Localized waves in lattices of Fermi-Pasta-Ulam type

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see G.I., G.J. Chaos 15, 05113 (2005) 1-15

Fermi-Pasta-Ulam lattice

particles nonlinearly coupled with their neighbors (V : interaction potential)

$$\ddot{u}_n = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1})$$

$$n \in \mathbb{Z}, \quad V'(0) = 0, V''(0) = 1 \ (> 0)$$

$$u_n(t), \quad t \in \mathbb{R}$$

For p fixed integer, look for solutions such that (τ is a parameter)

$$u_n(t) = u_{n-p}(t - p\tau)$$

for $p = 1$ travelling waves of velocity $1/\tau$

for $p \geq 2$ pulsating travelling waves (a time shift of $p\tau$ gives the displacement translated by p sites).

spatial localization if $u_n(t) \to cte_{\pm}$ as $n \to \pm\infty$

for $p = 1$: solitary wave of velocity $1/\tau$

for $p \geq 2$: "exact travelling breather" (velocity $1/\tau$)

Math results on classical lattices

G.Friesecke, J.Wattis, R.Pego,..

R.MacKay, S.Aubry, J.A.Sépulchre,...

G.I., K.Kirchgässner, G.James, P.Noble, Y.Sire,..

A.Mielke, J.Giannoulis,...

Today we only consider local methods using center manifold and normal form reductions

Formulation with a new variable

$$y_n(x) = u_n(\tau(n-x)),\ x = n - t/\tau$$

$$y_{n+p}(x) = y_n(x)$$

$$\frac{1}{\tau^2}\frac{d^2 y_n}{dx^2} = V'[y_{n+1}(x+1)-y_n(x)]-V'[y_n(x)-y_{n-1}(x-1)]$$

p-dim system of advanced-delay diff. equations

Invariances and symmetries:

$$\begin{aligned} y_n(x) &\mapsto y_{n+1}(x) \\ y_n(x) &\mapsto y_n(x)+b \\ y_n(x) &\mapsto -y_{-n}(-x) \end{aligned}$$

particular solution:

$$\begin{aligned} y_n(x) &= ax+b \\ a &> 0 \text{ or } <0: \text{ stretched or compressed state} \end{aligned}$$

First integral:

$$I = \frac{1}{p}\sum_{1\le n\le p} J_n$$

$$J_n = \frac{dy_n}{dx} - \tau^2 \int_0^1 V'[y_{n+1}(x+v)-y_n(x+v-1)]dv$$

F-P-U as a reversible Dynamical system

$$U_n(x)(v) = (y_n(x), \xi_n(x), Y_n(x,v))$$

$$\begin{aligned} \mathbb{H} &= \mathbb{R}^2 \times C^0[-1,1] \\ \mathbb{D} &= \{U = (y,\xi,Y) \in \mathbb{R}^2 \times C^1[-1,1]; Y(0) = y\} \\ \mathbb{D}_p &= \{U \in \mathbb{D}^{\mathbb{Z}}; U_{n+p} = U_n \text{ for all } n \in \mathbb{Z}\} \end{aligned}$$

$$\frac{dU}{dx} = L_\tau U + \tau^2 M(U) \text{ in } \mathbb{H}_p, \quad U \in \mathbb{D}_p$$

$$(L_\tau U)_n = \begin{pmatrix} \xi_n \\ \tau^2(\delta_1 Y_{n+1} - 2y_n + \delta_{-1} Y_{n-1}) \\ \frac{\partial Y_n}{\partial v} \end{pmatrix}$$

$$\begin{aligned} (M(U))_n &= (0, N(\delta_1 Y_{n+1} - y_n) - N(y_n - \delta_{-1} Y_{n-1}), 0) \\ N(y) &= V'(y) - y = O(y^2), \quad \delta_{\pm 1} Y = Y(\pm 1) \end{aligned}$$

Symmetry properties:

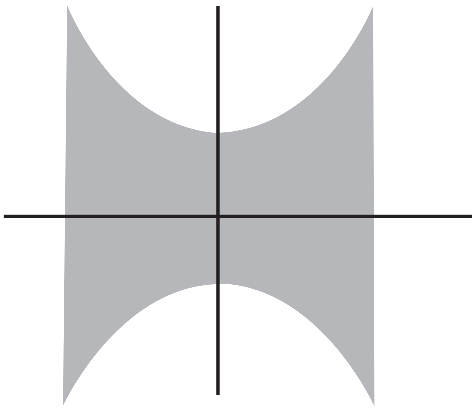
$$\begin{aligned} & L_\tau \text{ and } M \text{ commute with } \sigma \text{ and } \mathcal{T}_b \\ (\sigma U)_n &= U_{n+1} \text{ (shift)}, \\ \mathcal{T}_b U &= U + b\chi_0, \quad (\chi_0)_n = (1,0,1) \end{aligned}$$

$$\begin{aligned} & L_\tau \text{ and } M \text{ anticommute with } \mathcal{R} \\ (\mathcal{R}U)_n &= (-y_{-n}, \xi_{-n}, -Y_{-n}(-v)) \end{aligned}$$

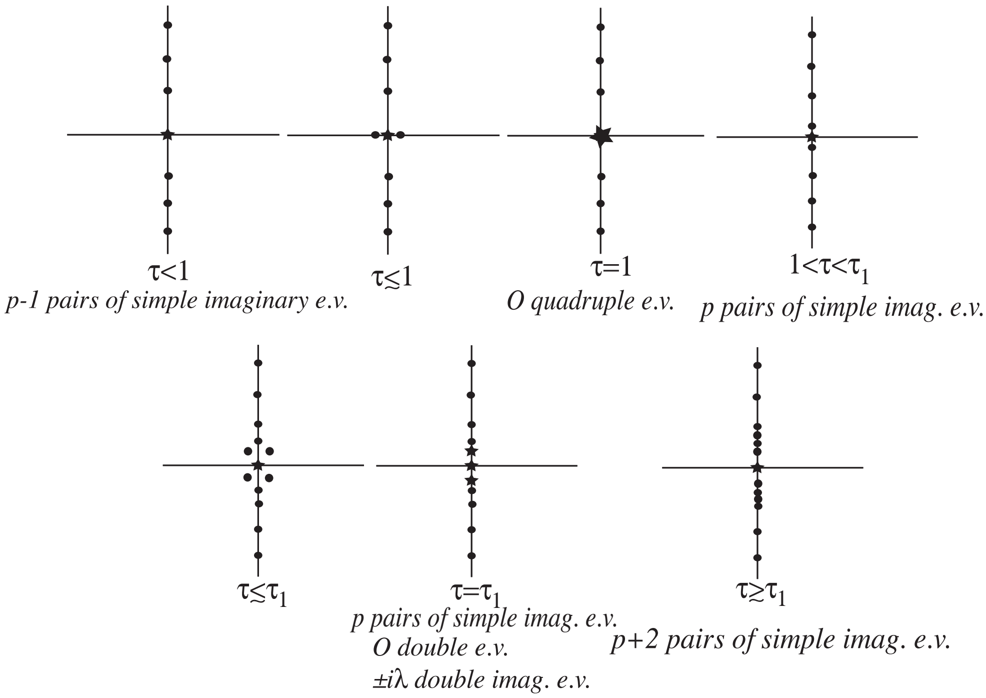
Linearized system

Spectrum of L_τ : set of $z \in \mathbb{C}$ such that $\exists\ m \in \{0, ...p-1\}$ with

$$\frac{z^2}{\tau^2} + 2(1 - \cosh(z - \frac{2i\pi m}{p})) = 0$$



location of eigenvalues of L_τ



Reduction of the system (1)

For $\tau = 1$, 0 is quadruple eigenvalue

$$\begin{aligned} L_1\chi_0 &= 0, \quad L_1\chi_j = \chi_{j-1}, \; j = 1,2,3 \\ \mathcal{R}\chi_j &= (-1)^{j+1}\chi_j, \quad j = 0,1,2,3 \end{aligned}$$

Projection on the 4-dim invariant subspace:

$$P_0 U = \sum_{0 \le j \le 3} \chi_j^*(U)\chi_j, \quad \chi_j^*(U) = \chi_{j-1}^*(L_1 U)$$

$$\chi_0^*(U) = \frac{2}{5p} \sum_{1 \le n \le p} \left(y_n - \int_{-1}^{1} (1 - |v| - 5(1 - |v|)^3) Y_n dv \right)$$

Use of the invariance under $\mathcal{T}_b$: $\mathcal{T}_b U = U + b\chi_0$

$$U = W + q\chi_0, \quad \chi_0^*(W) = 0$$

$$\begin{aligned} \frac{dq}{dx} &= \chi_1^*(W) \\ \frac{dW}{dx} &= \widetilde{L}_\tau W + \tau^2 M(W) \end{aligned}$$

0 *triple eigenvalue of* $\widetilde{L}_1$, eigenvector χ_1

Center manifold reduction

$$\begin{aligned}\frac{dW}{dx} &= \widetilde{L}_\tau W + \tau^2 M(W) \\ W &\in C_b^0(\mathbb{R};\mathbb{D}_p)\cap C_b^1(\mathbb{R};\mathbb{H}_p)\cap \ker \chi_0^*\end{aligned}$$

$W_h = \Psi(W_c,\tau)$, *center manifold* $\mathcal{M}_0$ ($= dim E_c$)

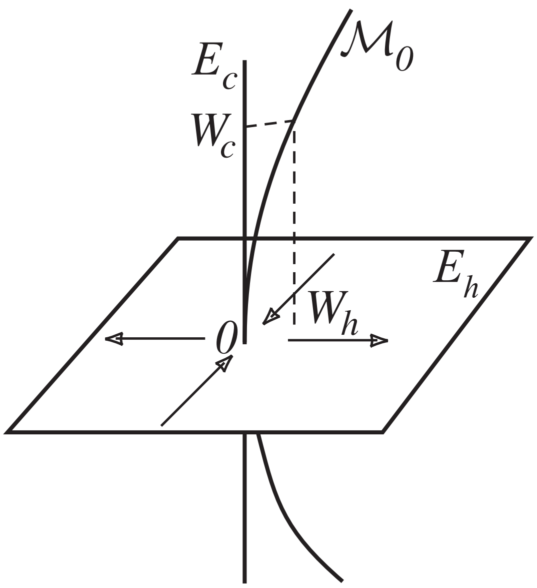
E_c central subspace is $2(p-1)+3$ - dim

$$W = W_c + W_h \text{ near } 0, \quad \tau \text{ near } 1$$

all "small" solutions lie in $\mathcal{M}_0$

Reduced system commutes with σ, anticommutes with $\mathcal{R}$

$$\begin{aligned}\frac{dW_c}{dx} &= \widetilde{L}W_c + F_\tau(W_c), \quad W_c(x)\in E_c \\ F_\tau(W_c) &= O(||W_c||^2 + |\tau-1|||W_c||)\end{aligned}$$



Normal form reduction ($\tau \approx 1$)

$\pm i\lambda_j, \quad j = 1, 2, ..p{-}1$ imag. eigenvalues for $\tau = 1$

Non strong resonance assumption:

$$\sum_{1 \le j \le p-1} r_j \lambda_j \neq 0$$

for all $r_j \in \mathbb{Z}$, such that $0 < \sum_{1 \le j \le p-1} |r_j| \le 4$

$\exists$ a polynomial Φ of degree 4 in $(A, B, C, D, \bar{D}, \tau - 1)$, $D = (D_1, ...D_{p-1})$ $D_j \in \mathbb{C}$, such that

$$\begin{aligned} W_c &= A\chi_1 + B\chi_2 + C\chi_3 + \\ &+ \sum_{1 \le j \le p-1} D_j \zeta_j + \bar{D}_j \bar{\zeta}_j + \Phi(A, B, C, D, \bar{D}, \tau) \end{aligned}$$

$$\begin{aligned} \frac{dA}{dx} &= B, \qquad \frac{dC}{dx} = 0 \\ \frac{dB}{dx} &= C + A\phi(A, C, Q, \tau) + h.o.t. \\ \frac{dD_j}{dx} &= i\lambda_j D_j + iD_j R_j(A, C, Q, \tau) + h.o.t. \\ j &= 1, ...p - 1, \quad \phi \text{ and } R_j \text{ real polynomials} \\ Q &= (|D_1|^2, ...|D_{p-1}|^2) \end{aligned}$$

$$
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j &= 1, ...p - 1
\end{aligned}
$$

$$
\begin{aligned}
Q &= (|D_1|^2, ...|D_{p-1}|^2) \\
C &: \text{ first integral } I \text{ of the original system}
\end{aligned}
$$

$$\mathcal{R} \; : \; (A, B, C, D_j) \mapsto (A, -B, C, \bar{D}_j) \text{ anticommutes}$$

$$\sigma \; : \; (A, B, C, D_j) \mapsto (A, B, C, e^{-2i\pi\frac{m_j}{p}} D_j) \text{ commutes}$$

$$
\begin{aligned}
\phi(A, C, Q, \tau) &= \nu + aA + bA^2 + \sum_{1 \le j \le p-1} b_j |D_j|^2 \\
\nu(C, \tau) &= 24(1 - \tau)\{1 + h.o.t.\} \\
a(C, \tau) &= -12\alpha + O(|C|) + h.o.t. \\
b(C, \tau) &= -12\beta + O(|C|) + h.o.t., \quad \text{if } \alpha = 0
\end{aligned}
$$

$$V(u) = \frac{1}{2}u^2 + \frac{\alpha}{3}u^3 + \frac{\beta}{4}u^4 + O(|u|^5)$$

"Physical" interpretation of solutions

$$\begin{aligned} U &= q\chi_0 + W \\ W &= W_c + \Psi(W_c, \tau) \end{aligned}$$

$$\begin{aligned} W_c &= A\chi_1 + B\chi_2 + C\chi_3 + \\ &+ \sum_{1\le j\le p-1} D_j\zeta_j + \bar{D}_j\bar{\zeta}_j + \Phi(A, B, C, D, \bar{D}, \tau) \end{aligned}$$

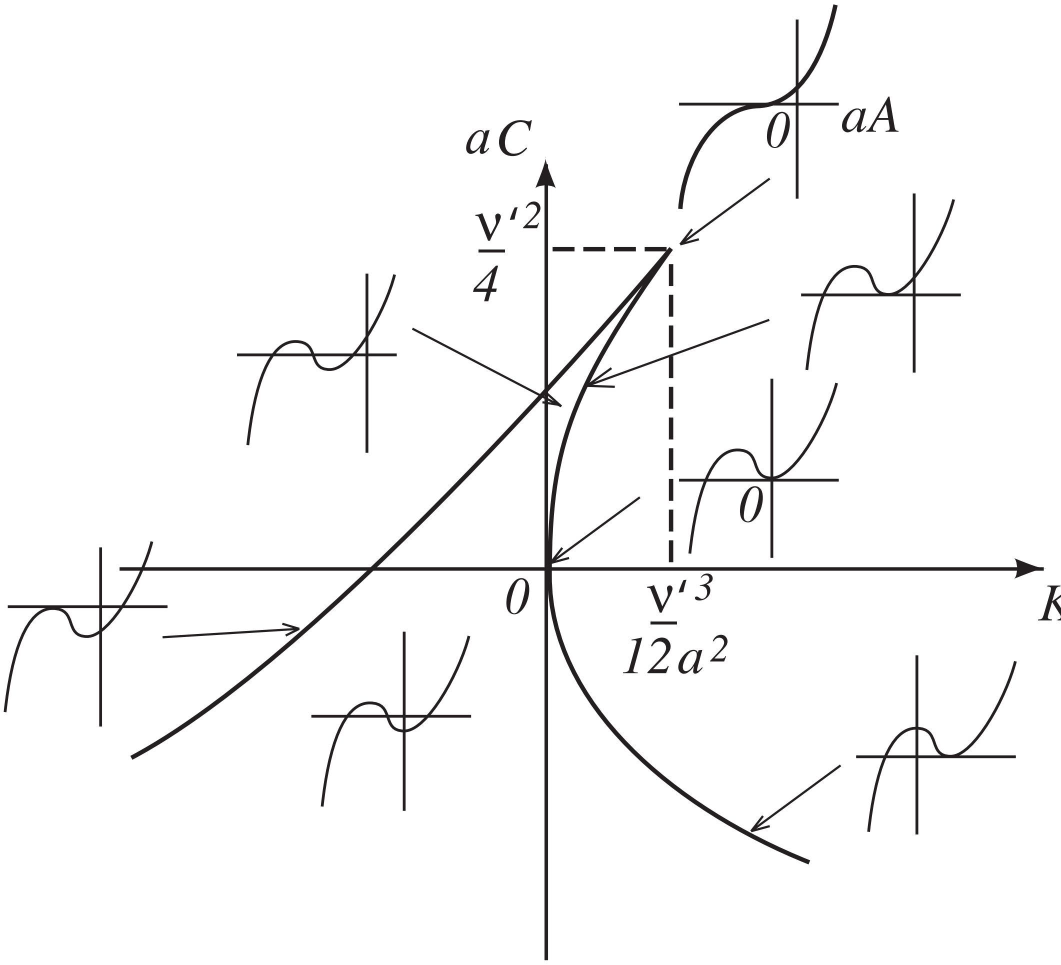
$$\begin{aligned} \frac{dq}{dx} &= A + \chi_1^*(\Phi + \Psi) \\ y_n(x) &= q(x) + \sum_{1\le j\le p-1} \Big(ie^{-2in\pi m_j/p} D_j + \\ & -ie^{2in\pi m_j/p}\bar{D}_j \Big) + [\Phi_n + \Psi_n]_y \end{aligned}$$

For equilibria in W:

A and C constants, $B = 0$, $D_j = 0$

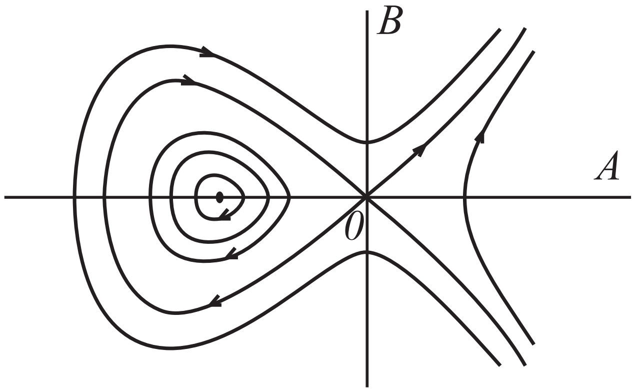
$A < 0 \Rightarrow$ uniform compression of the lattice

$A > 0 \Rightarrow$ uniform dilatation of the lattice

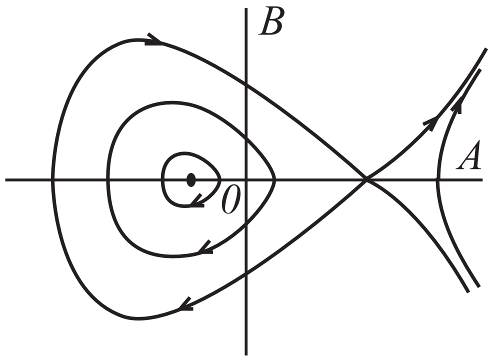


Graphs of $B^2 = f_{C,K}(A)$ for $\nu' > 0$ (for example $\tau < 1$ and $|D_j|$ small), $\alpha \neq 0$

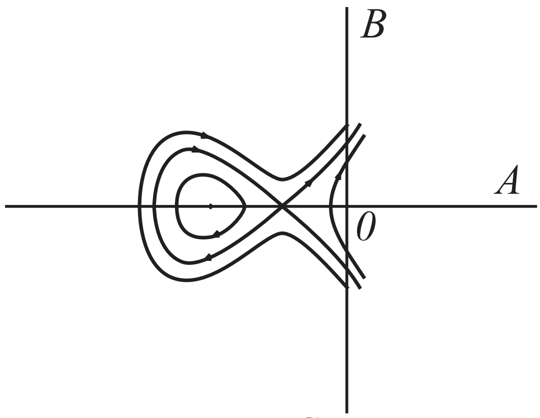
$$f_{C,K}(A) = \frac{2a}{3}A^3 + \nu' A^2 + 2CA + K$$
$$\nu' = \nu + \sum_{1 \leq j \leq p-1} b_j |D_j|^2$$



$C=0\ ,\ \alpha<0$



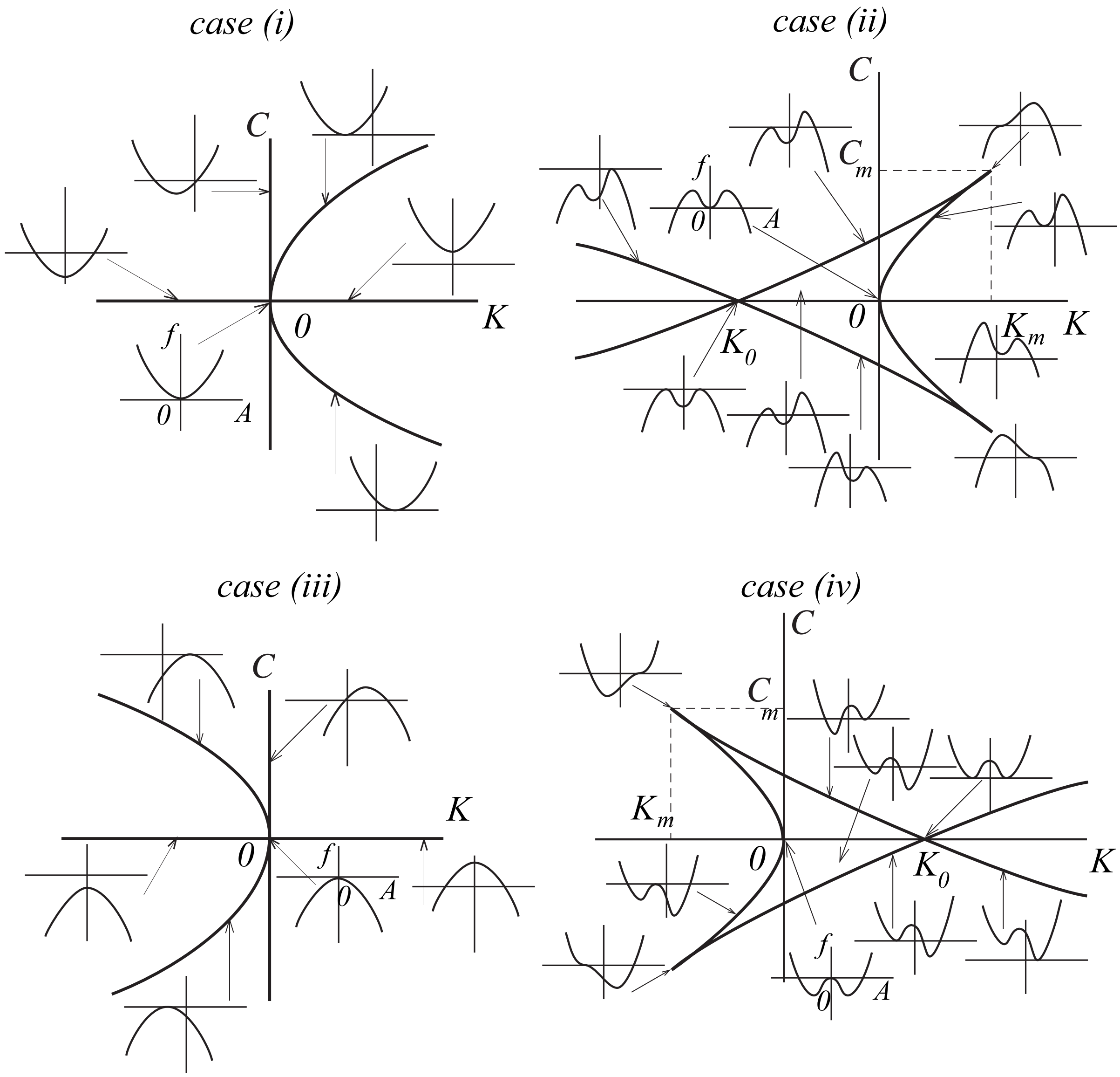
$C<0,\ \alpha<0$



$0<C<C_m\ ,\ \alpha<0$

$A<0$ corresponds to a *compression of the lattice*.

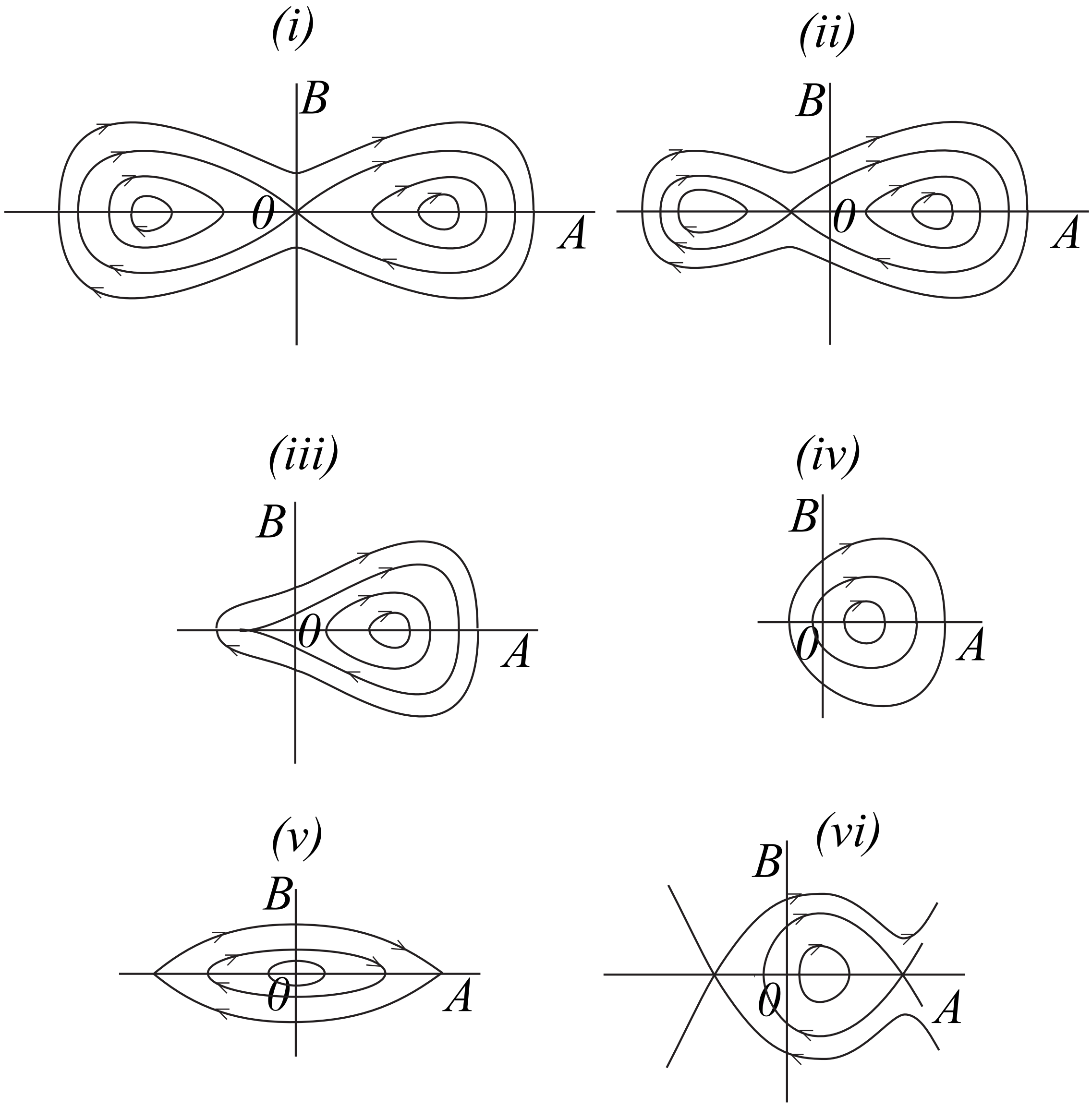
Here $\alpha<0$ in the cubic term of the potential V



Graphs of $B^2 = f_{C,K}(A)$ for $\alpha = 0$ and $\beta \neq 0$

case(i): $\nu' > 0$, $\beta < 0$, case (ii): $\nu' > 0$, $\beta > 0$,

case (iii): $\nu' < 0$, $\beta > 0$, case (iv): $\nu' < 0$, $\beta < 0$.



Phase portraits in the (A, B) plane for $\alpha = 0$

$\nu' > 0$, $\beta > 0$, (i) $C = 0$, (ii) $0 < C < C_m$

(iii) $C = C_m$, (iv) $C_m < C$

$\nu' < 0$, $\beta < 0$, (v) $C = 0$, (vi) $0 < C < C_m$

Persistence results

Problem: persistence of the above phase portraits when one takes into account the full system (the rest is not in normal form)

- Homoclinics and fronts persist (invariance under symmetry σ). They correspond to solitary (or front) travelling waves.

- For $p = 2$ (only one pair $\pm i\lambda_1$ for $\tau = 1$), symmetry σ acts as $D_1 \mapsto -D_1$. There exists a family of two reversible orbits homoclinic to periodic orbits, provided that their size is larger than $O(e^{-\frac{c}{|\tau-1|}})$. (see general results of E.Lombardi, Lecture Notes in Maths, 1741, 2000).

- For $p > 2$ No persistence proof on the market for homoclinics to quasi-periodic solutions for the normal form ($p-1$ frequencies).

These solutions would correspond to solitary waves or fronts, superposed to small oscillatory pulsating travelling waves, and to uniformly stretched ($A > 0$) or compressed ($A < 0$) states at infinity.

- The same (local) method applies for the study near other critical values of τ, also for the Klein-Gordon lattice, and for more general lattices.