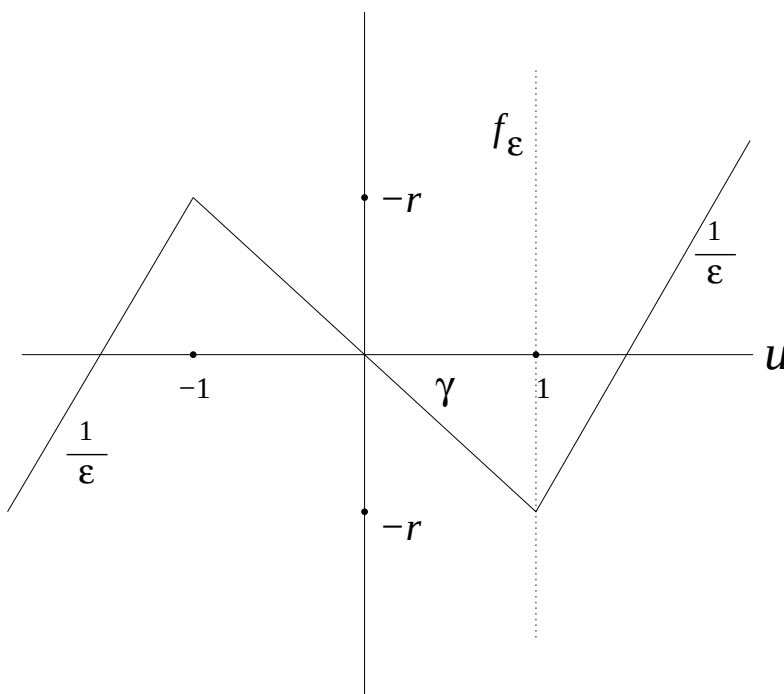
$(\Sigma, \mathbb{Z}^d)$ – full shift symbolic lattice system

— Chaotic a. a. dynamics largely exist when $d = 1$ (Furstenberg 1967, Jacobs-Keane 1969, Markley 1970, Paul 1976, Markley-Paul 1979, Williams 1984)

• **Theorem** (Berger-Siegmund-Y. 2002). Given $\rho \in [0, \log m)$, $\exists$ “residually many” a. a. minimal sets which are non-uniquely ergodic and have topological entropies $\geq \rho$.

- Pattern formation and spatial chaos – Discrete Allen-Cahn equation (Chow-Mallet-Paret-Van Vleck 95, 96):

$$\begin{cases} \dot{u}_{l,k} = \beta^+ \Delta^+ u_{l,k} + \beta^\times \Delta^\times u_{l,k} - f_\varepsilon(u_{l,k}) \\ (l,k) \in \mathbb{Z}^2. \end{cases}$$



$\varepsilon \rightarrow 0 \Rightarrow$:

$$\begin{cases} \dot{u}_{l,k} \in \beta^+ \Delta^+ u_{l,k} + \beta^\times \Delta^\times u_{l,k} - f_0(u_{l,k}) \\ (l, k) \in Z^2. \end{cases}$$

Equilibria: $\{u_{l,k}\}$ s.t.

$$\begin{cases} 0 \in \beta^+ \Delta^+ u_{l,k} + \beta^\times \Delta^\times u_{l,k} - f_0(u_{l,k}) \\ |u_{l,k}| \leq 1. \end{cases}$$

$\mathcal{S}$ solutions: Attracting equilibria assumed values $\{-1, 0, 1\}$.

Consider the full shift dynamics $(\Sigma, Z^2) = (\{-1, 0, 1\}^{Z^2}, Z^2)$. Then $\mathcal{S}$ is an invariant set.

Pattern formation: $h(\mathcal{S}) = 0$.

Spatial chaos: $h(\mathcal{S}) > 0$.

- Theorem: The following holds for some β^+, β^-, γ .
 - a) $\exists$ a. a. minimal sets which are almost 1-cover of 2-torus, non-uniquely ergodic, and admits nearly maximal entropy;
 - b) $\exists$ a family of a. a. minimal sets which are almost 1-cover of 2-solenoid, non-uniquely ergodic, whose entropy approaches to the maximal entropy.

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