Perturbation theory, KAM theory and Celestial Mechanics
4. Perturbation theory

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Outline

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Introduction

• Perturbation theory is an efficient tool to investigate nearly–integrable Hamiltonian systems, like the restricted three–body problem: the integrable part is Keplerian, the perturbation is due to the gravitational influence of the other primary, the perturbing parameter is the mass–ratio.

• Asteroid–Sun–Jupiter: m_A is so small that Sun and Jupiter move on Keplerian orbits ("restricted" problem); Jupiter–Sun mass–ratio: 10^{-3}. The solution of the restricted three–body problem can be investigated through perturbation theories and are used nowadays from ephemeris computations to astrodynamics.

• Perturbation theory in Celestial Mechanics is based on the implementation of a canonical transformation, which allows to find the solution of a nearly–integrable system within a better degree of approximation.

Outline

Nearly–integrable Hamiltonian systems

• Let us consider an n–dimensional Hamiltonian system described in terms of a set of conjugated action–angle variables $(\underline{I}, \underline{\varphi})$ with $\underline{I} \in V$, V being an open set of $\mathbb{R}^n$, and $\underline{\varphi} \in \mathbb{T}^n$.
• A nearly–integrable Hamiltonian function $\mathcal{H}(\underline{I}, \underline{\varphi})$ can be written in the form

$$\mathcal{H}(\underline{I}, \underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I}, \underline{\varphi}) \ , \qquad (1)$$

where h and f are analytic functions called, respectively, the unperturbed (or integrable) Hamiltonian and the perturbing function, while ε is a small parameter measuring the strength of the perturbation.
• For $\varepsilon = 0$ the Hamiltonian function reduces to

$$\mathcal{H}(\underline{I}, \underline{\varphi}) = h(\underline{I}) \ .$$

• The associated Hamilton's equations are simply

$$\begin{aligned} \dot{\underline{I}} &= \underline{0} \\ \dot{\underline{\varphi}} &= \underline{\omega}(\underline{I}) \ , \end{aligned} \tag{2}$$

where we have introduced the *frequency* or *rotation number:*

$$\underline{\omega}(\underline{I}) \equiv \frac{\partial h(\underline{I})}{\partial \underline{I}} \ .$$

• Equations (2) can be trivially integrated as

$$\begin{aligned} \underline{I}(t) &= \underline{I}(0) \\ \underline{\varphi}(t) &= \underline{\omega}(\underline{I}(0))t + \underline{\varphi}(0) \ , \end{aligned}$$

thus showing that the actions are constants, while the angle variables vary linearly with the time.

• For $\varepsilon \neq 0$ the equations of motion

$$\begin{aligned} \dot{\underline{I}} &= -\varepsilon \frac{\partial f}{\partial \underline{\varphi}}(\underline{I}, \underline{\varphi}) \\ \dot{\underline{\varphi}} &= \underline{\omega}(\underline{I}) + \varepsilon \frac{\partial f}{\partial \underline{I}}(\underline{I}, \underline{\varphi}) \end{aligned}$$

might no longer be integrable and chaotic motions could appear.

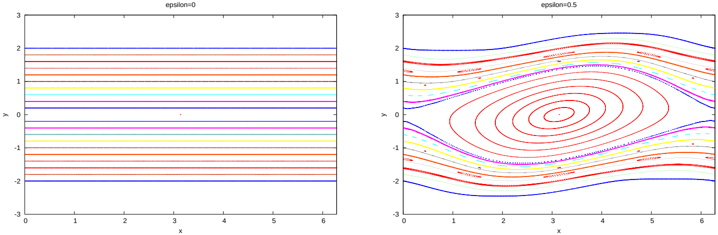

Figure: Portrait of the classical standard map, starting with $x_0 = \pi$ and varying 100 initial conditions y_0 within the interval $[0, 3]$. $a)$ Case $\varepsilon = 0$; $b)$ case $\varepsilon = 0.5$.

Outline

Classical perturbation theory

• The aim of *classical perturbation theory* is to construct a canonical transformation, which allows to push the perturbation to higher orders in ε.
• We introduce a canonical change of variables $\mathcal{C} : (\underline{I}, \underline{\varphi}) \to (\underline{I}', \underline{\varphi}')$, such that

$$\mathcal{H}(\underline{I}, \underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I}, \underline{\varphi})$$

in the transformed variables becomes

$$\mathcal{H}'(\underline{I}', \underline{\varphi}') \;=\; \mathcal{H} \circ \mathcal{C}(\underline{I}, \underline{\varphi}) \;\equiv\; h'(\underline{I}') + \varepsilon^2 f'(\underline{I}', \underline{\varphi}') \;, \qquad (3)$$

where h' and f' denote, respectively, the new unperturbed Hamiltonian and the new perturbing function.

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where h' and f' denote, respectively, the new unperturbed Hamiltonian and the new perturbing function.

• The proof is completely constructive and allows to obtain the new unperturbed Hamiltonian, the new perturbing function, the canonical transformation.
• One can iterate the algorithm to higher orders, say to obtain

$$\mathcal{H}''(\underline{I}'', \underline{\varphi}'') \ = \ \mathcal{H} \circ \mathcal{C}'(\underline{I}, \underline{\varphi}) \ \equiv \ h''(\underline{I}'') + \varepsilon^3 f''(\underline{I}'', \underline{\varphi}'') \ ,$$

and so on, provided one checks for the convergence!

• The result is obtained through the following steps:

⋄ define a suitable canonical transformation close to the identity,

⋄ perform a Taylor series expansion in ε,

⋄ require that the change of variables removes the dependence on the angles up to 2^{nd} order terms,

⋄ solve a normal form (homological) equation for the generating function,

⋄ expand in Fourier series to construct the explicit form of the canonical transformation.

• Consider two cases:

(i) the perturbing function f is a trigonometric function, namely there exists $N > 0$ such that

$$f(\underline{I}, \underline{\varphi}) = \sum_{\underline{k}\in\mathbb{Z}^n,\ 0\le|\underline{k}|\le N} \widehat{f}_{\underline{k}}(\underline{I})\ e^{i\underline{k}\cdot\underline{\varphi}}\ ;$$

(ii) the unperturbed Hamiltonian is a harmonic oscillator with frequency $\underline{\omega}_0 \in \mathbb{R}^n$:

$$h(\underline{I}) = \underline{\omega}_0 \cdot (\underline{I} - \underline{I}_0)\ .$$

Trigonometric case

Proposition (case (i)).

Let $\mathcal{H}(\underline{I},\underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I},\underline{\varphi})$ with $(\underline{I},\underline{\varphi}) \in V \times \mathbb{T}^n$ for $V \subset \mathbb{R}^n$ open and f analytic and trigonometric on $V \times \mathbb{T}^n$. Assume that for any $\underline{I}_0 \in V$, the frequency satisfies

$$|\underline{\omega}(\underline{I}_0) \cdot \underline{k}| > 0 \quad \text{for all } 0 < |\underline{k}| \le N \ .$$

Then, there exists $\rho_0 > 0$, $\varepsilon_0 > 0$ and for $|\varepsilon| < \varepsilon_0$ there exists a canonical transformation $(\underline{I},\underline{\varphi}) \to (\underline{I}',\underline{\varphi}')$ defined in $S_{\frac{\rho_0}{2}}(\underline{I}_0) \times \mathbb{T}^n \subset V \times \mathbb{T}^n$ and with values in $S_{\rho_0}(\underline{I}_0) \times \mathbb{T}^n$, which transforms $\mathcal{H}$ as

$$\mathcal{H}'(\underline{I}',\underline{\varphi}') \ = \ h'(\underline{I}') + \varepsilon^2 f'(\underline{I}',\underline{\varphi})' \ .$$

Proof.

• Define a change of variables through a close–to–identity generating function of the form $\underline{I}' \cdot \underline{\varphi} + \varepsilon\Phi(\underline{I}', \underline{\varphi})$ providing

$$\begin{aligned} \underline{I} &= \underline{I}' + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} \\ \underline{\varphi}' &= \underline{\varphi} + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{I}'} \ , \end{aligned} \tag{4}$$

where $\Phi = \Phi(\underline{I}', \underline{\varphi})$ is an unknown function, which is determined in order that $\mathcal{H}$ is transformed to $\mathcal{H}'$.

Proof.

• Define a change of variables through a close–to–identity generating function of the form $\underline{I}' \cdot \underline{\varphi} + \varepsilon \Phi(\underline{I}', \underline{\varphi})$ providing

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where $\Phi = \Phi(\underline{I}', \underline{\varphi})$ is an unknown function, which is determined in order that $\mathcal{H}$ is transformed to $\mathcal{H}'$.

• Split the perturbing function as

$$f(\underline{I}, \underline{\varphi}) = \bar{f}(\underline{I}) + \tilde{f}(\underline{I}, \underline{\varphi}) \ ,$$

where

◇ $\bar{f}(\underline{I})$ is the average over the angle variables,

◇ $\tilde{f}(\underline{I}, \underline{\varphi})$ is the remainder function defined as $\tilde{f}(\underline{I}, \underline{\varphi}) \equiv f(\underline{I}, \underline{\varphi}) - \bar{f}(\underline{I})$.

• Inserting the transformation in $\mathcal{H}$ and expanding in Taylor series around $\varepsilon = 0$ up to the second order, one gets

$$
\begin{aligned}
& h(\underline{I}' + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}}) + \varepsilon f(\underline{I}' + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}}, \varphi) \\
= \ & h(\underline{I}') + \underline{\omega}(\underline{I}') \cdot \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \varepsilon \bar{f}(\underline{I}') + \varepsilon \tilde{f}(\underline{I}', \underline{\varphi}) + O(\varepsilon^2) \ .
\end{aligned}
$$

• The transformed Hamiltonian is integrable up to the second order in ε provided that the function Φ satisfies the normal form equation:

$$
\underline{\omega}(\underline{I}') \cdot \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \tilde{f}(\underline{I}', \underline{\varphi}) = \underline{0} \ . \tag{5}
$$

• The new unperturbed Hamiltonian becomes

$$
h'(\underline{I}') \ = \ h(\underline{I}') + \varepsilon \bar{f}(\underline{I}') \ ,
$$

which provides a better integrable approximation with respect to that associated to $\mathcal{H}$.

• Explicit expression of the generating function: obtained solving the normal form equation as follows. Expand Φ and $\tilde{f}$ in Fourier series as

$$\begin{aligned} \Phi(\underline{I}', \underline{\varphi}) &= \sum_{\underline{m} \in \mathbb{Z}^n \setminus \{\underline{0}\}} \hat{\Phi}_{\underline{m}}(\underline{I}')\, e^{i\underline{m}\cdot\underline{\varphi}} , \\ \tilde{f}(\underline{I}', \underline{\varphi}) &= \sum_{0<|\underline{m}|\leq N} \hat{f}_{\underline{m}}(\underline{I}')\, e^{i\underline{m}\cdot\underline{\varphi}} . \end{aligned} \tag{6}$$

• Inserting in $\underline{\omega}(\underline{I}') \cdot \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \tilde{f}(\underline{I}', \underline{\varphi}) = \underline{0}$ one obtains

$$i \sum_{\underline{m} \in \mathbb{Z}^n \setminus \{\underline{0}\}} \underline{\omega}(\underline{I}') \cdot \underline{m}\, \hat{\Phi}_{\underline{m}}(\underline{I}')\, e^{i\underline{m}\cdot\underline{\varphi}} = - \sum_{0<|\underline{m}|\leq N} \hat{f}_{\underline{m}}(\underline{I}')\, e^{i\underline{m}\cdot\underline{\varphi}} ,$$

which provides

$$\hat{\Phi}_{\underline{m}}(\underline{I}') = - \frac{\hat{f}_{\underline{m}}(\underline{I}')}{i\, \underline{\omega}(\underline{I}') \cdot \underline{m}} . \tag{7}$$

• Summing over the Fourier coefficients, the generating function is given by

$$\Phi(\underline{I}', \underline{\varphi}) = i \sum_{0<|\underline{m}|\leq N} \frac{\hat{f}_{\underline{m}}(\underline{I}')}{\underline{\omega}(\underline{I}') \cdot \underline{m}} e^{i\underline{m}\cdot\underline{\varphi}} . \tag{8}$$

The normal form equation is solvable, provided $|\underline{I}' - \underline{I}_0| \leq \rho_1$ with ρ_1 small such that $\overline{S_{\rho_1}(\underline{I}_0)} \subset V$ and therefore

$$\underline{\omega}(\underline{I}') \cdot \underline{k} \neq 0 \quad \text{for all } 0 < |\underline{k}| \leq N .$$

From the implicit function theorem, if $|\varepsilon| < \varepsilon_0$ small, we can uniquely invert $\underline{\varphi}' = \underline{\varphi} + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{I}'}$ w.r.t. $\underline{\varphi}$ and $\underline{I} = \underline{I}' + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}}$ w.r.t. $\underline{I}'$ to get

$$\begin{aligned} \underline{I}' &= \underline{I} + \Xi'(\underline{I}, \underline{\varphi}) \\ \underline{\varphi} &= \underline{\varphi}' + \Delta(\underline{I}', \underline{\varphi}') \end{aligned}$$

with Ξ', Δ regular in $\overline{S_{\rho_1}(\underline{I}_0)} \times \mathbb{T}^n$. This ends the proof.

Remarks.

• The algorithm described above is constructive in the sense that it provides an explicit expression for the generating function and for the transformed Hamiltonian.

• We stress that (8) is well defined unless there exists an integer vector $0 < |\underline{m}| \leq N$ such that

$$\underline{\omega}(\underline{I}') \cdot \underline{m} = 0 \ . \tag{9}$$

On the contrary if, for a given value of the actions, $\underline{\omega} = \underline{\omega}(\underline{I})$ is rationally independent (which means that (9) is satisfied only for $\underline{m} = \underline{0}$), then there do not appear zero divisors, though the divisors can become arbitrarily small with a proper choice of the vector $\underline{m}$.

• For this reason, terms of the form $\underline{\omega}(\underline{I}') \cdot \underline{m}$ are called *small divisors* and they can prevent the implementation of perturbation theory.

• Moreover, the new Hamiltonian $\mathcal{H}'$ has no longer the trigonometric form and therefore the Proposition might not be applicable.

Harmonic oscillator case

Proposition (case (ii)).

Let $\mathcal{H}(\underline{I},\underline{\varphi}) = \underline{\omega}_0 \cdot (\underline{I} - \underline{I}_0) + \varepsilon f(\underline{I},\underline{\varphi})$ with $(\underline{I},\underline{\varphi}) \in V \times \mathbb{T}^n$ for $V \subset \mathbb{R}^n$ open and f analytic on $V \times \mathbb{T}^n$.

Assume that $\underline{\omega}_0$ satisfies the Diophantine condition:

$$|\underline{\omega}_0 \cdot \underline{k}|^{-1} \leq C|\underline{k}|^{\alpha} \quad \text{for all } \underline{k} \in \mathbb{Z}^n \backslash \{\underline{0}\}$$

for some $C > 0$, $\alpha > 0$.

Then, for any j there exists $\rho_0 > 0$, $\varepsilon_j > 0$ and for $|\varepsilon| < \varepsilon_j$ there exists a canonical transformation $\Phi_{\varepsilon,j}$ with $(\underline{I},\underline{\varphi}) \to (\underline{I}',\underline{\varphi}')$ defined in $S_{\frac{\rho_0}{2}}(\underline{I}_0) \times \mathbb{T}^n \subset V \times \mathbb{T}^n$, which transforms $\mathcal{H}$ into the Birkhoff normal form;

$$\mathcal{H}'(\underline{I}',\underline{\varphi}') \ = \ h_{\varepsilon,j}(\underline{I}') + \varepsilon^j f_{\varepsilon,j}(\underline{I}',\underline{\varphi}') \ ,$$

where $h_{\varepsilon,j}, f_{\varepsilon,j}$ are analytic in ε, $\underline{I}'$, $\underline{\varphi}'$.

Proof.
Define

$$\Phi_{\varepsilon,j}(\underline{I}',\underline{\varphi}) = \sum_{\ell=1}^{j} \varepsilon^\ell \Phi^{(\ell)}(\underline{I}',\underline{\varphi}) ,$$

so that the transformed Hamiltonian is

$$h(\underline{I}' + \frac{\partial \Phi_{\varepsilon,j}(\underline{I}',\underline{\varphi})}{\partial \underline{\varphi}}) + \varepsilon f(\underline{I}' + \frac{\partial \Phi_{\varepsilon,j}(\underline{I}',\underline{\varphi})}{\partial \underline{\varphi}}, \underline{\varphi})$$

with $h(\underline{I}) = \underline{\omega}_0 \cdot (\underline{I} - \underline{I}_0)$.

• Expanding in ε and using the analyticity of h, f, one needs to impose that the resulting series does not depend on $\underline{\varphi}$ up to the order j. This amounts to solve j normal form equations, which determine $\Phi^{(1)}$, ..., $\Phi^{(n)}$ like in case (i).

• Using the implicit function theorem, one can invert the transformation with invertibility conditions depending on j.

- Let us show how the $\Phi^{(1)}$, ..., $\Phi^{(n)}$ can be determined.
- Since f is analytic, we can expand in Taylor series to get:

$$
\begin{aligned}
& \underline{\omega}_0 \cdot \left(\underline{I}' + \varepsilon \frac{\partial \Phi^{(1)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \varepsilon^2 \frac{\partial \Phi^{(2)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \ldots + \varepsilon^j \frac{\partial \Phi^{(j)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} - \underline{I}_0 \right) \\
+ \quad & \varepsilon f(\underline{I}', \underline{\varphi}) + \varepsilon \frac{\partial f(\underline{I}', \underline{\varphi})}{\partial \underline{I}} \left(\varepsilon \frac{\partial \Phi^{(1)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \ldots + \varepsilon^j \frac{\partial \Phi^{(j)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} \right) \\
+ \quad & \frac{1}{2} \varepsilon \frac{\partial^2 f(\underline{I}', \underline{\varphi})}{\partial \underline{I}^2} \left(\varepsilon \frac{\partial \Phi^{(1)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \ldots + \varepsilon^j \frac{\partial \Phi^{(j)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} \right)^2 + \ldots
\end{aligned}
$$

• Order in powers of ε:

$$
\begin{aligned}
& \underline{\omega}_0 \cdot (\underline{I}' - \underline{I}_0) + \varepsilon f_0(\underline{I}') \\
+ \quad & \varepsilon \Big(\underline{\omega}_0 \cdot \frac{\partial \Phi^{(1)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \widetilde{f}(\underline{I}', \underline{\varphi}) \Big) \\
+ \quad & \varepsilon^2 \Big(\underline{\omega}_0 \cdot \frac{\partial \Phi^{(2)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \frac{\partial f(\underline{I}', \underline{\varphi})}{\partial \underline{I}} \frac{\partial \Phi^{(1)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} \Big) \\
+ \quad & \varepsilon^3 \Big(\underline{\omega}_0 \cdot \frac{\partial \Phi^{(3)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \frac{\partial f(\underline{I}', \underline{\varphi})}{\partial \underline{I}} \frac{\partial \Phi^{(2)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} \\
+ \quad & \frac{1}{2} \frac{\partial^2 f(\underline{I}', \underline{\varphi})}{\partial \underline{I}^2} \big(\frac{\partial \Phi^{(1)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} \big)^2 \Big) + \dots
\end{aligned}
$$

• Equate same orders of ε. First order:

$$\underline{\omega}_0 \cdot \frac{\partial \Phi^{(1)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \widetilde{f}(\underline{I}', \underline{\varphi}) = 0 \, .$$

Generic order ℓ:

$$\underline{\omega}_0 \cdot \frac{\partial \Phi^{(\ell)}(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \widetilde{R}_\ell(\underline{I}', \underline{\varphi}) = 0 \, ,$$

where $\widetilde{R}_\ell$ depends on $\Phi^{(1)}$, ..., $\Phi^{(\ell-1)}$ (the average is part of the new unperturbed Hamiltonian!). This equation can be solved as

$$\Phi^{(\ell)}(\underline{I}', \underline{\varphi}) = - \sum_{\underline{k} \in \mathbb{Z}^n \setminus \{0\}} \frac{\widehat{R}_{\ell, \underline{k}}(\underline{I}')}{i \, \underline{\omega}_0 \cdot \underline{k}} \, e^{i \underline{k} \cdot \underline{\varphi}},$$

which is well defined provided $\underline{\omega}_0$ is Diophantine.