MULTI-FREQUENCY OSCILLATIONS

IN DYNAMICAL SYSTEMS

Session 2. Multi-frequency Dynamics in Monotone Systems

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1. Totally Monotone Skew-product Semi-flows

$(X \times Y, \mathbb{R}) = (X \times Y, \Pi^t)$:

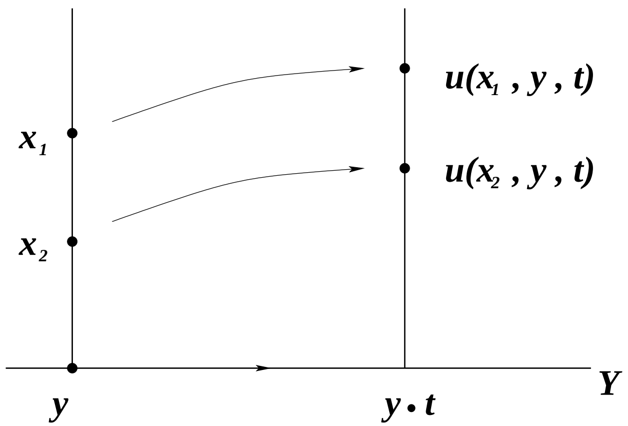
$$\Pi^t(x, y) = (u(x, y, t), y \cdot t), \ t \geq 0,$$

$--$ $(Y, \mathbb{R})$ is a. p. minimal.

$--$ $\exists$ a total ordering "$\geq$" on X s.t.

$$x_1 > x_2 \ \Rightarrow \ u(x_1, y, t) > u(x_2, y, t)$$

$\forall\, y \in Y$, $t > 0$.



- Example (parabolic PDE in one space dimension)

$$\begin{cases} u_t = u_{xx} + f(u, u_x, x, t), \quad t > 0, \quad 0 < x < 1 \\ u_x(0, t) = u_x(1, t) = 0, \quad t > 0, \end{cases}$$

where f is smooth and a. p. in t.

– The equation generates a totally monotone skew-product semiflow $\Pi = (X \times H(f), \mathbb{R}^+)$, where $X \hookrightarrow C^1(0, 1)$:

Using zero number property (Matano (1982), Angenent (1988), X. Y. Chen (1995)), define "$\geq$" on $X \times \{g\}$:

$$(U_1, g) > (U_2, g) \iff u(U_1, g, 0, t) > u(U_2, g, 0, t) \quad \text{for} \quad t \gg 1.$$

– The scalar ODE $u' = f(u, t), u \in R^1$ is a special case.

– If f is T-periodic in t, then all ω-limit set of Π is periodic minimal with period T (Chen & Matano (1989), Brunovský, Poláčik & Sandstede (1992)).

• Theorem (Shen & Yi, 1994-1996): Consider the a. p. totally monotone skew-product semiflow Π^t.

1) (ω-limit set) Each ω-limit set contains at most two minimal set;

2) (a. a.) Each minimal set is a. a.;

3) (module containment) The frequency module of any a. a. orbit is contained in that of f;

4) (ergodicity) A minimal set E is uniquely ergodic iff the residual set $Y_0 \subset H(f)$ has full Haar measure. Moreover, If E is ergodic, then $(E, \mathbb{R}) \simeq$ subflow of $(R^1 \times H(f), \mathbb{R})$;

5) (a. p.) An ω-limit set or a minimal set is a. p. if one of the following holds:

– It is uniformly stable;

– It is hyperbolic;

– $f_u \leq 0$.

• 'Proof' for scalar ODE:

$$\dot{u} = f(u,t), \quad u \in R^1$$

$\Longleftrightarrow \Pi = (R^1 \times H(f), \mathbb{R})$.

Proof of 2): Let $E \subset R^1 \times H(f)$ be a minimal set

$$p : R^1 \times H(f) \to H(f)$$

Consider $h : H(f) \to 2^E : g \mapsto E \cap p^{-1}(g)$.

h upper semi-continuous $\Longrightarrow Y_0 = \{g \in H(f) | h \text{ is continuous at } g\}$ is residual in $H(f)$.

Denote

$$a(g) = \max h(g), \quad b(g) = \min h(g), \quad g \in Y_0.$$

Let $t_n \to \infty$ be such that

$$u(a(g), g, t_n) \to b(g).$$

Lower-semicontinuity $\Longrightarrow \exists\ (u_n, g) \in E \cap P^{-1}(g)$ s. t.

$$u(u_n, g, t_n) \to a(g).$$

$u(a(g), g, t_n) \geq u(u_n, g, t_n) \Longrightarrow$

$$b(g) \geq a(g) \Longrightarrow E \cap P^{-1}(g) = \{\text{singleton}\}.$$

Proof of 4): Let μ -Haar measure on $H(f)$.

If $\mu(Y_0) = 1$, then $(E, \mathbb{R})$ is uniquely ergodic.

If $\mu(Y_0) = 0$, define $v_{a,b} \in C(E, \mathbb{R})'$

$$v_a(f) = \int_{H(f)} f(a(g), g) d\mu,$$

$$v_b(f) = \int_{H(f)} f(b(g), g) d\mu,$$

$\Longrightarrow v_a \neq v_b$, and v_a, v_b are invariant.

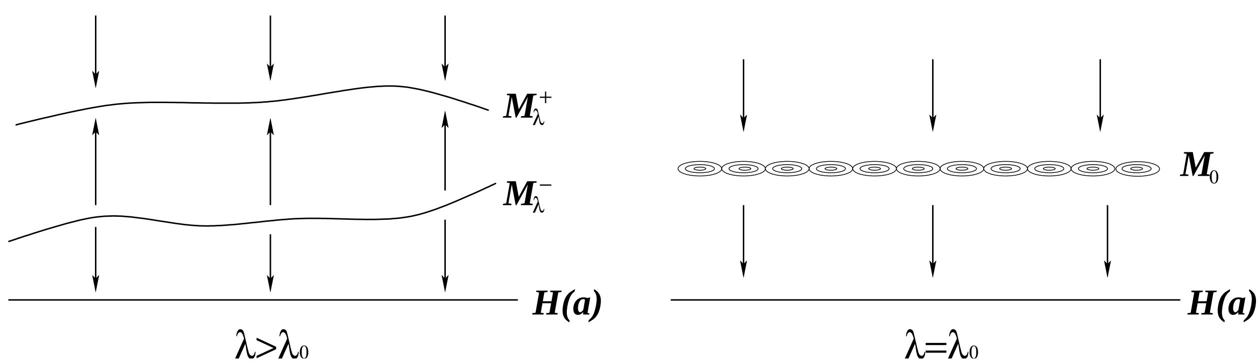
• a. a. dynamics as intermittency of bifurcations:

$$\dot{x} = x^2 - \lambda + a(t)$$

where $a(t)$ is a. p.

−− Skew-product flows: $\pi_\lambda = (R^1 \times H(a), \mathbb{R})$.

−− $\exists! \lambda_0$ s. t.



2. Strongly Monotone Skew-product Semiflows

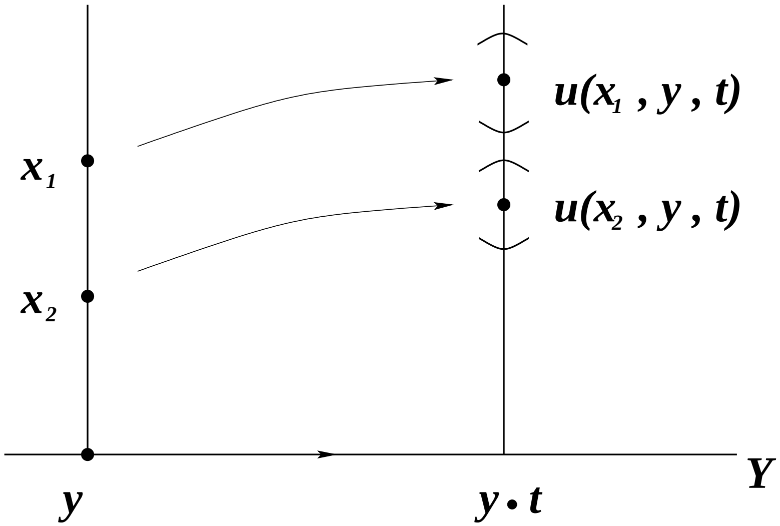
- Strong monotonicity: $(X \times Y, \mathbb{R}^+) = (X \times Y, \Pi^t)$:

$$\Pi^t(x, y) = (u(x, y, t), y \cdot t),\ t \geq 0,$$

is *strongly monotone* if $\exists$ a partial ordering "$\geq$" on X s.t.

$$x_1 > x_2 \ \Rightarrow\ u(x_1, y, t) \gg u(x_2, y, t)$$

$\forall\, y \in Y,\ t > 0.$



- Examples:

−− Cooperative system of ODEs and FDEs with a. p. time dependence

−− Parabolic PDE in higher space dimension with a. p. time dependence

$$\begin{cases} u_t = \Delta u + f(u, \nabla u, x, t), \quad x \in \Omega, \quad t > 0, \\ u|_{\partial\Omega} = 0 \quad or \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad t > 0. \end{cases}$$

$\Longrightarrow$: Strongly monotone skew-product semiflow $(X \times H(f), \mathbb{R}^+)$, where $X \hookrightarrow C^1(\bar{\Omega})$.

• Theorem (Shen & Y. 1998): Consider the a. p. strongly monotone skew-product semiflow $(X \times Y, \mathbb{R}^+)$.

1) (a. a.) Each linearly stable minimal set E is a. a. and in fact an almost N-cover of Y for some N;

2) (a. p.) Each uniformly stable minimal set is a. p.;

3) (module containment) $N\mathcal{M}(E) \subset \mathcal{M}(Y)$ for some N.

4) (spatial homogeneity) Assume that $f \equiv f(u, \bigtriangledown u, t)$ and the Neumann boundary condition on a convex domain Ω. Then any linearly stable a. a. solution u_g is spatially homogenous and satisfies

$$u^{'} = g(u, 0, t).$$

Moreover, $\mathcal{M}(u_g) \subset \mathcal{M}(f)$.

5) (global attractor) If there is a $\delta > 0$ such that $f_u \leq -\delta$ and if bounded solution exists, then $\exists|$ a. p. solution $u(U_0, f, \cdot, t)$ which attracts all bounded solutions, and moreover, $\mathcal{M}(u) \subset \mathcal{M}(f)$.

– Related works: Novo-Obaya-Sanz 05, Novo-Obaya 04, Shen-Zhao 04, Hetzer-Shen 02, 05, Jiang-Zhao 02, Chuechov 01 ...

• Theorem (Huang & Y. 2008): Each a. a. minimal set in both totally monotone and strongly monotone cases has zero topological entropy.

3. Algebraic Theory of Topological Dynamics

Let (X, Π^t) be a compact flow. For any $t \in \mathbb{R}$,

$$\Pi^t : X \to X$$

is a homeomorphism.

• Ellis semigroup: $E(X) = \mathrm{cl}\{\Pi^t\}$ – pointwise topology.

– $E(X)$ is a sub-semigroup of X^X under composition of maps.

– Π^t induces a flow $\tilde{\Pi}^t$ on $E(X)$: $\tilde{\Pi}^t \gamma = \Pi^t \circ \gamma$.

• Minimal ideal: $I \subset E(X)$ is an *ideal* if $E(X)I \subset I$. It is a minimal ideal if it does not contain any non-empty proper subideal.

• Idempotent: $u \in E(X)$ such that $u^2 = u$.

• Ellis Theorem:

1) $I \subset E(X)$ is a minimal ideal iff I is a minimal set of $(E(X), \tilde{\Pi}^t)$;

2) For any minimal ideal $I \subset E(X)$,

– the set $J(I)$ of idempotent points of I is non-empty,

– $\forall u \in J(I)$, uI is a group with identity u,

– $I = \bigcup_{u \in J(I)} uI$.

• Proximal and distal:

– $x_1, x_2 \in X$ is *distal* if

$$\inf_{t\in\mathbb{R}} d(\Pi^t x_1, \Pi^t x_2) > 0.$$

Otherwise, x_1, x_2 is said to be *proximal*.

– $x \in X$ is a *distal point* if it is only proximal to itself.

• Point-distal flow: A flow is *point-distal* if it is minimal and contains a distal point.

– An a. a. minimal flow is point-distal.

• Distal flow: A flow is *distal* if all points in it are distal points.

– $(X, \mathbb{T})$ is distal iff $E(X)$ is a group.

– An a . p. minimal flow is distal.

– One can define positive, negative distal and distal pairs, as well as positive and negative distal flows.

- Proximal relation:

$$P(X) = \{(x_1, x_2) \in X \times X : x_1, x_2 \text{ are proximal}\}.$$

– $P(X)$ need not be an equivalence relation;

– If $P(X)$ is an equivalence relation, then any two proximal pair is both positive and negative proximal;

– For an a. a. minimal flow, $P(X)$ is a closed equivalence relation.

- Theorem (Sacker-Sell 74). If $(X, \mathbb{R})$ is either positive or negative distal, then it is distal.

Proof: Suppose $(X, \mathbb{T})$ is negatively distal. Then $\alpha(e)$, where e is the identity of $E(X)$, is compact invariant. Hence it contains a minimal set I, i.e., a minimal ideal of $E(X)$.

– I is a group: $\forall u = \lim_{t_n \to -\infty} \Pi^{t_n} \in J(I)$, $x \in X$, denote $x^* = ux$. We have

$$(x^*, x^*) = (ux, ux^*) = u(x, x^*),$$

i.e., x, x^* are negatively proximal. Hence

$$x = x^* = ux \implies u = e \implies I = eI \text{ is a group.}$$

– $E(X) = I$: $E(X) = E(X)e \subset E(X)I \subset I$.

4. Application of Topological Dynamics

Consider the a. p. strongly monotone skew-product semiflow $(X \times Y, \mathbb{R}^+)$.

• Theorem: Each linearly stable minimal set E is a. a. and in fact an almost N-cover of Y for some N.

Proof: Consider the proximal relation $P(E)$ and the order relation

$$O(E) = \{(x_1, y), (x_2, y) \in E : x_1, x_2 \text{ are ordered}\}.$$

– $O(E)$ is a closed relation;

– $P(E)$ is an equivalence and invariant relation;

– $O(E) \subset P(E)$: Strong monotonicity $\Longrightarrow \exists Y_0 \subset Y$ such that $\forall y \in Y_0$ no two pointed on $p^{-1} \cap E$ are ordered. Let $((x_1, y), (x_2, y)) \in O(E) \setminus P(E)$. Take $y_0 \in Y_0$ and $t_n \to +\infty$ such that $y \cdot t_n \to y_0$. WLOG, let $u(x_i, y, t_n) \to x_i^*$, $i = 1, 2$.

$(x_1, y), (x_2, y)$ distal $\Longrightarrow x_1^* \neq x_2^2$. $(x_1, y), (x_2, y)$ are ordered $\Longrightarrow$ $(x_1^*, y_0), (x_2^*, y_0)$ are also ordered, a contradiction.

– $P(E) \subset O(E)$: Linear stability $\Longrightarrow$ any two points on the same fiber of $P(E) \setminus P(O)$ are negatively distal - a contradiction because $P(E)$ is an equivalence relation and any two proximal points are negatively proximal;

– Let $Y^* = E/P(E) = E/O(E)$. Then $(E, \mathbb{R})$ induces a minimal distal flow on Y^*.

– Y^* is an N-cover of Y for some N, hence a. p. minimal.

– E is an almost 1-cover of Y^*, hence a. a.

- Theorem: Each uniformly stable minimal set E is a. p.

Proof: Uniform stability $\Longrightarrow E$ is negatively distal, hence distal. Therefore, $E = Y^*$.

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