Perturbation theory, KAM theory and Celestial Mechanics
7. KAM theory

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Outline

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1. Introduction

2. Non–degeneracy conditions

3. Diophantine conditions

4. Quasi-periodic motions

5. Conservative and conformally symplectic KAM theorems

6. Computer–assisted proofs

Introduction

• At the ICM in 1954 A.N. Kolmogorov gave the closing lecture titled *"The general theory of dynamical systems and classical mechanics"* on the persistence of quasi–periodic motions under small perturbations of an integrable system. V.I. Arnold (1963) used a different approach and generalized to Hamiltonian systems with degeneracies, while J.K. Moser (1962) covered the finitely differentiable case.

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• KAM theory was motivated by stability problems in Celestial Mechanics, following the works of Laplace, Lagrange, Poincaré, etc.

• KAM theory applies to *nearly–integrable* systems of the form

$$\mathcal{H}(y,x) = h(y) + \varepsilon f(y,x) \ ,$$

where $y \in \mathbb{R}^n$ (actions), $x \in \mathbb{T}^n$ (angles), $\varepsilon > 0$ is a small parameter.

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• In the *integrable* approximation $\varepsilon = 0$ Hamilton's equations are solved as

$$\begin{aligned} \dot{y} &= -\frac{\partial h(y)}{\partial x} = 0 \qquad \Rightarrow \qquad y(t) = y(0) = const. \\ \dot{x} &= \frac{\partial h(y)}{\partial y} \equiv \omega(y) \qquad \Rightarrow \qquad x(t) = \omega\left(y(0)\right) t + x(0) \ , \end{aligned}$$

where $(y(0), x(0))$ are the initial conditions. The solution takes place on a torus with frequency $\omega = \omega(y(0))$; we look for its persistence as $\varepsilon \neq 0$.

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\end{aligned}$$

where $(y(0), x(0))$ are the initial conditions. The solution takes place on a torus with frequency $\omega = \omega(y(0))$; we look for its persistence as $\varepsilon \neq 0$.
• We shall consider also *nearly–integrable dissipative* systems, like ($\lambda > 0$ dissipative constant, μ drift term):

$$\begin{aligned}
\dot y &= -\varepsilon \frac{\partial f(y,x)}{\partial x} - \lambda(y-\mu)\ , \\
\dot x &= \omega(y) + \varepsilon \frac{\partial f(y,x)}{\partial y}\ .
\end{aligned}$$

• An application to the N–body problem in Celestial Mechanics was given by Arnold, who proved the existence of a positive measure set of initial data providing quasi–periodic tori for nearly circular and nearly coplanar orbits.

• Quantitative estimates on a three–body model were given by M. Hénon, based on the original versions; the results were far from reality (at best for primaries mass-ratio 10^{-48} vs. Jupiter-Sun 10^{-3}) and Hénon concluded:

"Ainsi, ces théorèmes, bien que d'un très grand intérêt théorique, ne semblent pas pouvoir en leur état actuel être appliqués á des problèmes pratiques".

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• A change came with *computer–assisted* proofs and *a–posteriori* approach:
▷ long computations performed by a computer; rounding–off and propagation errors controlled through *interval arithmetic;*
▷ *near a nondegenerate approximately invariant torus, there is a true invariant torus* (started in [Llave-Gonzalez-Jorba-Villanueva 2005] in the context of symplectic systems).
▷ ▷ ▷ One obtains KAM results consistent with the numerical (or physical) expectation.

Remarques sur ARNOLD.

On a (ARNOLD, 1963, Usp. Mat. Nauk, n° 5, p. 21/16):

$$\delta^{(3)} \leq e^{2n} (32 n^2 + 100 n)^{-2n} ;$$

$$\delta^{(5)} \leq \delta^{(3)} ;$$

et, p. 27 :

$$\delta_1 < \delta^{(5)} ;$$

$$M = \delta_1^{8n+24} .$$

d'où :

$$M < \left(\frac{e}{32 n^2 + 100 n} \right)^{2n(8n+24)}$$

M est donc extrêmement petit : pour n = 2, plus petite valeur intéressante, on a :

$$M < \left(\frac{e}{328} \right)^{160} < 10^{-320} \quad !!$$

Donc le théorème n'a qu'un intérêt théorique et n'est absolument pas utilisable dans la pratique, du moins sous la forme présentée (cf. p. 18, §1.5, 2°) : il faut en effet que la perturbation soit extrêmement petite.

Par exemple, la stabilité du système solaire n'est pas démontrée

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Non–degeneracy conditions

i) An n–dimensional Hamiltonian function $h = h(\underline{y})$, $\underline{y} \in V$, being V an open subset of $\mathbb{R}^n$, is said to be *non–degenerate* if

$$\det\left(\frac{\partial^2 h(\underline{y})}{\partial \underline{y}^2}\right) \neq 0 \qquad \text{for any } \ \underline{y} \in V \subset \mathbb{R}^n \ . \tag{1}$$

Condition (1) is equivalent to require that the frequencies vary with the actions as

$$\det\left(\frac{\partial \underline{\omega}(\underline{y})}{\partial \underline{y}}\right) \neq 0 \qquad \text{for any } \underline{y} \in V \ .$$

The non–degeneracy condition guarantees the persistence of invariant tori with fixed frequency.

ii) $h = h(\underline{y})$, $\underline{y} \in V \subset \mathbb{R}^n$, is *isoenergetically non–degenerate* if

$$\det \begin{pmatrix} \frac{\partial^2 h(\underline{y})}{\partial \underline{y}^2} & \frac{\partial h(\underline{y})}{\partial \underline{y}} \\ \frac{\partial h(\underline{y})}{\partial \underline{y}}^\top & \underline{0} \end{pmatrix} \neq 0 \qquad \text{for any } \underline{y} \in V \subset \mathbb{R}^n . \tag{2}$$

This condition can be written as

$$\det \begin{pmatrix} \frac{\partial \underline{\omega}(\underline{y})}{\partial \underline{y}} & \underline{\omega} \\ \underline{\omega}^\top & \underline{0} \end{pmatrix} \neq 0 \qquad \text{for any } \underline{y} \in V \subset \mathbb{R}^n .$$

The isoenergetic non–degeneracy condition (independent from *i*)) guarantees that the frequency ratio varies as one crosses the tori on fixed energy surfaces.
iii) An n–dimensional Hamiltonian function $\mathcal{H}(\underline{y}, \underline{x}) = h(\underline{y}) + \varepsilon f(\underline{y}, \underline{x})$, $\underline{y} \in \mathbb{R}^n$, $\underline{x} \in \mathbb{T}^n$, is said to be *properly degenerate* if $h(\underline{y})$ does not depend explicitly on some action variables. In this case, the perturbation $f(\underline{y}, \underline{x})$ is said to remove the degeneracy if $f(\underline{y}, \underline{x}) = \bar{f}(\underline{y}) + \varepsilon f_1(\underline{y}, \underline{x})$ with the property that $h(\underline{y}) + \varepsilon \bar{f}(\underline{y})$ is non–degenerate.

Examples:

- Degenerate Hamiltonian:

$$h(\underline{y}) = \underline{\omega}_0 \cdot \underline{y} \; .$$

- Non-degenerate Hamiltonian:

$$h(\underline{y}) = \frac{\underline{y}^2}{2} \; ,$$

which implies $\frac{\partial^2 h(\underline{y})}{\partial \underline{y}^2} = \mathrm{Id}$.

- Isoenergetically non-degenerate Hamiltonian:

$$h(y_1, y_2) = \frac{y_1^2}{2} + y_2 \; ,$$

which does not satisfy the non-degeneracy, since $\frac{\partial^2 h(\underline{y})}{\partial \underline{y}^2} = 0$, while

$$\det \begin{pmatrix} \frac{\partial^2 h(\underline{y})}{\partial \underline{y}^2} & \frac{\partial h(\underline{y})}{\partial \underline{y}} \\ \frac{\partial h(\underline{y})}{\partial \underline{y}}^T & \underline{0} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & y_1 \\ 0 & 0 & 1 \\ y_1 & 1 & 0 \end{pmatrix} \neq 0 \; .$$

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Diophantine conditions

• To apply KAM theory assume that the unperturbed Hamiltonian satisfies a non-degeneracy condition.
• The second requirement is that the frequency $\underline{\omega}$ satisfies a strong irrationality assumption, namely the *diophantine condition.*

Definition

The frequency vector $\underline{\omega} \in \mathbb{R}^n$ satisfies a Diophantine condition of type (C, τ) for some $C \in \mathbb{R}_+$, $\tau \geq 1$, if for any integer vector $\underline{m} \in \mathbb{R}^n \backslash \{\underline{0}\}$:

$$|\underline{\omega} \cdot \underline{m}|^{-1} \leq C|\underline{m}|^{\tau} \ . \tag{3}$$

For maps $\omega \in \mathbb{R}^n$ satisfies the Diophantine condition if

$$|\frac{\underline{\omega} \cdot \underline{q}}{2\pi} - p|^{-1} \ \leq \ C|q|^{\tau} \ , \qquad p \in \mathbb{Z} \ , \quad q \in \mathbb{Z}^n \backslash \{0\} \quad C > 0 \ , \quad \tau > 0 \ .$$

SOME PROPERTIES OF DIOPHANTINE NUMBERS:

• The size of the sets $\mathcal{D}(C, \tau)$ increases as C or τ increases.

• There are no Diophantine vectors in $\mathbb{R}^n$ with $\tau < n - 1$.

• The set of Diophantine vectors with $\tau = n - 1$ in $\mathbb{R}^n$ has zero Lebesgue measure (but it is everywhere dense).

• For any $\tau > n - 1$ almost every vector in $\mathbb{R}^n$ is τ-Diophantine, namely the complement has zero Lebesgue measure although it is everywhere dense.

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Quasi-periodic motions

• A conditionally periodic motion is given by a function $t \mapsto f(\omega_1 t, \ldots, \omega_n t)$, where $f(x_1, \ldots, x_n)$ is periodic in all variables; the vector $\underline{\omega} = (\omega_1, \ldots, \omega_n)$ is called frequency.

• Conditionally periodic motions with incommensurable frequencies are called quasi-periodic motions.

• An invariant torus is an invariant manifold diffeomorphic to the standard torus $\mathbb{T}^n$. Any trajectory on an invariant torus carrying quasi-periodic motions is dense on the torus.

Theorem (Kolmogorov)

Given the Hamiltonian system

$$\mathcal{H}(\underline{y},\underline{x}) = h(\underline{y}) + \varepsilon f(\underline{y},\underline{x}) \ , \qquad \underline{y} \in \mathbb{R}^n \ , \ \ \underline{x} \in \mathbb{T}^n \ ; \tag{4}$$

satisfying the non–degeneracy condition

$$\det \left(\frac{\partial^2 h(\underline{y})}{\partial \underline{y}^2} \right) \neq 0 \qquad \text{for any } \ \underline{y} \in V \subset \mathbb{R}^n \ , \tag{5}$$

having fixed a diophantine frequency $\underline{\omega}$ for the unperturbed system, if ε is *sufficiently small* there still exists an invariant torus on which the motion is quasi–periodic with frequency $\underline{\omega}$.

• Later extended in different settings by V.I. Arnold and J. Moser: KAM theorem.
• For low values of ε there exists an invariant surface with diophantine frequency $\underline{\omega}$; as far as ε increases the invariant torus with frequency $\underline{\omega}$ is more and more distorted and displaced, until ε reaches a critical value at which the torus breaks down (Figure 1).
• KAM theorem provides a lower bound on the breakdown threshold; *computer–assisted* KAM estimates provide, in simple examples, results on the parameters which are consistent with the physical values.

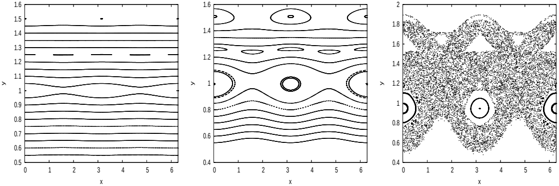

Figure: The Poincarè section of the spin–orbit problem for 20 different initial conditions and for $e = 0.1$. Left: $\varepsilon = 10^{-3}$, center: $\varepsilon = 10^{-2}$, right: $\varepsilon = 10^{-1}$.

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Conservative and conformally symplectic KAM theorems

- We state the KAM theorems in a general setting, precisely:

◇ symplectic maps (e.g. the conservative standard map),

◇ conformally symplectic maps (e.g. the dissipative standard map).

- Many physical problems are described by conformally symplectic systems, characterized by the property that they transform the symplectic form into a multiple of itself.
- Example of conformally symplectic systems:

(i) Hamiltonian systems with a dissipation proportional to the velocity, like in the spin–orbit problem with tidal torque;

(ii) Euler-Lagrange equations of exponentially discounted systems, which are models typically found in finance, when inflation is present and one needs to minimize the cost in present money;

(iii) Gaussian thermostats (mechanical systems with forcing and a thermostating term based on the Gauss Least Constraint Principle for nonholonomic constraints).

• Let $\mathcal{M} = U \times \mathbb{T}^n$ be the phase space with $U \subseteq \mathbb{R}^n$ open, simply connected domain with smooth boundary; $\mathcal{M}$ is endowed with the standard scalar product and a symplectic form Ω.

Definition

A diffeomorphism f on $\mathcal{M}$ is *conformally symplectic*, if there exists a function $\lambda : \mathcal{M} \to \mathbb{R}$ such that (f^* denotes the pull–back via f)

$$f^*\Omega = \lambda\Omega \ .$$

• For $n = 1$ any diffeomorphism is conformally symplectic with λ depending on the coordinates; $\lambda =$constant for $n \geq 2$; $\lambda = 1$ in the *symplectic* case.

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Definition

We say that the frequency vector $\omega \in \mathbb{R}^n$ satisfies the Diophantine condition if

$$|\frac{\omega \cdot q}{2\pi} - p|^{-1} \ \leq \ C|q|^\tau \ , \qquad p \in \mathbb{Z} \ , \quad q \in \mathbb{Z}^n \backslash \{0\} \quad C > 0 \ , \quad \tau > 0 \ ;$$

$\mathcal{D}(C, \tau)$ = set of Diophantine vectors, which is of full Lebesgue measure in $\mathbb{R}^n$.

FLOWS:

Definition

We say that a vector field X is a *conformally symplectic flow* if, denoting by L_X the Lie derivative, there exists a function $\lambda : \mathbb{R}^{2n} \to \mathbb{R}$ such that

$$L_X \Omega = \lambda \Omega \ .$$

The time t-flow Φ_t satisfies

$$(\Phi_t)^* \Omega = e^{\lambda t} \Omega \ ,$$

where Ω is the symplectic form such that $\Omega_x(u, v) = (u, J(x)v)$.

Definition

In the case of *flows* the Diophantine condition is:

$$|\underline{\omega} \cdot \underline{k}|^{-1} \ \leq \ C|\underline{k}|^{\tau} \ , \qquad \underline{k} \in \mathbb{Z}^n \backslash \{0\} \ ,$$

for $C > 0$, $\tau > 0$.

EXAMPLE OF CONFORMALLY SYMPLECTIC MAP: the dissipative standard map.

$$\begin{aligned} y' &= \lambda y + \mu + \varepsilon \sin(x) \\ x' &= x + \lambda y + \mu + \varepsilon \sin(x) \ . \end{aligned}$$

- $f^* \Omega(\underline{u}, \underline{v}) = \Omega(Df\ \underline{u}, Df\ \underline{v})$? =? $\lambda \Omega(\underline{u}, \underline{v})$ with $\underline{u} = (u_1, u_2)$, $\underline{v} = (v_1, v_2)$.
- $\Omega(\underline{u}, \underline{v}) = (\underline{u}, J\underline{v})$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- $Df = \begin{pmatrix} \lambda & \varepsilon \cos x \\ \lambda & 1 + \varepsilon \cos x \end{pmatrix}$.

NOTATION: From now on drop the underline to denote vectors.

- A quasi–periodic solution is an orbit of the form $(A_n, \theta_n) = K(n \cdot \omega)$, ω irrational.

Definition

Let $\mathcal{M} \subseteq \mathbb{R}^n \times \mathbb{T}^n$ be a symplectic manifold and let $f : \mathcal{M} \to \mathcal{M}$ be a symplectic map. A *KAM surface* with frequency $\omega \in \mathcal{D}(C, \tau)$ is an n–dimensional invariant surface described parametrically by an embedding $K : \mathbb{T}^n \to \mathcal{M}$, which is the solution of the invariance equation:

$$f \circ K(\theta) = K(\theta + \omega) . \tag{6}$$

For a family f_μ of conformally symplectic **diffeomorphisms** depending on a real parameter μ, look for $\mu = \mu_*$ and an embedding K, such that

$$f_{\mu_*} \circ K(\theta) = K(\theta + \omega) .$$

For conformally symplectic **vector fields** X_μ look for μ_* and K, such that

$$X_{\mu_*} \circ K(\theta) = (\omega \cdot \partial_\theta)\, K(\theta) .$$

EXAMPLE OF PARAMETRIC REPRESENTATION: the dissipative standard map.

$$\begin{aligned} y' &= \lambda y + \mu + \varepsilon \sin(x) \\ x' &= x + y' . \end{aligned}$$

- $x = \theta + K_1(\theta)$, $y = K_2(\theta)$, $\theta' = \theta + \omega$.
- From the first equation:

$$K_2(\theta + \omega) - \lambda K_2(\theta) = \mu + \varepsilon \sin(\theta + K_1(\theta)) .$$

- From the second equation:

$$K_2(\theta) = \omega + K_1(\theta) - K_1(\theta - \omega) .$$

- Combining the last two equations:

$$K_1(\theta+\omega) - (1+\lambda)K_1(\theta) + \lambda K_1(\theta - \omega) = -(1-\lambda)\omega + \mu + \varepsilon \sin(\theta + K_1(\theta)) .$$

• Invariant tori are Lagrangian; if f is confor. symplectic, $|\lambda| \neq 1$ and K satisfies the invariance equation:

$$K^*\Omega = 0 \ . \tag{7}$$

If f is symplectic and ω is irrational, then the torus is Lagrangian (i.e. with maximal dimension and isotropic, namely the symplectic form on the manifold restricts to zero, i.e. each tangent space is an isotropic subspace of the ambient manifold's tangent space).

Definition

Analytic norm. Given $\rho > 0$, we define $\mathbb{T}^n_\rho$ as the set

$$\mathbb{T}^n_\rho = \{\theta \in \mathbb{C}^n/(2\pi\mathbb{Z})^n : \ \mathrm{Re}(\theta) \in \mathbb{T}^n, \ |\mathrm{Im}(\theta_j)| \leq \rho \ , \ \ j = 1, ..., n\} \ ;$$

we denote by $\mathcal{A}_\rho$ the set of analytic functions in $Int(\mathbb{T}^n_\rho)$ with the norm

$$\|f\|_\rho = \sup_{\theta \in \mathbb{T}^n_\rho} |f(\theta)| \ .$$

Sobolev norm. Expand in Fourier series $f(z) = \sum_{k \in \mathbb{Z}^n} \widehat{f}_k e^{2\pi i k z}$ and for $m > 0$:

$$H^m = \left\{ f : \mathbb{T}^n \to \mathbb{C} \ : \ \|f\|^2_m \equiv \sum_{k \in \mathbb{Z}^n} |\widehat{f}_k|^2 (1 + |k|^2)^m < \infty \right\} .$$

Theorem (conservative case, R. de la Llave et al.)

$\omega \in \mathcal{D}(C,\tau)$, $f : \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n$ symplectic and analytic, K_0 approximate solution: $f \circ K_0(\theta) - K_0(\theta+\omega) = E_0(\theta)$. Let $N(\theta) \equiv (DK_0(\theta)^T\, DK_0(\theta))^{-1}$; let $J = J(x)$ be the matrix representing Ω at x: $(\Omega_x(u,v) = (u, J(x)v))$ and let $S(\theta)$ be

$$
\begin{aligned}
S(\theta) \quad &\equiv N(\theta+\omega)^T DK_0(\theta+\omega)^T \Big[Df(K_0(\theta))\, J(K_0(\theta))^{-1} DK_0(\theta)\, N(\theta) \\
&\qquad -J(K_0(\theta+\omega))^{-1}\, DK_0(\theta+\omega)\, N(\theta+\omega)\, A(\theta)\Big]
\end{aligned}
$$

with $A(\theta) = \mathrm{Id}$. Assume that S satisfies the non–degeneracy condition

$$\det \langle S(\theta) \rangle \neq 0 \,,$$

($\langle \cdot \rangle$=average). Let $0 < \delta < \frac{\rho}{2}$; if the solution is sufficiently approximate:

$$\|E_0\|_\rho \le C_1\, C^{-4}\, \delta^{4\tau} \qquad (C_1 > 0) \,,$$

then there exists an exact solution $K_e = K_e(\theta)$ of (6), such that

$$\|K_e - K_0\|_{\rho-2\delta} < C_2\, C^2\, \delta^{-2\tau}\, \|E_0\|_\rho \qquad (C_2 > 0) \,.$$

Theorem (conformally sympl. case, R. Calleja, A.C., R. de la Llave)

Let $\omega \in \mathcal{D}(C,\tau)$, $f_\mu : \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{R}^n \times \mathbb{T}^n$ *conformally symplectic,* (K_0, μ_0) *approximate solution:* $f_{\mu_0} \circ K_0(\theta) - K_0(\theta+\omega) = E_0(\theta)$*. Let* $M(\theta)$ *be the* $2n \times 2n$ *matrix*

$$M(\theta) = [DK_0(\theta) \mid J(K_0(\theta))^{-1}\, DK_0(\theta)N(\theta)]\ .$$

Assume the following non–degeneracy condition:

$$\det \begin{pmatrix} \langle S \rangle & \langle SB^0 \rangle + \langle \widetilde{A}_1 \rangle \\ (\lambda-1)\mathrm{Id} & \langle \widetilde{A}_2 \rangle \end{pmatrix} \neq 0\ ,$$

with $A(\theta) = \lambda\,\mathrm{Id}$, $\widetilde{A}_1$, $\widetilde{A}_2$ *first and second n columns of* $\tilde{A} = M^{-1}(\theta+\omega)D_{\mu_0}f_{\mu_0} \circ K_0$, $B^0 = B - \langle B \rangle$ *solution of* $\lambda B^0(\theta) - B^0(\theta+\omega) = -(\widetilde{A}_2)^0(\theta)$*. Let* $0 < \delta < \frac{\rho}{2}$*; if the solution is sufficiently approximate, i.e.*

$$\|E_0\|_\rho \leq C_3\, C^{-4}\, \delta^{4\tau} \qquad (C_3 > 0)\ ,$$

there exists an exact solution (K_e, μ_e)*, such that*

$$\|K_e - K_0\|_{\rho-2\delta} \leq C_4\, C^2\, \delta^{-2\tau}\, \|E_0\|_\rho\ , \quad |\mu_e - \mu_0| \leq C_5\, \|E_0\|_\rho \quad (C_4, C_5 > 0)\ .$$

• A remark on the non–degeneracy conditions.

• For the conservative standard map

$$\begin{aligned} y' &= y + \varepsilon\, g(x) \\ x' &= x + y'\,, \end{aligned} \qquad\qquad \frac{\partial x'}{\partial y} \neq 0\,,$$

non–degeneracy equivalent to the *twist* condition, namely the lift transforms any vertical line always on the same side.

• For the (generalized) dissipative standard map

$$\begin{aligned} y' &= \lambda y + p(\mu) + \varepsilon\, g(x) \\ x' &= x + y'\,, \end{aligned} \qquad\qquad \frac{\partial x'}{\partial y} \neq 0 \quad \& \quad \frac{dp(\mu)}{d\mu} \neq 0\,,$$

non–degeneracy condition involves the twist condition and a non–degeneracy w.r.t. to the parameters.

Outline

Computer–assisted proofs

• The KAM proof requires very long computations (initial approximation, KAM algorithm, etc). A computer is needed due to the high number of operations involved, but it introduces rounding-off and propagation errors $\rightarrow$ *interval arithmetic.*

$\diamond$ The computer stores real numbers using a sign–exponent–fraction representation with a number of digits in the fraction and the exponent, varying with the machine. The result of any elementary operation (+,-,*,/) usually produces an approximation of the true result (other calculations can be reduced to a sequence of elementary operations).

$\diamond$ Interval arithmetic: represent any real number as an interval and perform elementary operations on intervals, rather than on real numbers (computer time increases).

$\diamond$ Example: $a+b$. Assume $a \in [a_1, a_2]$, $b \in [b_1, b_2]$. Then $c = a + b$ is $[c_1, c_2] \equiv [a_1 + b_1, a_2 + b_2]$. The end points c_1, c_2 are themselves produced by an elementary operation and therefore affected by rounding errors. Take a slightly smaller value for c_1 and slightly bigger for c_2, so that $a + b \in [c_1, c_2]$.