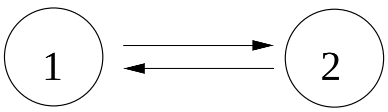
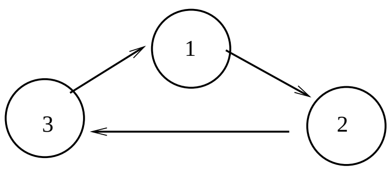
Coupled Systems of Differential Equations
Figures for DANCE Winter School, January 2012

Marty Golubitsky (mg@mbi.osu.edu)



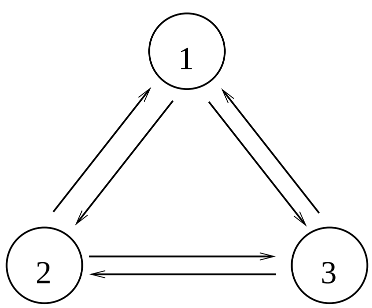
$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= f(x_2, x_1) \end{aligned} \qquad x_1, x_2 \in \mathbf{R}^k$$

Figure 1: Symmetric two-cell system: $\Gamma = \mathbf{Z}_2$



$$\begin{aligned} \dot{x}_1 &= f(x_1, x_3) \\ \dot{x}_2 &= f(x_2, x_1) \\ \dot{x}_3 &= f(x_3, x_2) \end{aligned}$$

Figure 2: Unidirectional ring with three cells: $\Gamma = \mathbf{Z}_3$



$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_2, x_3}) \\ \dot{x}_2 &= f(x_2, \overline{x_3, x_1}) \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_2}) \end{aligned}$$

Figure 3: Bidirectional ring with three cells: $\Gamma = \mathbf{D}_3$ [24]

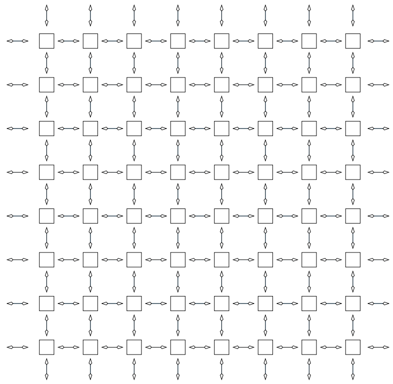


Figure 4: Square lattice array with nearest neighbor coupling [32]

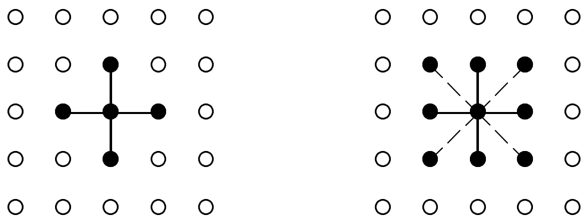


Figure 5: Nearest neighbor and next nearest neighbor couplings [32, 31]

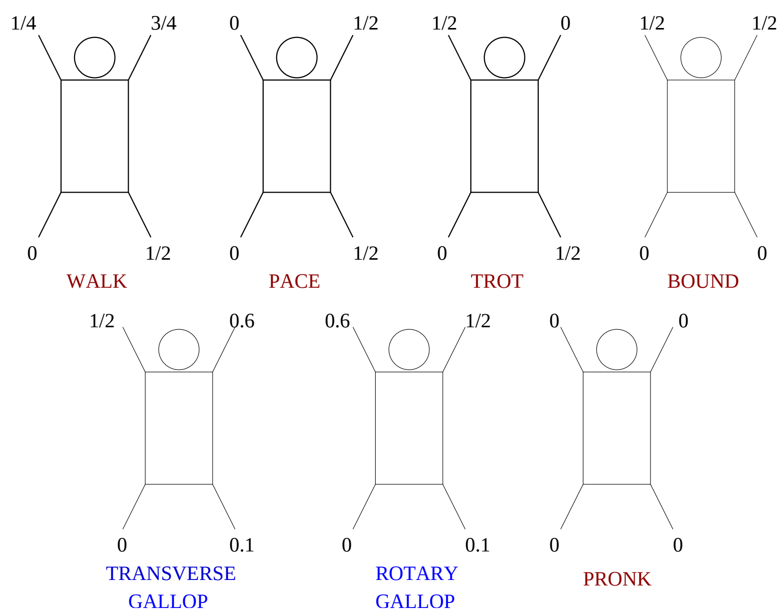
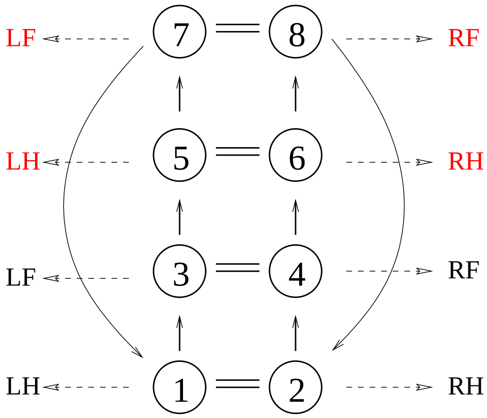


Figure 6: Standard gait phases [37, 38]



K	Phase Diagram	Gait
$\mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$	$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$	pronk
$\mathbf{Z}_4(\omega)$	$\begin{matrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{matrix}$	pace
$\mathbf{Z}_4(\kappa\omega)$	$\begin{matrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{matrix}$	trot
$\mathbf{Z}_2(\kappa) \times \mathbf{Z}_2(\omega^2)$	$\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{matrix}$	bound
$\mathbf{Z}_2(\kappa\omega^2)$	$\begin{matrix} \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{matrix}$	walk
$\mathbf{Z}_2(\kappa)$	$\begin{matrix} 0 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{matrix}$	jump

Figure 7: Central pattern generator for quadrupeds: $\Gamma = \mathbf{Z}_4 \times \mathbf{Z}_2$ [38, 34]

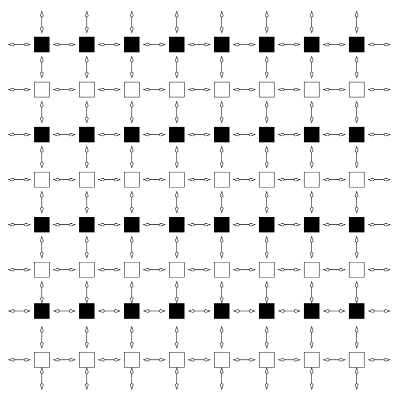


Figure 8: Periodic balanced 2-coloring

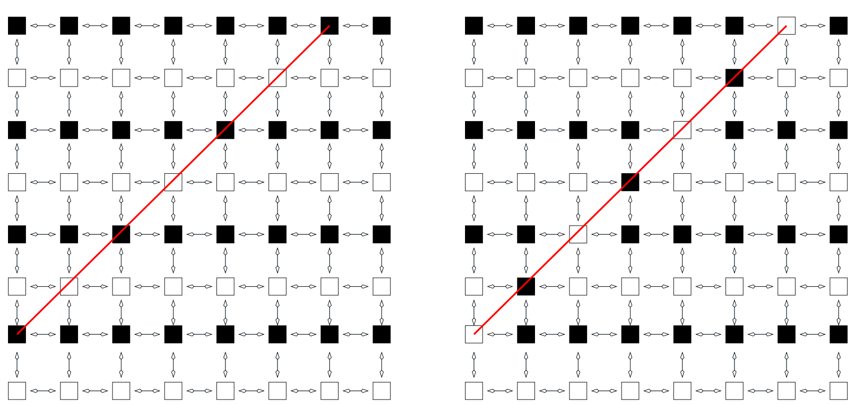


Figure 9: Periodic balanced 2-coloring; aperiodic balanced 2-coloring [2, 31]

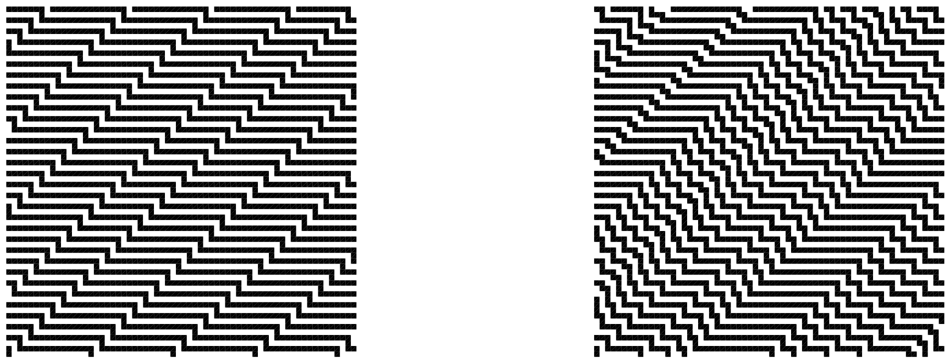
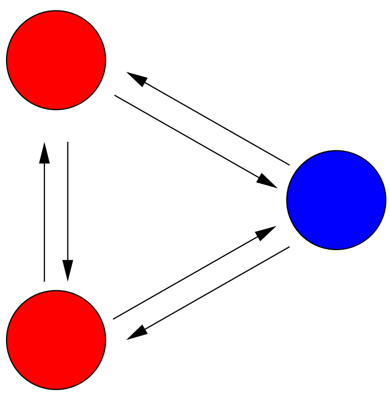
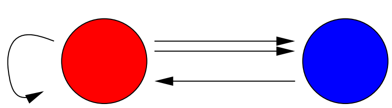


Figure 10: Sample of a continuum of balanced 2-colorings [2, 32]



$$\begin{aligned}\dot{\mathbf{x}}_\mathbf{1} &= \mathbf{f}(\mathbf{x}_\mathbf{1}, \overline{\mathbf{x}_\mathbf{2}, \mathbf{x}_\mathbf{3}})\\ \dot{\mathbf{x}}_\mathbf{2} &= \mathbf{f}(\mathbf{x}_\mathbf{2}, \overline{\mathbf{x}_\mathbf{3}, \mathbf{x}_\mathbf{1}})\\ \dot{\mathbf{x}}_\mathbf{3} &= \mathbf{f}(\mathbf{x}_\mathbf{3}, \overline{\mathbf{x}_\mathbf{1}, \mathbf{x}_\mathbf{2}})\end{aligned}$$

Figure 11: Balanced 2-coloring on bidirectional ring



$$\begin{aligned}\dot{\mathbf{x}}_\mathbf{1} &= \mathbf{f}(\mathbf{x}_\mathbf{1}, \overline{\mathbf{x}_\mathbf{1}, \mathbf{x}_\mathbf{3}})\\ \dot{\mathbf{x}}_\mathbf{3} &= \mathbf{f}(\mathbf{x}_\mathbf{3}, \overline{\mathbf{x}_\mathbf{1}, \mathbf{x}_\mathbf{1}})\end{aligned}$$

Figure 12: Quotient network with self-coupling and multiple arrows [12]

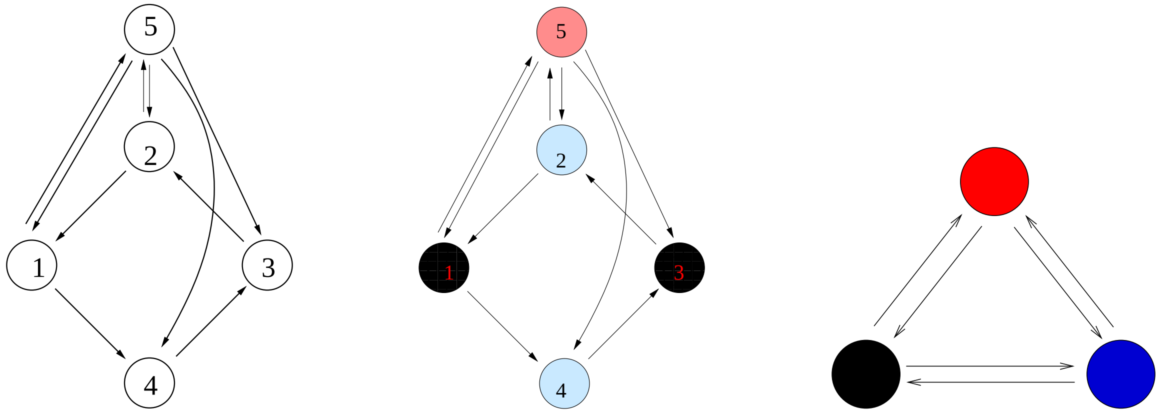


Figure 13: Asymmetric network with symmetric quotient network [12]

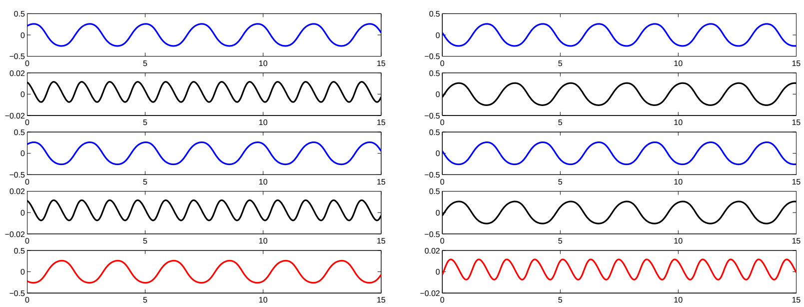


Figure 14: Phase-shift synchrony caused by symmetry on quotient network

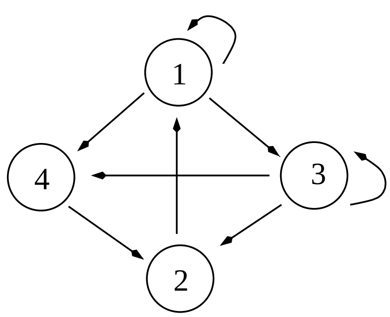
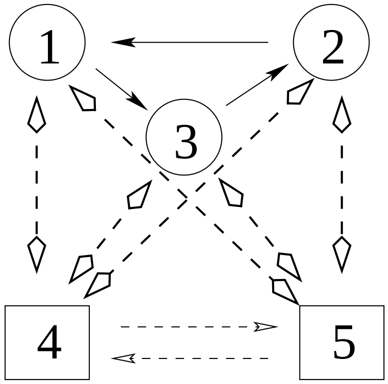


Figure 15: Four-cell network whose adjacency matrix has eigenvalues $2, 0, \pm i$ [20].



$$
\begin{aligned}
\dot{x_1} &= f(x_1, x_2, \overline{x_4, x_5}) \\
\dot{x_2} &= f(x_2, x_3, \overline{x_4, x_5}) \\
\dot{x_3} &= f(x_3, x_1, \overline{x_4, x_5}) \\
\dot{x_4} &= g(x_4, x_5, \overline{x_1, x_2, x_3}) \\
\dot{x_5} &= g(x_5, x_4, \overline{x_1, x_2, x_3})
\end{aligned}
$$

Figure 16: All-to-all coupled rings [3, 11]

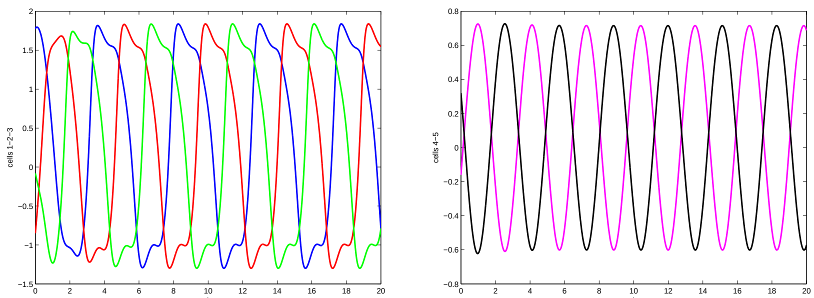


Figure 17: Polyrhythms in all-to-all coupled rings [3]

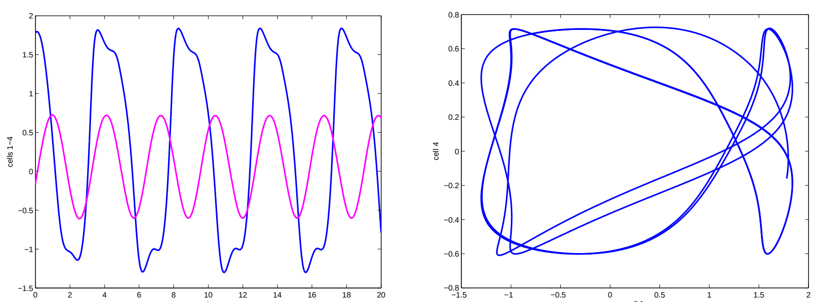
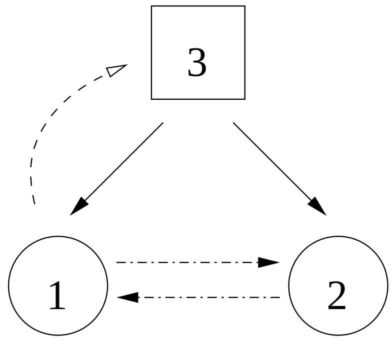
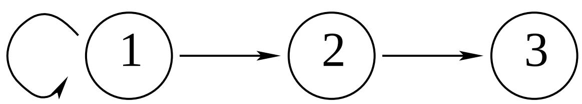


Figure 18: Pictorial test of multifrequencies



$$
\begin{aligned}
\dot{x}_1 &= f(x_1, x_2, x_3) & x_1 \in \mathbf{R}^k \\
\dot{x}_2 &= f(x_2, x_1, x_3) & x_2 \in \mathbf{R}^k \\
\dot{x}_3 &= g(x_3, x_1) & x_3 \in \mathbf{R}^\ell
\end{aligned}
$$

Figure 19: Three cell network with two cell types



$$
\begin{aligned}
\dot{x}_1 &= f(x_1, x_1, \lambda) \\
\dot{x}_2 &= f(x_2, x_1, \lambda) \\
\dot{x}_3 &= f(x_3, x_2, \lambda)
\end{aligned}
$$

Figure 20: Feedforward network [2, 18, 22]

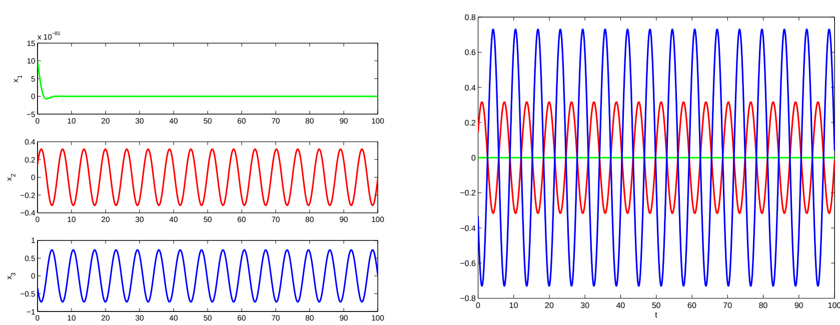


Figure 21: $\lambda^{1/6}$ growth rate solution from Hopf bifurcation in feedforward network [2]

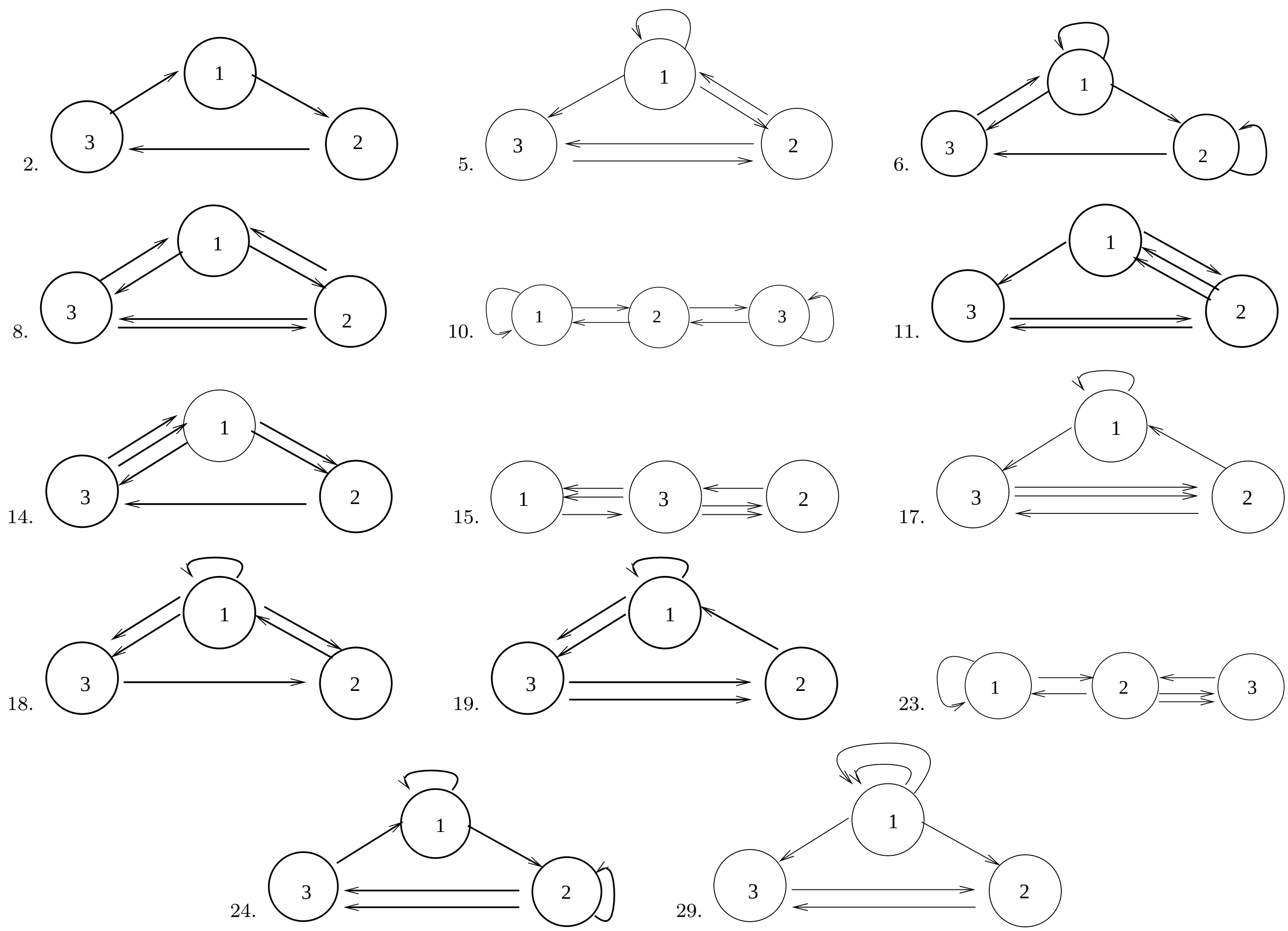


Figure 22: 14 Regular three-cell transitive networks with valency 1 or 2 [23]

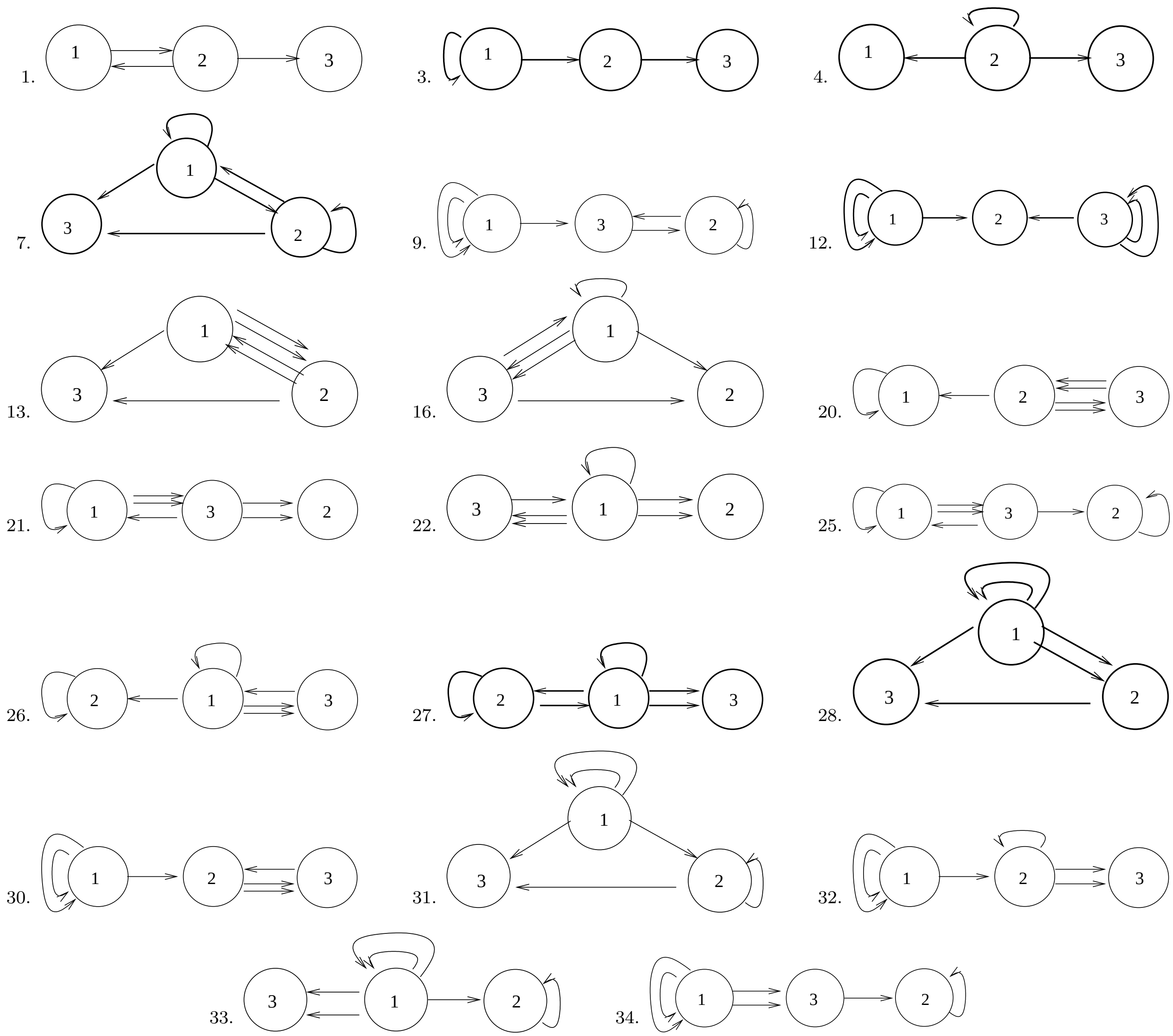
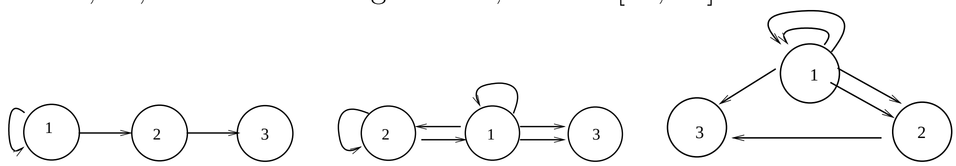
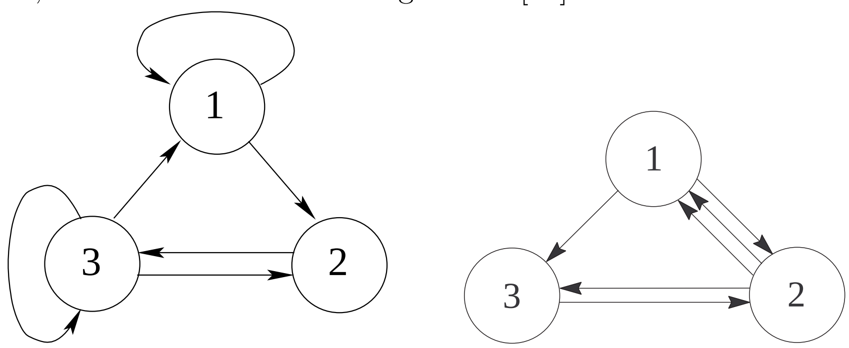


Figure 23: 20 Regular three-cell feed-forward networks with valency 1 or 2 [23]

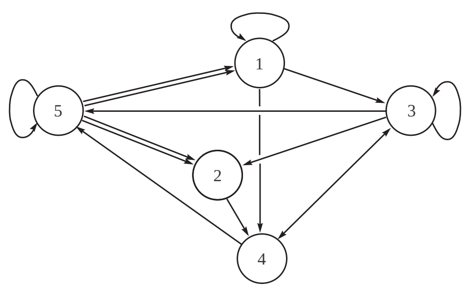
- Networks 3, 27, 28: one branch grows $\lambda^{\frac{1}{6}}$; one $\lambda^{\frac{1}{2}}$ [18, 22]



- Networks 6, 11: two or four branches grow $\lambda^{\frac{1}{2}}$ [18]



- Regular five-cell network: two branches grow λ



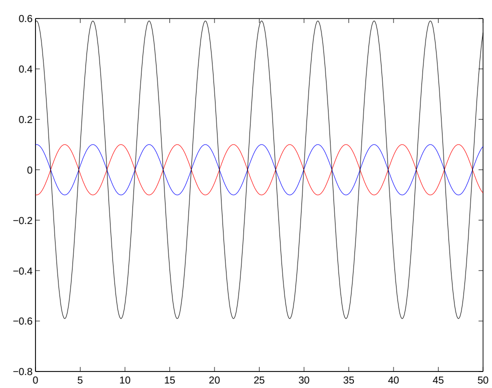


Figure 24: Nilpotent Hopf in Network 27

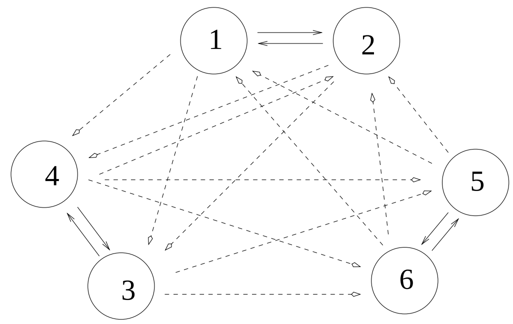


Figure 25: Coupled cell version of Guckenheimer-Holmes heteroclinic cycle: $\Gamma = \mathbf{Z}_2 \wr \mathbf{Z}_3$ [29]

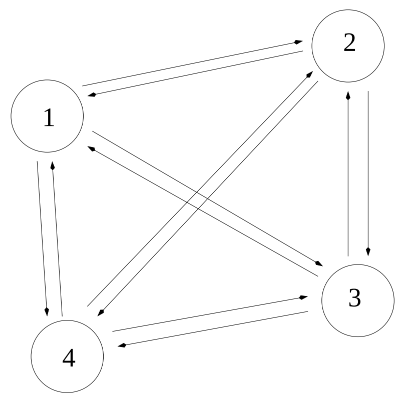


Figure 26: Four-cell all-to-all coupled simplex: $\Gamma = \mathbf{S}_4$

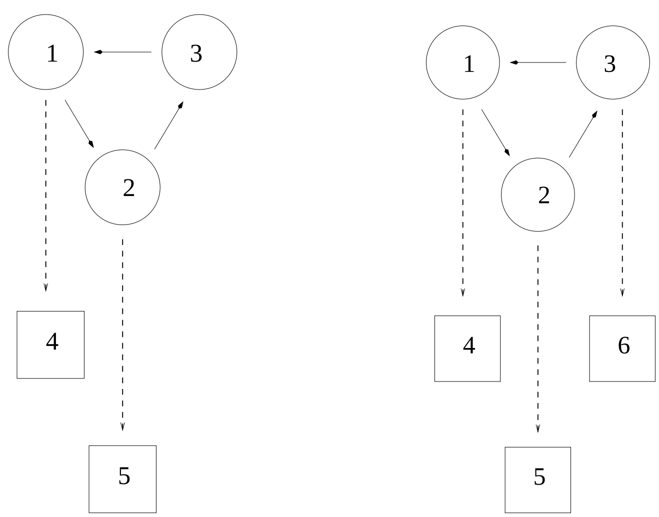


Figure 27: Three-cell ring forcing two hanging nodes; completion with $\mathbf{Z}_3$ symmetry [11]

References

General Background: this list of references is not meant to be complete

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