MULTI-FREQUENCY OSCILLATIONS

IN DYNAMICAL SYSTEMS

Session 3. Multi-frequency Dynamics in Damped Oscillatory Systems

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1. Integral Manifold Reduction

Consider damped oscillators

$$x_i'' + h(x_i, x_i') = 0, \qquad i = 1, 2, \cdots, n.$$

Assume that each oscillator admits an asymptotically stable invariant cycle C_i.

Adding small couplings and a. p. forcing, the coupled oscillators has the form

$$X' = F(X) + \varepsilon G(X, t), \qquad X \in R^{2n},$$

where

– G is a. p. in t;

– When $\varepsilon = 0$, the unperturbed system has an asymptotically stable quasi-periodic invariant n-torus $M \simeq T^n$ with frequency $\omega \in R^n$.

• Theorem (Hale 61, Y. 91): For $0 < \varepsilon \ll 1$, M persists and gives rise to an asymptotically stable integral manifold $\simeq T^n \times H(G)$ whose dynamics can be described by an a. p. forced toral flow

$$\theta' = \omega + \varepsilon f(\theta, t, \varepsilon), \;\; \theta \in T^n, \;\; \text{where } \mathcal{M}(f) = \mathcal{M}(G).$$

Proof. In a neighborhood of M, consider change of coordinate $X = \Theta(\theta) + H(\theta, y)$, $\theta \in T^n, y \in R^n$, $H(\theta, 0) \equiv 0$. Then

$$\begin{cases} y' = (A_0 + \varepsilon A(\theta))y + \varepsilon \tilde{G}(y, \theta, t, \varepsilon), \\ \theta' = \omega + \varepsilon \tilde{F}(y, \theta, t, \varepsilon). \end{cases}$$

$\Longrightarrow$: Integral manifold

$$M_\varepsilon = \{(h_\varepsilon(\theta, t), \theta) : \theta \in T^n\} \simeq T^n \times H(G).$$

– Even without the a .p. forcing, dynamics on a toral flow is not quasi-periodic or a. p. in general. They can be chaotic (depending on the nature of rotation set, Franks-Misiurewicz 90).

2. A. P. Forced Circle Flow

Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, $(Y, \mathbb{T})$ = an almost periodic (a. p.) minimal flow.

- **a. p. forced circle flow** : $(S^1 \times Y, \mathbb{T})$:

$$(\phi_0, y_0) \cdot t = (\phi(\phi_0, y_0, t), y_0 \cdot t), \quad \phi_0 \in S^1, \; y_0 \in Y.$$

Strong conditions needed for the existence of a. p. dynamics

e.g.: $\phi' = \lambda + \varepsilon \Psi(\phi, \omega t)$, where Ψ is sufficiently smooth, ω Diophantine, and λ, ε, $|\varepsilon| \ll 1$ are parameters (Arnold-Moser Theorem).

- **Problems**:

$--$ General dynamical complexity;

$--$ Topological and dynamical properties of minimal sets.

I. Models

- **Damped, a. p. forced nonlinear oscillators:**

$$\ddot{x} = F(\dot{x}, x, y \cdot t), \quad x \in R^1, \, y \in Y, \, (Y, \mathbb{R}) \text{ a. p.}$$

$\Longleftrightarrow$ Skew-product flow: $(R^2 \times Y, \mathbb{R})$.

$--$ *Hale integral manifold*: $S^1 \times Y$ (existence due to damping) which generates an a. p. forced circle flow containing global attractor, less smooth if damping is weak.

e.g.:

$$\ddot{u} - \alpha(1 - u^2)\dot{u} + u = \varepsilon b(u, \dot{u}, y \cdot t) \text{ (van der Pol)}$$

$$\ddot{u} + \beta\dot{u} + \sin u = F(y \cdot t) \text{ (Josephson junction)}$$

all admit Hale integral manifolds $S^1 \times Y$ for $\alpha > 0$, $|\varepsilon| \ll 1$, $\beta \geq 2$ (Shen 02, Y. 03).

Strange, non-chaotic attractors (SNAs) observed numerically (Ott, Yorke, etc 80s)

• **Damping-free, a. p. forced oscillators**:

$$\ddot{x} + V_x(x, y \cdot t) = 0,\ x \in S^1,\ y \in Y,\ (Y, \mathbb{R}) \text{ a. p.}$$

$\Longleftrightarrow$ Skew-product flow: $(R^1 \times S^1 \times Y, \mathbb{R})$.

$--$ *Minimizing measure*: $\forall \eta \in R$,

$$\int (L - \eta) d\mu_\eta = \inf_\mu \int (L - \eta) d\mu, \qquad L = \frac{\dot{x}^2}{2} - V(x, y \cdot t).$$

$--$ *Mather set*: $M_\eta = \overline{\cup_{\mu_\eta} \text{supp } \mu_\eta} \subset R^1 \times S^1 \times Y$.

$--$ *Projected Mather set*: $\tilde{M}_\eta = \pi M_\eta \subset S^1 \times Y$, $\pi|_{M_\eta} : M_\eta \to \tilde{M}_\eta$ bi-Lipschizian, and $\tilde{M}_\eta$ Lipschiz embedable into an a. p. forced circle flow (Mather 89, 91, Iturriaga 96).

Mather sets can be be both dynamically and topologically complicated (e.g., Aubry-Mather sets)

• **a. p. projective flows**:

$$x' = A(y \cdot t)x, \; x \in R^2, \; tr A = 0.$$

Let $\phi = Arg\ x \Longrightarrow$ a. p. forced circle flow.

Almost automorphic (a. a.) Floquet theory (Johnson 82)

Spectrum of Schödinger operator with a. p. potential (Johnson-Moser 82)

• **Oceanic flows and climate models**: Quasi-periodically forced circle flow naturally arises in the global attractor via invariant manifolds reduction and high frequency averaging (Pliss-Sell 06).

Regular dynamics and turbulence in oceanic flows need to be understood

• **Toral flow with thin rotation set**: A quasi-periodically forced circle flow is a special case of a toral flow with point rotation set.

Dynamics of a toral flow with "thin" rotation set

II. Circle Homeomorphism

• **General dynamics**: Never chaotic by any means.

• **Minimal sets**:

$--\ \rho \in Q$: Each minimal set consists of finite points and is periodic.

$--\ \rho \notin Q$: Each minimal set is either the whole circle and conjugates to a pure rotation, or is a Denjoy Cantor set with a. a. dynamics.

• **Rotation number, topology, and dynamics**:

a) Nature of rotation number and regularity of the map determine dynamics and topology of minimal sets.

b) Topology of a minimal set completely determines its dynamics.

An a. p. forced circle flow can be chaotic

For an a. p. forced circle flow, neither a) nor b) is true

III. Chaos in an a. p. Forced Circle Flow

• **Theorem 1.** (Huang-Y. 09) $h_{\text{top}}(S^1 \times Y, \mathbb{T}) = 0.$

$--$ A general result: If an extension $\pi : (X, \mathbb{T}) \to (Y, \mathbb{T})$ preserves an n-partial order relation, then $h_{\text{top}}(X, \mathbb{T}) = h_{\text{top}}(Y, \mathbb{T})$.

• Li-Yorke chaos:

$--$ $\{x_1, x_2\} \subset X$ is a *Li-Yorke pair* if

$$\overline{\lim}_{t\to\infty} d(x_1 \cdot t, x_2 \cdot t) > 0, \quad \underline{\lim}_{t\to\infty} d(x_1 \cdot t, x_2 \cdot t) = 0.$$

$--$ $(X, \mathbb{T})$ is *Li-Yorke chaotic* if $\exists$ uncountable set $\mathcal{A} \subset X$ s. t. $\forall \{x_1, x_2\} \subset \mathcal{A}$ is a Li-Yorke pair.

$--$ $h(X) > 0 \implies (X, \mathbb{T})$ is Li-Yorke chaotic (Blanchard et al 2002). The converse is not true.

• **Theorem 2.** (Huang-Y. 09) Each minimal set of $(S^1 \times Y, \mathbb{T})$ is either residually Li-Yorke chaotic or point-distal.

—– Recall that point-distal flow is general than a. a. minimal flow.

—– *Residually Li-Yorke chaos*: $\exists$ a residual subset $Y_0 \subset Y$ s. t. each fiber over Y_0 contains an uncountable set in which every two points form a Li-Yorke pair.

—– Li-Yorke chaos does occur in an a. p. forced circle flow (Johnson 82, Bjerklov-Johnson 07, Huang-Y. 09).

—– A general result: If $\pi : (X, \mathbb{T}) \to (Y, \mathbb{T})$ is a proximal extension which is not almost 1-1, then $(X, \mathbb{T})$ is residually Li-Yorke chaotic.

IV. Topological Classification of Minimal Sets

- **Theorem 3.** (Huang-Y. 09) Let M be a minimal set of $(S^1 \times Y, \mathbb{T})$. Then precisely one of the following holds:

  a) M is an almost N-1 extension of Y for some positive integer N;

  b) $M = S^1 \times Y$;

  c) M is a Cantorian.

$--$ *Cantorian*: M is a Cantorian if $\exists$ a residual subset $Y_0 \subset Y$ s. t. each fiber over Y_0 is a Cantor set.

$--$ All cases a)-c) above are independent of resonant type of the rotation number.

$--$ a. a. dynamics can occur in all cases a)-c).

$--$ Residual Li-Yorke chaos can occur in case b) and should also occur in case c).

V. Almost N-1 Extensions and Local Connectivity

• **Theorem 4**. (Huang-Y. 09) Suppose that an a. p. forced circle flow $(S^1 \times Y, \mathbb{T})$ has more than one minimal sets. Then

1) $\exists$ a positive integer N such that each minimal set is an almost N-1 extension of Y.

2) If one minimal set is a. a., then so are others.

3) If Y is locally connected, then all minimal sets are a. a..

−− 1), 2) above originally known for projective flows (Johnson 82).

• **Theorem 5**. (Huang-Y. 09) Let M be a minimal set of an a. p. forced circle flow $(S^1 \times Y, \mathbb{T})$ which is an almost N-1 extension of Y and has a locally connected point. Then M is a. a. and each fiber over Y consists of exactly N connected components.

−− Theorem 5 holds in general.

VI. Mean Motion and Dynamics

- *Rotation number*:

$$\rho = \lim_{t \to \infty} \frac{\phi(t)}{t}$$

is well defined, where $\phi(t)$ denotes the lift of an orbit $\phi(\phi_0, y_0, t)$.

- *Mean motion*: $(S^1 \times Y, \mathbb{T})$ is said to have mean motion if

$$\phi(\phi_0, y_0, t) - \rho t$$

is bounded for some (ϕ_0, y_0).

- Mean motion property

−− holds for circle maps and periodically forced circle $\mathbb{R}$-flows.

−− holds if $\exists$ an a. p. motion.

−− does not always hold (several examples by Johnson in the 80s).

• **Theorem 6**. (Shen 01, Y. 04) Suppose that an a. p. forced circle flow $(S^1 \times Y, \mathbb{T})$ admits mean motion. Then

1) Each minimal set E is a. a. and $\mathcal{M}(E) = gl\{\rho, \mathcal{M}(Y)\}$.

2) A minimal set is an almost N-1 extension of Y iff $N\rho \subset \mathcal{M}(Y)$.

• **Theorem 7**. (Huang-Y. 09) Suppose that an a. p. forced circle flow $(S^1 \times Y, \mathbb{T})$ admits no mean motion. Then

1) Each minimal set is either the whole space or is everywhere non-locally connected.

2) If Y is locally connected, then $(S^1 \times Y, \mathbb{T})$ is positively transitive and has only one minimal set.

−− Theorem 7 2) was first proved for one-frequency-forced circle map (Jäger-Keller 06, Jäger-Stark 06).

−− Cantorians exist in both cases with or without mean motion (Béguin-Croviser-Jäger-Le Roux 07).

VII. An Extended Denjoy Theorem

• **Theorem 8**. (Huang-Y. 09) Suppose that an a. p. forced circle flow $(S^1 \times Y, \mathbb{T})$ is quasi-periodically forced and the rotation number is rationally independent of the forcing frequencies. Then

a) $\exists$ a unique minimal set M.

b) Either $M = S^1 \times Y$ or M is everywhere non-locally connected.

c) If, in addition, $(S^1 \times Y, \mathbb{T})$ admits mean motion, then M is a. a., and moreover, either $M = S^1 \times Y$ or M is an everywhere non-locally connected Cantorian.

VIII. Projective flow

More detailed topological, measure-theoretic, and dynamical characterizations on minimal sets can be made for a. p. projective flows (Johnson 82, Novo-Obaya 96, Jorba-Núñez-Obaya-Tatjer 06, Bjerklov-Johnson 07, Huang-Y. 09).

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