

Chapter 9: SR3BP (Spatial Restricted 3 Body Problem)

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Outline

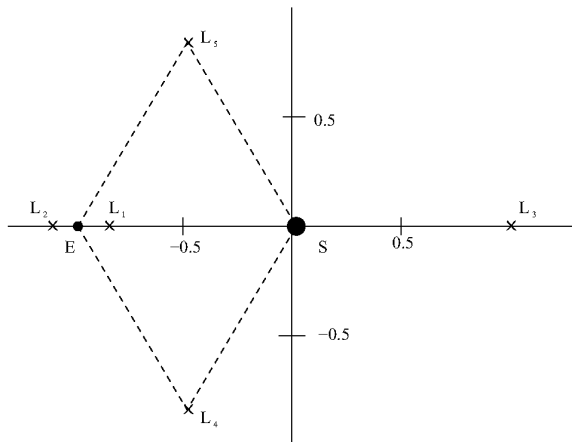
Dynamics of the SR3BP Near L_1

Standard Scattering Map

Existence of Drift Orbits

Application: Shortest-Time Drift Trajectories

The Spatial R3BP



$$H = \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + YP_X - XP_Y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} - \frac{1}{2}\mu(1-\mu)$$

$$C = -2H \quad \text{Jacobi constant} \iff H = C/2$$

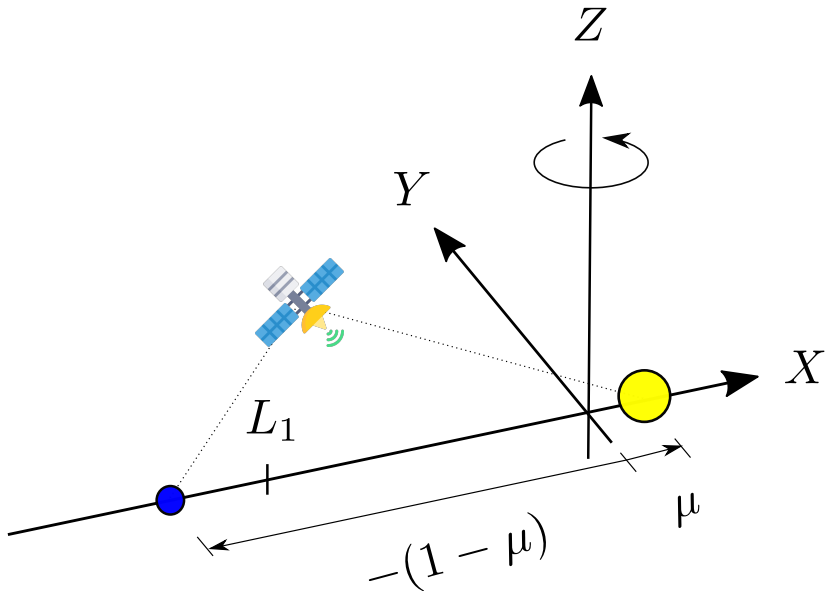


Figure: Schematic location of the Sun (in yellow), the Earth (in blue), and the equilibrium point L_1 .

Dynamics Near L_1

- ▶ L_1 is of type center $\times$ center $\times$ saddle. Denote by $\pm\lambda, \pm i\nu_p, \pm i\nu_v$, with λ, ν_p, ν_v real and positive the eigenvalues of the linearized system near L_1 .
- ▶ Fix the energy level to $C = c$ close to the one of L_1 . In our computations, $C = 3.00088$ (in the Sun-Earth system $\mu = 3.040423398444176 \times 10^{-6}$, $C_{L_1} = 3.00090$, $C_{\text{halo}} = 3.00082$, $\lambda = 2.5326591740529678$, $\nu_p = 2.0864535642231075$ and $\nu_v = 2.0152106629966386$.)
- ▶ By the Center Manifold theorem there exists a 4-dimensional invariant center manifold $\Lambda = W^c(L_1)$ that is tangent at L_1 to the generalized eigenspace corresponding to $\pm i\nu_p, \pm i\nu_v$. The manifold Λ is a normally hyperbolic invariant manifold (NHIM) for the flow Φ_H .

- ▶ If we fix the energy level, the restriction of the center manifold $\Lambda = W^c(L_1)$ to the energy level $M_c = \{H = -c/2\}$ is a 3-dimensional sphere

$$\Lambda_c = \Lambda \cap M_c,$$

which is a NHIM for the flow Φ_H restricted to the energy manifold M_c .

- ▶ The Hamiltonian H can be expanded about L_1 , via a symplectic coordinate change

$$(P_X, P_Y, P_Z, X, Y, Z) \mapsto (y_h, J_p, J_v, x_h, \phi_p, \phi_v),$$

as a Birkhoff normal form

$$\mathcal{H}^{(N)} = H_2 + \mathcal{Z}^{(N)} + \mathcal{R}^{(N)},$$

where H_2 denotes the quadratic part of H expressed in terms of the new variables, $\mathcal{Z}^{(N)}$ is a polynomial of degree N that Poisson-commutes with H_2 , and the remainder $\mathcal{R}^{(N)}$ is small in a neighborhood of L_1 (of the order of the $(N+1)$ -th power of the distance to L_1). The coordinates x_h, y_h represent the saddle directions of motion.

- ▶ We refer to J_p, J_v as the action variables, and to ϕ_p, ϕ_v as the angle variables. The *planar variables* (J_p, ϕ_p) correspond to the *planar* component of the motion, while the *vertical variables* (J_v, ϕ_v) corresponds to the *vertical* component.
- ▶ The truncated Birkhoff normal form depends only on $x_h y_h, J_p, J_v$:

$$\mathcal{H}_{\text{trunc}}^{(N)} = H_2 + \mathcal{Z}^{(N)} = \lambda x_h y_h + \nu_p J_p + \nu_v J_v + \mathcal{Z}^{(N)}(x_h y_h, J_p, J_v),$$

which are integrals of motion for $\mathcal{H}_{\text{trunc}}^{(N)}$. We denote by Φ_N the flow associated to $\mathcal{H}^{(N)}$.

- ▶ The truncated Birkhoff normal form $\mathcal{H}_{\text{trunc}}^{(N)}$ represents an approximation of the original Hamiltonian H , and therefore the invariant objects for $\mathcal{H}_{\text{trunc}}^{(N)}$ give approximations of the corresponding invariant objects of H .

- ▶ Truncating the normal form at order $N = 16$ ensures that the error between the true dynamics in the NHIM and the dynamics induced by the Birkhoff normal form is within machine precision. In particular, we consider the center manifold Λ^N for the truncated Birkhoff normal $H_{\text{trunc}}^{(N)}$, which represents an approximation of Λ .
- ▶ In the action-angle coordinates, we have that

$$\Lambda^N = \{(y_h, J_p, J_v, x_h, \phi_p, \phi_v) \mid y_h = 0, x_h = 0, \\ J_p, J_v \geq 0, \phi_p, \phi_v \in \mathbb{T}\},$$

is parameterized by the symplectic action-angle coordinates $(J_p, J_v, \phi_p, \phi_v)$.

- ▶ For a fixed energy $\mathcal{H}_{\text{trunc}}^{(N)} = h = -c/2$, J_p is given as an implicit function of J_v, ϕ_p, ϕ_v, c . The corresponding NHIM for the truncated system is given by

$$\Lambda_c^N = \Lambda^N \cap \left\{ \mathcal{H}_{\text{trunc}}^{(N)} = -c/2 \right\},$$

and is therefore parameterized by the 3 variables (J_v, ϕ_p, ϕ_v) .

- ▶ For each fixed value of the vertical action $J_v = \bar{J}_v$, there is a unique invariant torus for the Birkhoff normal form

$$T_{\bar{J}_v}^N = \Lambda_c^N \cap \{J_v = \bar{J}_v\}$$

which is therefore parameterized by the 2 angles (ϕ_p, ϕ_v) .

- ▶ Thus, the NHIM Λ_c^N is foliated by a family of 2-dimensional invariant tori. By comparison, the NHIM Λ_c corresponding to the full Hamiltonian H contains a KAM family of 2-dimensional invariant tori, with gaps between the tori that are exponentially small in the action.
- ▶ Each of the objects Λ^N , Λ_c^N , $T_{\bar{J}_v}^N$, have associated stable and unstable (or asymptotic) manifolds of one more dimension than the object itself. The truncated Birkhoff normal form provides an accurate approximation of the local asymptotic manifolds in a neighborhood of L_1 .

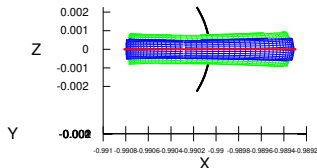
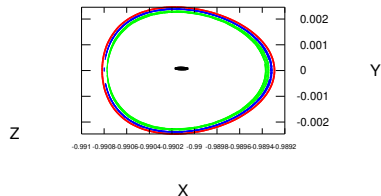
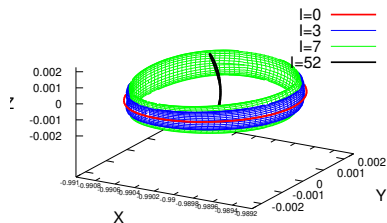
- ▶ In particular, the 4-dimensional local stable and unstable manifolds of Λ_c^N are given by

$$W_{\text{loc}}^{s,u}(\Lambda_c^N) = \Lambda_c^N \oplus \Upsilon^{s,u},$$

where $\Upsilon^{s,u}$ is a small interval in the stable (unstable) direction. In practice, we take this interval to be of length $\delta = 10^{-3}$.

- ▶ Once the local stable and unstable manifolds are obtained, the global manifolds are computed by integrating the local ones by the flow Φ_H of the Hamiltonian H .

NHIM Λ (radius: 300, 000 km, distance Earth to Sun = 148.34×10^6 km=1 unity of length)



Inner Flow

The SR3BP flow on the center manifold Λ consists simply on a translation on the 2-torus,

$$\begin{aligned} \dot{J}_p &= 0, & \dot{\phi}_p &= \frac{\partial H}{\partial J_p} =: \nu_p(J_p, J_v), \\ \dot{J}_v &= 0, & \dot{\phi}_v &= \frac{\partial H}{\partial J_v} =: \nu_v(J_p, J_v). \end{aligned}$$

The value of the planar and vertical frequencies ν_p, ν_v of the torus are obtained differentiating the Hamiltonian in Birkhoff normal form.

Upon restriction to the NHIM Λ_c , we get rid of the planar action J_p , which can be recovered if necessary using the energy condition, so the inner flow in Λ_c is

$$\dot{J}_v = 0, \quad \dot{\phi}_p = \nu_p(J_v), \quad \dot{\phi}_v = \nu_v(J_v).$$

Inner Map

To reduce dimensionality we consider the Poincaré section $\Sigma = \{\phi_p = 0\}$, with associated first return map $\mathcal{F}: \Sigma \rightarrow \Sigma$. Let $\Lambda_c^{N,\Sigma}$ denote the intersection of the NHIM Λ_c^N with Σ :

$$\Lambda_c^{N,\Sigma} = \Lambda_c^N \cap \Sigma.$$

$\Lambda_c^{N,\Sigma}$ is a normally hyperbolic invariant manifold for $\mathcal{F}$, which we call the *reduced NHIM*. This is a 2-dimensional manifold, that can be parameterized in terms of (J_v, ϕ_v) .

The first return map of the inner flow to the section Σ is

$$J'_v = J_v, \quad \phi'_v = \phi_v + \frac{2\pi\nu_v(J_v)}{\nu_p(J_v)}.$$

For ease of notation, we will simply write (J, ϕ) for (J_v, ϕ_v) .

From now on, we simply write (J, ϕ) for (J_v, ϕ_v) .

We note that J and ϕ have different scales: J is of order 10^{-3} , while ϕ is of order 1. To improve numerical conditioning, we scale J as follows:

$$I = 1000J. \quad (1)$$

From now on, we will work with the scaled coordinate I instead of J .

In terms of the scaled coordinates I, ϕ , the inner map

$$\tau: \Lambda_c^{N, \Sigma} \rightarrow \Lambda_c^{N, \Sigma}$$

is given by

$$I' = I, \quad \phi' = \phi + \nu(I), \quad (2)$$

where we have introduced the new function

$$\nu(I) := \frac{2\pi\nu_v(I/1000)}{\nu_p(I/1000)}.$$

As seen in this figure, the inner shift $\nu(I)$ decreases almost linearly with I . In particular, this shows that the inner map τ is a twist map.

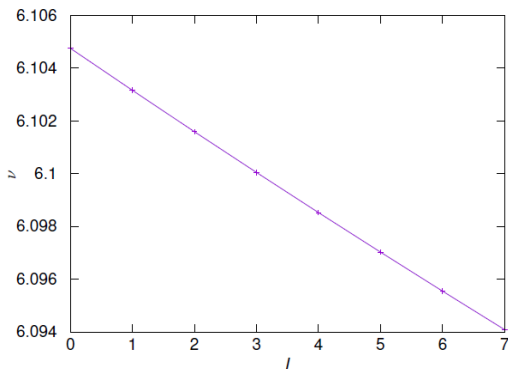


Figure: The inner shift $\nu(I)$.

Planar Lyapunov Orbit and its Two Homoclinics

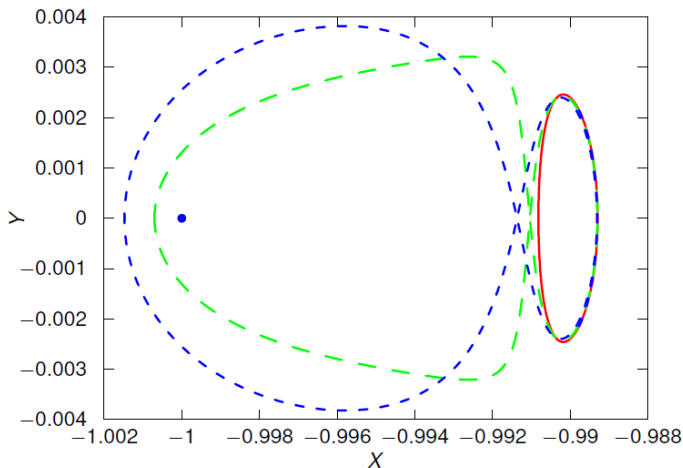


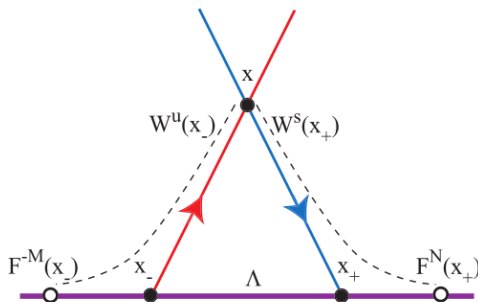
Figure: Planar Lyapunov orbit (red line), and its two 'primary' homoclinics (dashed). Both homoclinics travel around the Earth once.

Invariant Tori and their Homoclinics

- ▶ If we consider an invariant torus instead, we increase the dimension of the manifolds, giving rise to $4+4$ homoclinics. (Morse theory).
- ▶ The family of green homoclinics will be encoded by scattering map σ_1 .
- ▶ The family of blue homoclinics will be encoded by scattering map σ_2 .

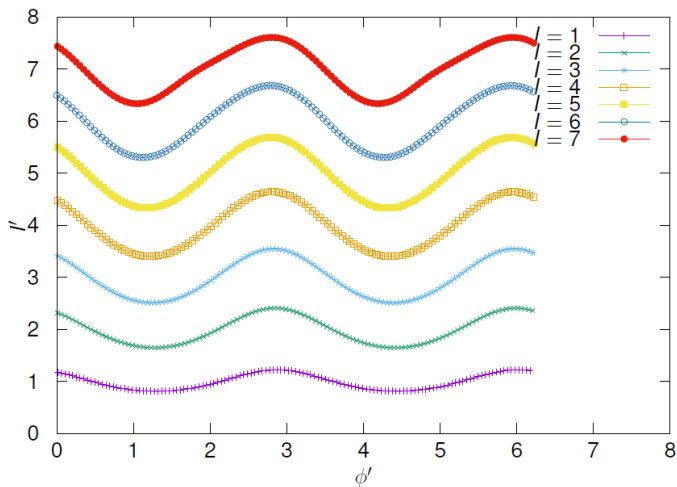
Scattering Map [D, Llave & Seara, 2008]

- ▶ A scattering map σ encodes homoclinic trajectories to Λ_C^Σ .



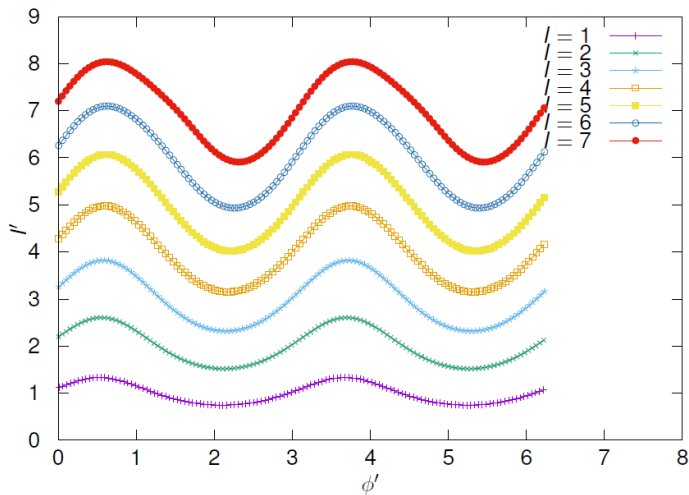
- ▶ $\sigma: \Lambda_C^\Sigma \rightarrow \Lambda_C^\Sigma, \quad \sigma(x_-) = x_+.$
- ▶ σ is exact symplectic.

Numerical SM σ_1 [D, Gidea & Roldan, 2016]



- ▶ When $l > 7$, the SM ceases to be globally defined.
- ▶ The SM is π -periodic, due to the vertical symmetry.

Numerical SM σ_2



► Notice that σ_2 makes larger jumps than σ_1 in the l variable.

Generating Function for the SM

- ▶ We represent the symplectic map $\sigma: (I, \phi) \mapsto (I', \phi')$ by a generating function

$$\mathcal{L}(I, \phi') = I\phi' + \Omega(I) + \tilde{\mathcal{L}}(I, \phi').$$

- ▶ Thus, the equations for σ are given implicitly by

$$\phi = \frac{\partial \mathcal{L}}{\partial I} = \phi' + \omega(I) + \frac{\partial \tilde{\mathcal{L}}}{\partial I}(I, \phi') \quad (3a)$$

$$I' = \frac{\partial \mathcal{L}}{\partial \phi'} = I + \frac{\partial \tilde{\mathcal{L}}}{\partial \phi'}(I, \phi'). \quad (3b)$$

Fourier-Taylor Representation

- We approximate the function $\tilde{\mathcal{L}}(I, \phi')$ using a Fourier-Taylor expansion

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \phi'} = \sum_{n=1}^N B_n(I) \sin n\phi' + \sum_{n=1}^N A_n(I) \cos n\phi'. \quad (4)$$

where

$$A_n(I) = \sum_{l=0}^L a_l^{(n)} I^l \quad \text{and} \quad B_n(I) = \sum_{l=0}^L b_l^{(n)} I^l. \quad (5)$$

- We approximate the function $\omega(I)$ using a Taylor expansion

$$\omega(I) = \sum_{l=0}^L c_l I^l. \quad (6)$$

Fourier-Taylor Coefficients

- Given the numerical SM, we use the FFT and Newton's divided differences to fit the data and obtain the Fourier-Taylor coefficients $a_l^{(n)}$, $b_l^{(n)}$, c_l . For instance, for σ_1 :

n	$\tilde{a}_1^{(n)}$	$\tilde{a}_2^{(n)}$	$\tilde{a}_3^{(n)}$	$\tilde{a}_4^{(n)}$	$\tilde{a}_5^{(n)}$	$\tilde{a}_6^{(n)}$	$\tilde{a}_7^{(n)}$
2	0.178180	-0.020025	0.000401	-0.000251	0.000046	0.000019	-0.000022
4	0.011726	-0.004878	0.001114	-0.000129	0.000039	-0.000006	0.000001
6	0.000712	-0.000519	0.000295	-0.000065	0.000004	-0.000000	-0.000001
8	0.000147	-0.000064	0.000027	-0.000012	0.000002	0.000000	0.000000

n	$\tilde{b}_1^{(n)}$	$\tilde{b}_2^{(n)}$	$\tilde{b}_3^{(n)}$	$\tilde{b}_4^{(n)}$	$\tilde{b}_5^{(n)}$	$\tilde{b}_6^{(n)}$	$\tilde{b}_7^{(n)}$
2	-0.097275	-0.005156	0.002856	-0.000574	0.000143	0.000003	-0.000016
4	-0.017677	0.003403	0.000242	-0.000067	0.000004	-0.000000	-0.000003
6	-0.002965	0.001408	-0.000332	0.000035	-0.000008	0.000002	-0.000000
8	-0.001045	0.000566	-0.000206	0.000053	-0.000009	0.000002	-0.000000

$\tilde{c}_0$	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{c}_3$	$\tilde{c}_4$	$\tilde{c}_5$	$\tilde{c}_6$
2.027748	0.046673	0.001164	0.000489	-0.000046	0.000016	0.000003

Local Approximation Error

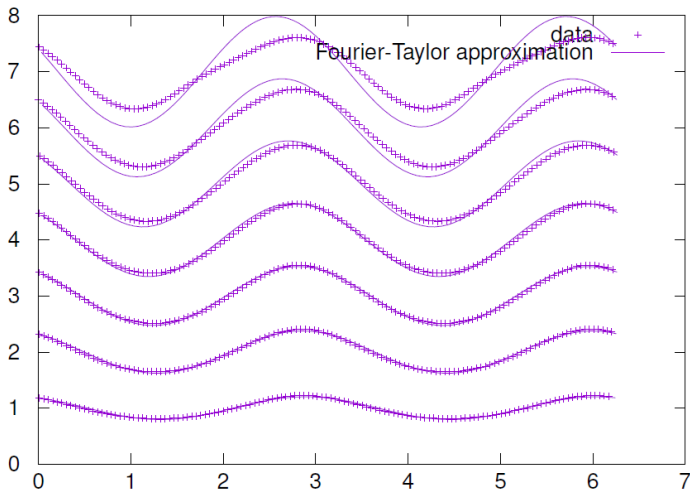


Figure: Action of the numerical scattering map (points) versus the scattering map series of degree $N = L = 2$ (lines).

Global Approximation Error

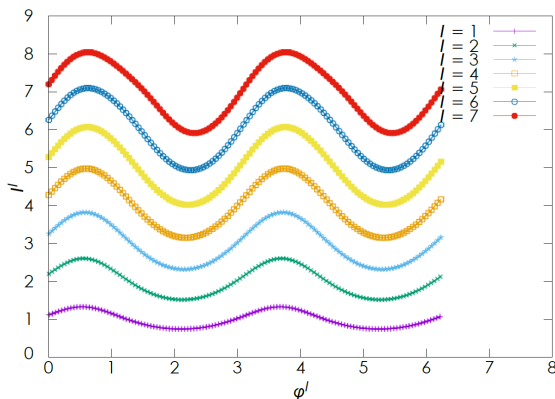


Figure: Action of the numerical scattering map (points) versus the scattering map series of degree $N = 4, L = 5$ (lines).

Standard Scattering Map

Definition

The *Standard Scattering Map* (SSM) is the map

$$\phi' = \phi - \omega(I) - \frac{\partial \tilde{\mathcal{L}}}{\partial I}(I, \phi') \quad (7a)$$

$$I' = I + \frac{\partial \tilde{\mathcal{L}}}{\partial \phi'}(I, \phi') \quad (7b)$$

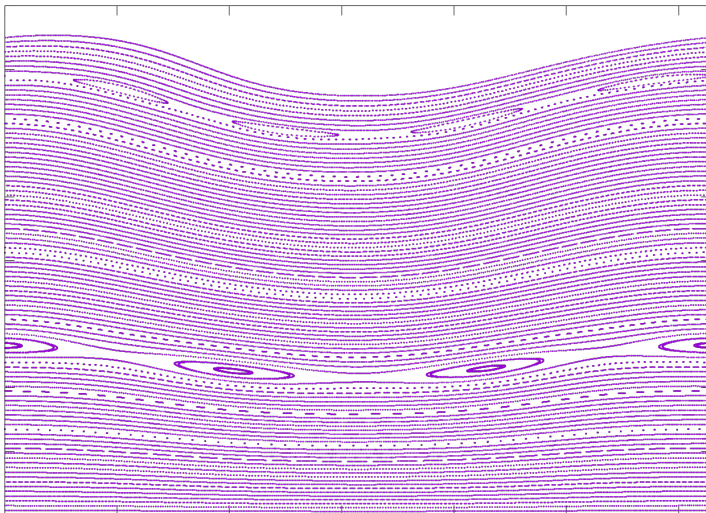
with *truncated* Fourier-Taylor series (4), (5), (6).

- ▶ Semi-analytical approximation of the exact scattering map.
- ▶ It is defined on the whole annulus

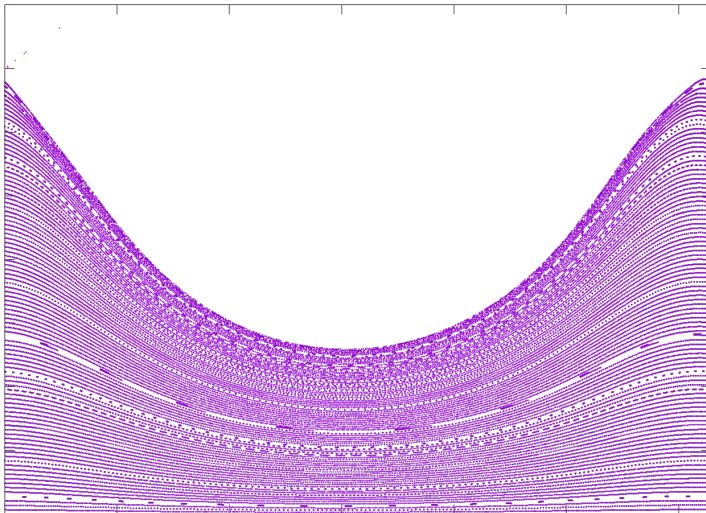
$$\mathcal{A} = \{(I, \phi) : J \in (0, 7] \text{ and } \phi \in [0, 2\pi)\}.$$

- ▶ We can iterate it!!

Phase Portrait of SSM σ_1



Phase Portrait of SSM σ_2



Existence of Drift Orbits

Theorem (Moeckel, 2002)

If the inner map τ and the SSM σ do not have a common invariant circle, then the iterated function system $\{\tau, \sigma\}$ drifts from the bottom to the top of $\mathcal{A}$.

- ▶ Invariant curves of the SSM can be represented as

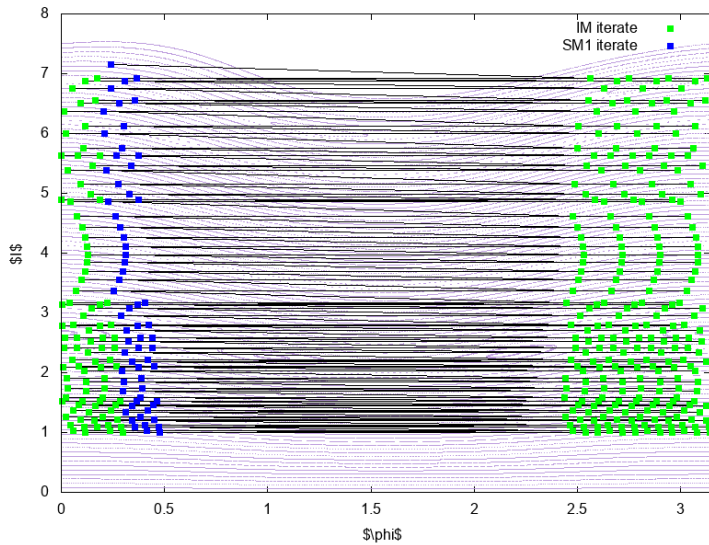
$$I = I_0 + h(\phi) \quad \text{with} \quad h(\phi) = \sum_{n=-N, n \neq 0}^N h_n e^{in\phi}.$$

- ▶ Fourier coefficients satisfy ($C_n = \frac{1}{2}(A_n - iB_n)$)

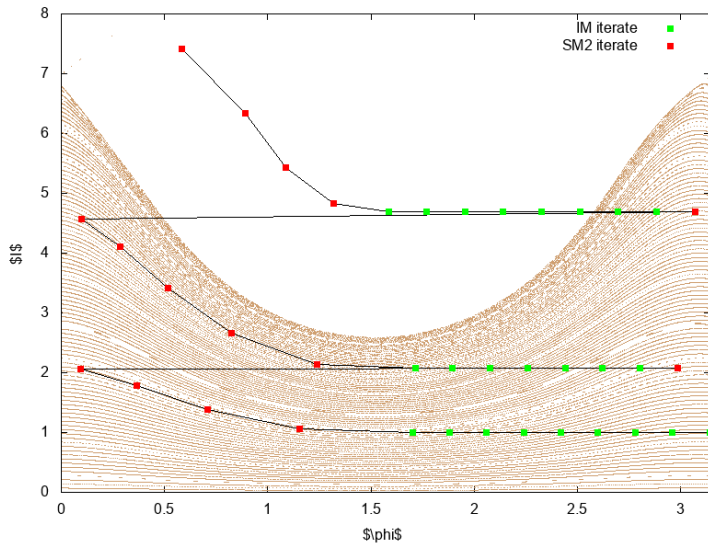
$$h_n = -\frac{C_n}{e^{in\omega_0} - 1} + O(C^2) \text{ for } 0 < |n| \leq N.$$

- ▶ Hence, the inner map and the SSM do not have any common invariant curves. $\square$

Drift Orbit of $\{\tau, \sigma_1\}$



Drift Orbit of $\{\tau, \sigma_2\}$



From Drift Orbits to Drift Trajectories

- ▶ For applications, we want the **R3BP trajectory** corresponding to the drift orbit.
- ▶ Each iterate of the inner map corresponds to a trajectory of the inner flow that takes time

$$3.0235 < t_{\text{in}} := \frac{2\pi}{\nu_p} < 3.0261.$$

- ▶ Each iterate of the outer map corresponds to a homoclinic trajectory that takes time

$$5.936738 \leq t_{\text{out}} \leq 6.000688.$$

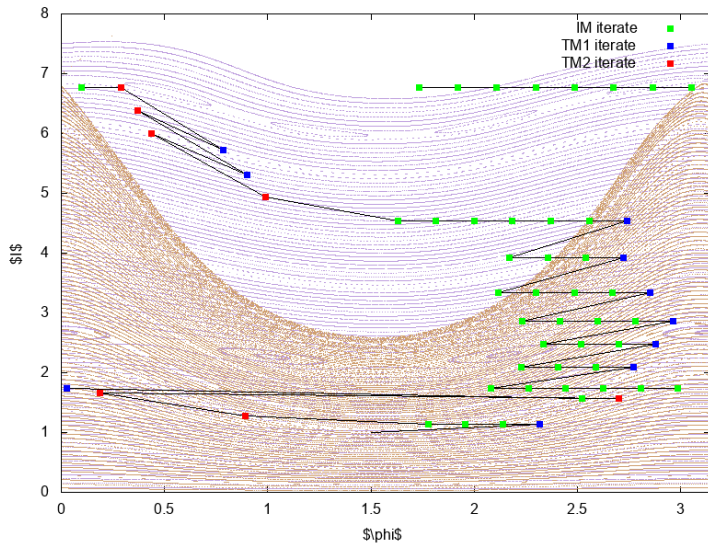
The unit of time in the Sun-Earth system satisfies $2\pi = 1$ year, so that 1 R3BP time unit ≈ 2 months, $t_{\text{out}} \approx 1$ year $\approx 2t_{\text{in}}$.

Shortest-Time Drift Trajectory: Algorithm

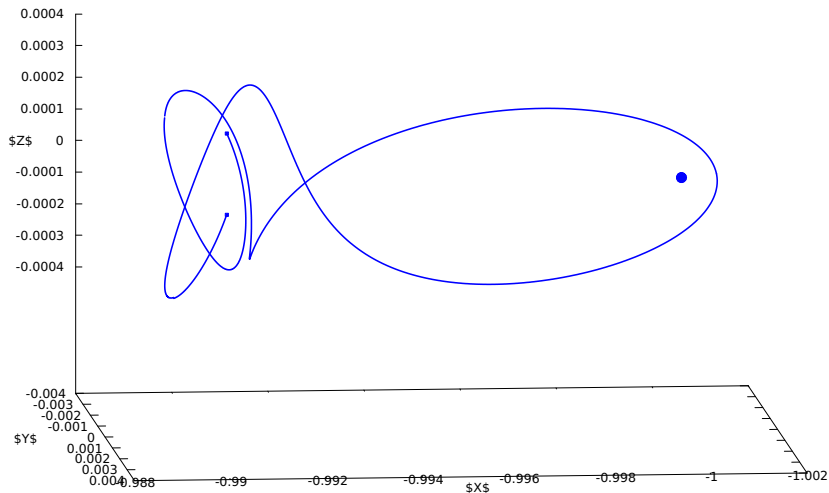
- ▶ Partition the domain $\mathcal{A}$ into a uniform grid of 30×30 rectangles of equal size.
- ▶ Introduce a directed graph $G = \langle V, E \rangle$ whose vertices V represent the different cells.
- ▶ An edge $(u, v) \in E$ means that the center point of cell u is mapped into cell v either by τ , σ_1 , or σ_2 .
- ▶ Every edge records the 'distance' between cells, defined as the time $t_{\text{in}}, t_{\text{out}}, t_{\text{out}}$ corresponding to applying τ, σ_1 , resp. σ_2 (otherwise $t = \infty$).
- ▶ Given a source cell s and a destination cell t , **Dijkstra's algorithm** provides the shortest path in the graph from s to t in terms of the distance defined above.

The corresponding pseudo-orbit for the R3BP flow takes time $34 t_{\text{in}} + 17 t_{\text{out}} \approx 204$ R3BP time units, i.e., about 32 years (optimal drift time).

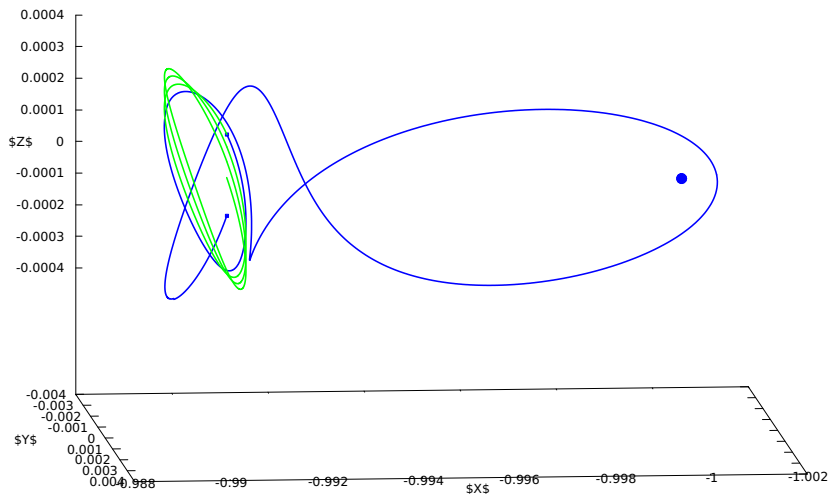
Shortest-Time Drift Trajectory



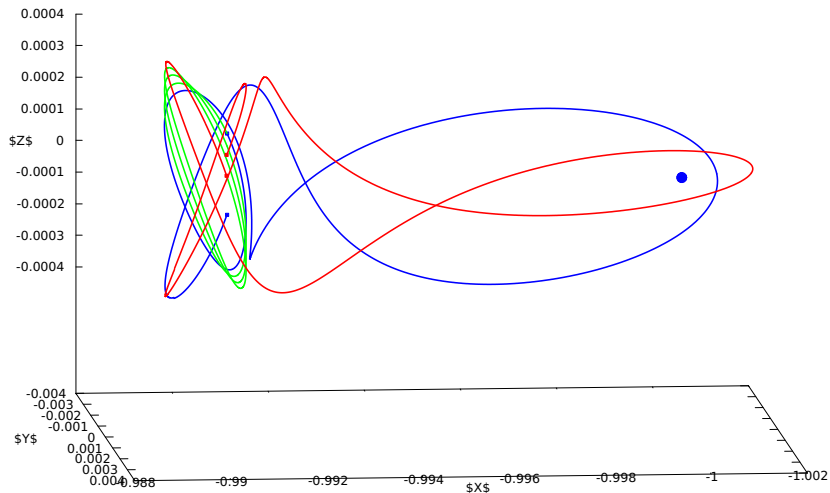
Shortest-Time Drift Trajectory: 3 first segments



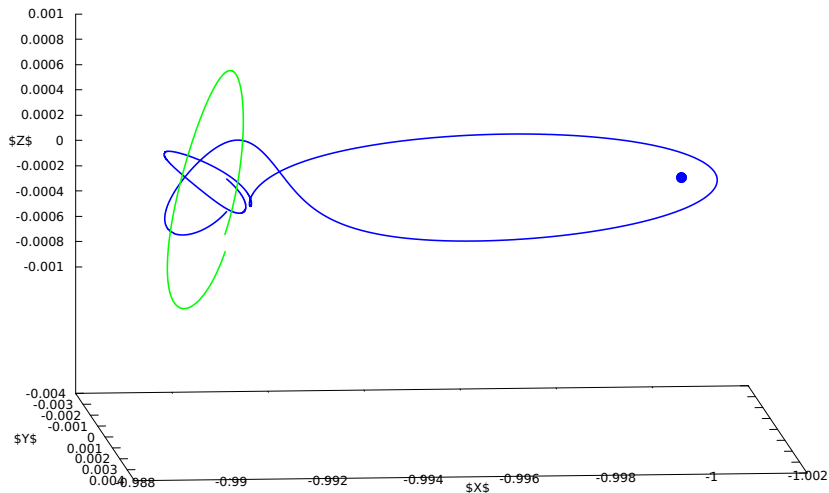
Shortest-Time Drift Trajectory: 3 first segments



Shortest-Time Drift Trajectory: 3 first segments



Initial and Final Segments



movie

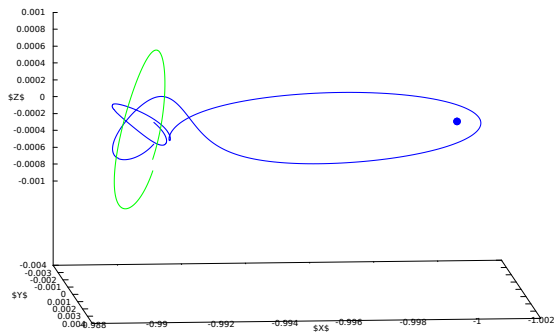


Figure: Initial segment in blue and the last segment in green