

Symbolic Dynamics in some Models of Celestial Mechanics

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Abstract

Symbolic dynamics is a powerful tool to understand chaotic behaviour in dynamical systems; in some sense, symbolic dynamics codes chaos. In this series of lessons we initially analyse what it means that a dynamic model presents a symbolic dynamic, proposing some basic examples. Then we will consider one, or possibly more, models arising from celestial mechanics for which the presence of a symbolic dynamics has been proved (anisotropic Kepler problem, N-center problem, Keplerian billiards).

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1 Chaotic Discrete Dynamical Systems

The first part of the present Note treats discrete dynamical systems (we will use the acronym DDSs). In such models, the system's state is observed at a sequence of discrete times, rather than continuously. The evolution of the system is described by a rule that dictates the evolution from one state to the next, starting from an initial condition. The state space can be very simple but also more complex, as we will see in §1.2 and 1.4. In some cases, DDSs can present predictable behaviour such as reaching equilibrium points, where the state no longer changes; nevertheless, complexity can definitely increase through the presence of multiple or intricate attracting sets, implying a strong dependence of the outcome on the initial states. A DDS can also reach the highest level of complexity, a dynamical chaos, exhibiting complex and unpredictable behaviours. Even with simple, deterministic rules (as the ones proposed in §1.2) the outcomes are highly sensitive to initial conditions and can be fractal in nature.

1.1 A bridge from continuous to discrete dynamical systems: the Poincaré section

Analytical solutions of nonlinear continuous systems can be really unmanageable and, possibly, impossible to find. A well-tested strategy to manage them is to look at their discrete counterpart, considering suitable fixed times in which a flow trajectory will be represented by a sequence of dots in the phase space. Henri Poincaré devised a technique, known as *Poincaré section* or *map*, which set up a bridge between continuous systems and their discrete analogue. The Poincaré section provides a glimpse about the nature of the dynamics and does not provide information on the time history of the evolution. Let us briefly recall the main contents of this theory, even if not in its generality.

Let us start with the definition of the main character of this Note.

Definition 1.1. A *dynamical system* is a triple (T, X, Φ) , where:

- T is the *time-set* ($\mathbb{R}$ when we study a continuous model, or $\mathbb{Z}$ when we consider a discrete one);
- X is the *state space*¹;
- $\Phi : E \subseteq T \times X \rightarrow X$ is the *evolution function* or *flow*. $\Phi(t, x_0)$ is the state of the system at time t , when the initial state is x_0 .

In the continuous case, when $T = \mathbb{R}$, we can deal with a finite dimensional state space, $X = \mathbb{R}^n$, developing the well-known Cauchy-Lipschitz theory of o.d.e.'s. Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set and $g : \Omega \rightarrow \mathbb{R}^n$ a continuous function, and (at least) locally lipshitz in the second variable². Fix an *initial time* $t_0 \in \mathbb{R}$ such that the corresponding *set of admissible initial points*

$$\Omega_0 = \{x \in \mathbb{R}^n \mid (t_0, x) \in \Omega\}$$

is not empty. Then, for any $x \in \Omega_0$ the Cauchy problem

$$\begin{cases} u'(t) = g(t, u(t)) \\ u(t_0) = x, \end{cases} \quad (1)$$

admits a unique maximal solution $u(\cdot; x)$ defined on its *maximal definition interval* $I(x)$. Let us now consider

$$E = \{(t, x) \in \Omega \mid x \in \Omega_0, t \in I(x)\},$$

¹X is often a metric space, but a topological space can be enough,

²We can think g of class $C^1(\Omega)$, which is a sufficient condition.

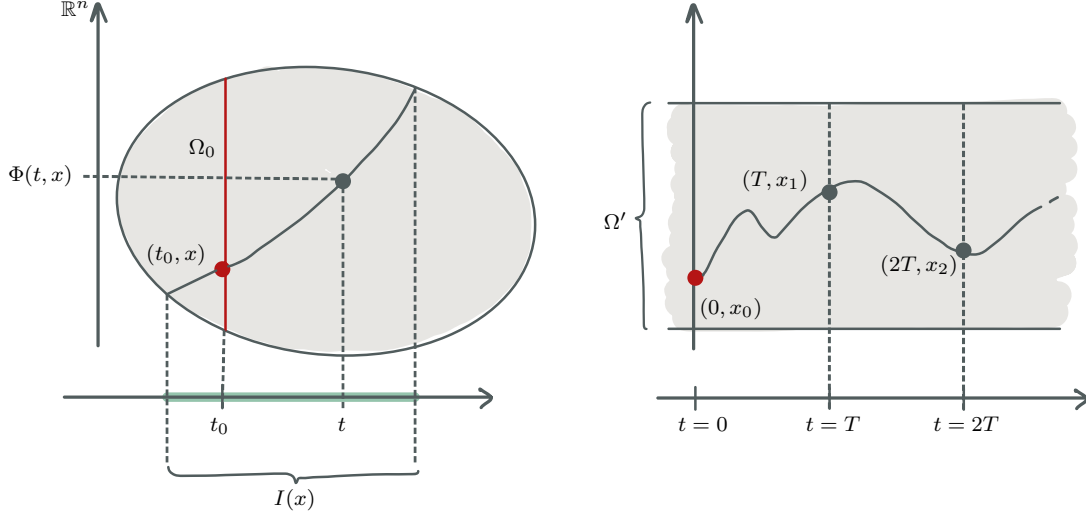


Figure 1: In both pictures the grey set is Ω . Left: the red line represents Ω_0 ; take $x \in \Omega_0$, its maximal definition interval is $I(x)$ (in green). The initial red point, (t_0, x) , move to the grey one, $(t, \Phi(t, x))$, following the flow.

Right: the function g is assumed to be T -periodic and $\Omega = \mathbb{R} \times \Omega'$. The initial condition $(0, x_0)$, red point, evolves to (T, x_1) , with $x_1 = \Phi(T, x_0)$ and then to $(2T, x_2)$, with $x_2 = \Phi(T, x_1)$, grey points.

and the *flow* associated to the differential equation in (1)

$$\Phi : E \rightarrow \mathbb{R}^n, \quad \Phi(t, x) = u(t; x).$$

Hence, the map Φ , once t_0 has been chosen, associates to any admissible initial condition $x \in \Omega_0$ and any $t \in I(x)$ the point $y \in \mathbb{R}^n$ reached at time t by the solution started at x (see Figure 1, left). Shorter notations for the flow are $\Phi^t(x)$ or $\Phi^t x$, as well as $\Phi_t(x)$ or $\Phi_t x$.

Assume now that the function g describing the dynamics is T -periodic in the time-variable, that is

$$g(t + T, u) = g(t, u), \quad \text{for any } (t, u) \in \Omega, \quad (2)$$

for some (minimal) $T > 0$. In this case $\Omega = \mathbb{R} \times \Omega'$ and $\Omega_0 = \Omega'$ does not depend on t_0 . It is then not restrictive to assume $t_0 = 0$. Furthermore, up to restrict Ω' , we are able to compute the evolution of any initial condition for at least a time T remaining in Ω' , i.e.

$$\text{for any } x \in \Omega', [0, T] \subset I(x) \text{ and } u(T; x) \in \Omega',$$

so that $\Phi^T(x)$ is defined for any $x \in \Omega'$ and $\Phi^T(x) \in \Omega'$.

Fixed $x_0 \in \Omega'$, let now $\xi(t) = u(t + T; x_0)$, for any $t + T \in I(x_0)$. It turns out, since g is T -periodic in time, that

$$\begin{cases} \dot{\xi}(t) = \dot{u}(t + T; x_0) = g(t + T, u(t + T; x_0)) = g(t, u(t + T; x_0)) = g(t, \xi(t)) \\ \xi(0) = u(T; x_0). \end{cases}$$

We have actually proved that the T -translation of a solution is still a solution as far as the periodicity assumption (2) is imposed and Ω' is appropriately chosen. We can then iterate this procedure and define a sequence in Ω'

$$x_1 = \Phi^T(x_0), \quad x_2 = \Phi^T(x_1), \quad \dots \quad x_{n+1} = \Phi^T(x_n), \quad \dots$$

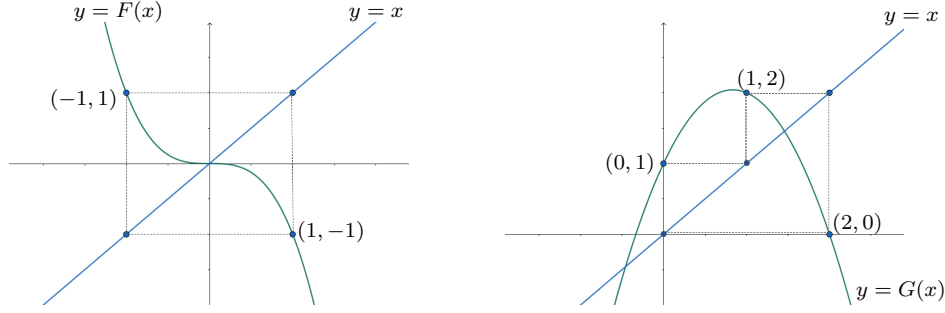


Figure 2: Cobweb diagram for some periodic points of the functions F and G in Example 1.5. Left: the two points on the graph of F are 2-periodic, indeed $F^2(1) = F(F(1)) = F(-1) = 1$. The three points on the graph of g are 3-periodic, indeed $G^3(0) = G^2(1) = G(2) = 0$.

deducing that the knowledge of the flow after a time T implies its knowledge at every nT , $n \in \mathbb{N}$, $n \geq 1$ (see Figure 1, right). This is basically the idea of the construction of the *Poincaré map*, which is defined as the flow $\Phi^T(\cdot)$ on Ω' . The regularity of the flow is transposed to the Poincaré map, that it is also invertible, by virtue of the uniqueness of the solution. Furthermore, if for some $x_0 \in \Omega$ we have $\Phi^T(x_0) = x_0$, then the orbit with initial condition x_0 is periodic.

We can then state that the Poincaré map encloses the main information of a continuous flow taking its pictures at periodic time-intervals and motivates the study of the discrete counterpart of dynamical systems.

Definition 1.2. A *discrete dynamical system* (DDS for short) is a pair (X, F) , where the state space, X , is a topological space, and F is the evolution map $F : X \rightarrow X$.

Definition 1.3. The *(positive) orbit* of a *seed* (or initial condition) $x_0 \in X$ is the sequence

$$(x_n)_{n \in \mathbb{N}} \subset X, \text{ where } x_{n+1} = F(x_n) \text{ for any } n \geq 1.$$

Definition 1.4. An *equilibrium point* for (X, F) is $x_0 \in X$ such that $F(x_0) = x_0$; a *periodic point of period k* or *k -periodic point* is $x_0 \in X$ such that $F^k(x_0) = x_0$. The orbit of a k -periodic point consists of at most k points

$$x_0, x_1 = F(x_0), \dots, x_{k-1} = F^{k-1}(x_0),$$

each of them being k -periodic. When k is the minimal period of x_0 , then the orbit of x_0 contains exactly k points and it is termed *k -periodic orbit* or *k -cycle*.

We indicate the number of k -periodic points of the map F with

$$P_k(F) = \#\{x \in X : x \text{ is } k\text{-periodic}\} \quad (3)$$

$P_k(F)$ counts all k -periodic points, not only the ones with minimal period k .

Example 1.5. The functions

$$F(x) = -x^3, \text{ and } G(x) = \frac{1}{2}(2-x)(3x+1), \quad x \in \mathbb{R},$$

have respectively a pair of 2-periodic points $x_0 = 1$, $x_1 = -1$, and a 3-cycle with seed $x_0 = 0$ (see Figure 2).

An equilibrium point x^* for (F, X) can be classified in terms of the behaviour of the orbits of seeds near it:

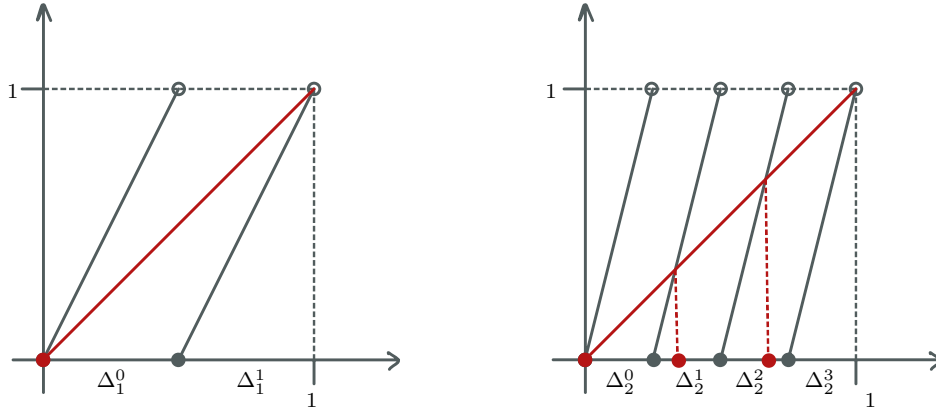


Figure 3: In black, the graphs of f_2 (left) and f_2^2 (right): the first one intersects the red line $y = x$ just at 0. The second one intersects it at 0 and also in other two points, each of them belonging respectively to the interior of Δ_2^1 and Δ_2^2 .

- x^* is an *attractive equilibrium point* or *sink* if there exists a neighbourhood U_{x^*} such that for any $y_0 \in U_{x^*}$, definitely $F^n(y_0) \in U_{x^*}$, and $F^n(y_0) \rightarrow x_0$ as $n \rightarrow \infty$.
- x^* is a *repulsive equilibrium point* or *source* if, fixed a neighbourhood U_{x^*} , for any $y_0 \in U_{x^*} \setminus \{x^*\}$ there exists $\bar{n} \in \mathbb{N}$ such that for any $n \geq \bar{n}$ the iterates $F^n(y_0) \notin U_{x^*}$.

When $X = \mathbb{R}^n$, and F is sufficiently smooth, the nature of an equilibrium point can be detected studying the linearization of F in x^* .

1.2 (Not so) Simple discrete models

1.2.1 Linear expanding maps

Let us start by considering the DDS $([0, 1), f_2)$, where f_2 is the map

$$f_2: [0, 1) \rightarrow [0, 1), \quad x \mapsto f_2(x) = 2x \pmod{1},$$

or, equivalently, the DDS $(\mathbb{S}^1, F_2)$ map that doubles the angle on the complex circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$

$$F_2: \mathbb{S}^1 \rightarrow \mathbb{S}^1, \quad F_2(z) = z^2.$$

We now investigate some interesting properties of f_2 (and F_2).

First, we deal with em existence, multiplicity and distribution of periodic points. Let $n \in \mathbb{N}$, $n \geq 1$; the n -th iterates of f_2 and F_2 are

$$f_2^n(x) = 2^n x \pmod{1} \quad \text{and} \quad F_2^n(z) = z^{2^n}.$$

The graph of f_2^n is composed of 2^n segments, each of them image of an interval

$$\Delta_n^k := \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right), \quad k = 0, \dots, 2^n - 1, \quad (4)$$

and each of them onto $[0, 1)$. Each of these segments (except for the last) intersects the line $y = x$ exactly at one point. This implies the existence of $2^n - 1$ n -periodic points for f_2 (see Figure 3). Equivalently, for any $n \geq 1$, equation $F_2^n(z) = z$, or $z^{2^n} = z$, admits exactly $2^n - 1$ solutions in $\mathbb{S}^1$. Such points are uniformly spread over the circle with equal intervals. Since, when n increases, the length of such intervals decreases and becomes infinitesimal, we have finally proved the next result.

Proposition 1.6. $P_n(f_2) = P_n(F_2) = 2^n - 1$ and periodic points for f_2 (resp. F_2) are dense in $[0, 1)$ (resp. in $\mathbb{S}^1$).

Second, we observe that under the iteration of the system's map, f_2 (or F_2), a point starting in any open set of $[0, 1)$ (or $\mathbb{S}^1$) can eventually move to any other open set of $[0, 1)$ (or $\mathbb{S}^1$). Indeed, let us take two open subsets $U, V \subset [0, 1)$. For n sufficiently large, there exists $k \in \{0, \dots, 2^n - 1\}$ such that

$$\Delta_n^k \subset U, \quad \text{for some } k \text{ and } n,$$

so that $[0, 1) \subset f^n(U)$ and $f^n(U) \cap V \neq \emptyset$.

This property, termed *transitivity* (see Definition 1.16 at page 12) implies that the system cannot be broken down into independent, non-interacting components: transitivity, together with the density of periodic points, features a chaotic dynamical system, as we will see in §1.3.

Remark 1.7. Proposition 1.6 suggests that a natural measure of asymptotic growth of the number of periodic points is the *exponential growth rate* $p(f)$ for the sequence $P_n(f)$

$$p(f) = \limsup_{n \rightarrow \infty} \frac{\log_+ P_n(f)}{n}, \quad \text{where } \log_+ s = \begin{cases} 0, & \text{if } s \in [0, 1) \\ \log s, & \text{if } s \geq 1. \end{cases} \quad (5)$$

With this definition we can compute

$$\begin{aligned} p(f_2) = p(F_2) &= \limsup_{n \rightarrow \infty} \frac{\log_+(2^n - 1)}{n} \\ &= \limsup_{n \rightarrow \infty} \frac{\log_+ 2^n + \log_+(1 - 2^{-n})}{n} = \limsup_{n \rightarrow \infty} \frac{\log 2^n}{n} = \log 2. \end{aligned}$$

Remark 1.8. A straightforward generalization of f_2 and F_2 are, respectively, the maps

$$f_m(x) = mx \pmod{1} \quad \text{and} \quad F_m(z) = z^m$$

for any integer $m \geq 2$. In this case $P_n(f_m) = P_n(F_m) = m^n - 1$, periodic points are still dense in $[0, 1)$ and $\mathbb{S}^1$, and $p(f_2) = p(F_2) = \log m$.

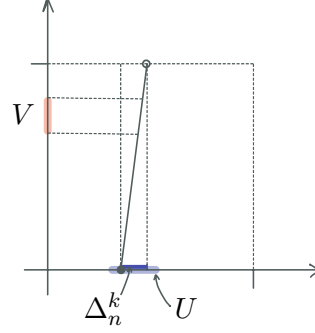
Remark 1.9. We observe another interesting property of f_m : f_m is length preserving. Indeed, let $I \subset [0, 1)$ be an interval, then its preimage under f_m consists of the union of m intervals of length $|I|/m$ each.

Coding the action of f_2 . We conclude this paragraph describing f_2 (and finally every f_m) from a different point of view. Fixed any $x \in [0, 1)$, let us consider the binary representation x , that is

$$0.s_0s_1\dots s_n\dots, \quad \text{with } s_n \in \{0, 1\}, \text{ and } n \in \mathbb{N}.$$

In other terms, to any point $x \in [0, 1)$ we associate an infinite sequence of 0s and 1s,

$$\underline{s} = (s_n)_{n \in \mathbb{N}}, \quad s_n \in \{0, 1\}.$$



To do that, when a number admits a finite representation, we complete it with a queue of 0s. Furthermore, we remark that some $x \in [0, 1)$ admits a double representation, indeed any two binary numbers of the form

$$0.(xxx)01111\dots \quad \text{and} \quad 0.(xxx)10000\dots$$

correspond to the same point in $[0, 1)$ (here (xxx) stays for any finite array of 0s and 1s). The lack of a unique representation for a point x implies that there is no surjective map from $[0, 1)$ to the space of infinite sequences with two symbols. Despite this, there exists a surjective (but not bijective) map from the space of infinite sequences to $[0, 1)$, provided that we identify $0.1111\dots$ with $1 \pmod{1} = 0$. Hence, we have the following situation

$$[0, 1) \rightarrow \{(s_n)_{n \in \mathbb{N}}, s_n \in \{0, 1\}\} \quad \text{is not surjective,}$$

since, for instance, sequences ending with a queue of 1s are not in its image, and

$$\{(s_n)_{n \in \mathbb{N}}, s_n \in \{0, 1\}\} \rightarrow [0, 1) \quad \text{is surjective, but not bijective.}$$

Let us conclude this paragraph observing that the action of f_2 on $[0, 1)$ can be understood in terms of sequences. Indeed, f_2 definitely associates to $x \in [0, 1)$ the fractional part of its double, so its binary representation is mapped in to its right-shift, that is

$$f_2(0.s_0s_1\dots s_n\dots) = 0.s_1s_2\dots s_{n+1}\dots$$

What is interesting is that a binary representation of a number in $[0, 1)$, $0.s_0s_1\dots s_n\dots$, encloses information about its orbit under the action of f_2 . This is the main idea of *coding* a dynamical system. We will return on it, considering the space of sequences of symbols, as well as the shift-map on it in §1.4.

Remark 1.10. Similarly, for any $m \geq 2$, the action of f_m on $[0, 1)$ can be understood as a right shift on the set of infinite sequences of symbols in $\{0, 1, \dots, m-1\}$.

Remark 1.11. Using the binary representation of a number in $[0, 1)$, it is fairly straightforward to prove the presence of a dense orbit of f_2 in $[0, 1)$. Indeed, let us consider an integer $k \in \{0, \dots, 2^n-1\}$ with binary representation

$$k = k_0k_1\dots k_{n-1}, \quad k_i \in \{0, 1\}, \quad i = 0, \dots, n-1.$$

Recalling that $\Delta_n^k = [k/2^n, (k+1)/2^n)$, Eq. (4), it turns out that

$$x \in \Delta_n^k \iff \text{in binary repr. } x = 0.x_0x_1\dots \text{ with } x_i = k_i, \quad i = 0, \dots, n-1.$$

Let now ω_n be the concatenation of all possible n -sequences of 0s and 1s, for instance

$$\omega_1 = 01 \quad \omega_2 = 00011011 \quad \omega_3 = 000001010100011101110111.$$

Finally, let ω be the infinite sequence

$$\omega = \omega_1\omega_2\omega_3\dots,$$

and let $x = 0.\omega$. Then the orbit of x intersect any Δ_n^k , hence it is dense in $[0, 1)$.

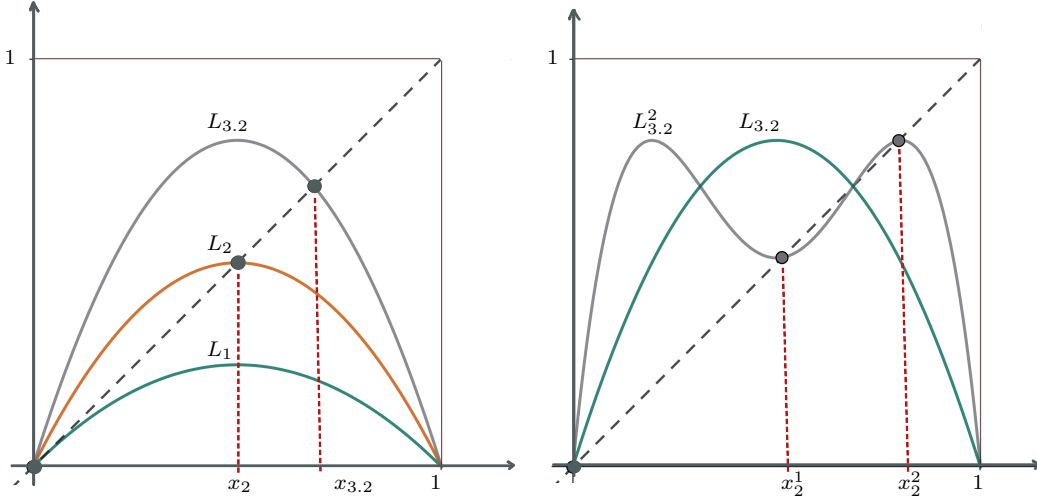


Figure 4: Left: graphs of the quadratic functions L_1 , L_2 and $L_{3.2}$ on $[0, 1]$; in the second and third a fixed point different from 0 appears, respectively x_2 and $x_{3.2}$. Right: graphs of $L_{3.2}$ and its iterate $L_{3.2}^2$; the second has two intersections with the line $y = x$ that corresponds to a pair of 2-periodic points, x_2^1 and x_2^2 .

1.2.2 Quadratic maps

Consider the *discrete-time logistic map*

$$L_\lambda(x) = \lambda x(1 - x), \quad x \geq 0. \quad (6)$$

The real and positive parameter λ represents the “fertility” of the population x . Eq. (6) is a prototype model for a wider class of dynamical systems described by a quadratic map, see [33, Chapter 7] and [41, §2.5 §7.1.2 and §11]. These maps share the property of changing monotonicity at $x = 1/2$, having a maximum point at $x = 1/2$.

Here we summarize, without providing any proof, how the dynamics of (6) depends on λ and becomes more and more complex as it increases. Let us start by considering small values of λ (see Figure 4, left). More precisely we consider $\lambda \in (0, 4]$ and we focus on the domain interval $[0, 1]$; in this case $L_\lambda([0, 1]) \subseteq [0, 1]$, even if it is not surjective if $\lambda < 4$. The following results hold (for a proof we refer the interested reader to [41, Propositions 2.5.1. and 2.5.2.]):

- When $\lambda \in [0, 1]$, all orbits of $L_\lambda(x) = \lambda x(1 - x)$ on $[0, 1]$, are asymptotic to the unique fixed point 0.
- When $\lambda \in (1, 3]$, all orbits of $L_\lambda(x) = \lambda x(1 - x)$ on $[0, 1]$, except for 0 and 1, are asymptotic to the fixed point.

$$x_\lambda = \frac{\lambda - 1}{\lambda}.$$

Hence, when $\lambda \in [0, 3]$ the map L_λ admits at most two periodic points, and both of them are fixed points. With some effort this analysis can be extended as far as $\lambda = 1 + \sqrt{6}$ ([41, Proposition 11.2.1., p. 303 and following]). It turns out that $\lambda = 3$ is a *period doubling bifurcation* value: the fixed point x_λ becomes a repulsive equilibrium, and an attractive 2-cycle appears:

- When $\lambda \in (3, 1 + \sqrt{6}]$, all orbits of $L_\lambda(x) = \lambda x(1 - x)$ on $[0, 1]$, except for 0, x_λ and their preimages, are asymptotic to a unique periodic orbit of period 2 (see Figure 4, right, the case $\lambda = 3.2$).

Let us define the basins of attraction of the two periodic orbits when $\lambda \in (3, 1 + \sqrt{6})$. The only preimage of 0 is 1, while 1 do not have preimages, since the maximal level is below 1. Concerning the unstable fixed point x_λ , it has a unique preimage distinct from itself, that is

$$x_{\lambda,1} = \frac{1}{\lambda},$$

which in turn has two preimages,

$$x_{\lambda,2}^\pm = \frac{1}{2} \pm \frac{\sqrt{\lambda^2 - 4}}{2\lambda}.$$

Going further, preimages of x_λ form two sequences that converge to the endpoints 0 and 1, respectively. Those preimages separate the basin of attraction of the stable orbit of period 2 into countably many disjoint intervals. Hence, the basin of attraction of each of the two stable fixed points of L_λ^2 , consists of countably many disjoint intervals. The central observation that makes it possible to carry this analysis further is based on a *renormalization* argument: if we restrict the map L_λ^2 , $\lambda \in (3, 1 + \sqrt{6})$, to the immediate basin of attraction of either of its two attracting fixed points, we obtain a map that looks qualitatively like the original map L_λ on the whole interval $[0, 1]$ for $\lambda < 3$: it has a unique critical point, a repelling fixed point at one end, the preimage of that point on the other end, and an attracting fixed point with negative derivative. As the parameter increases, the attracting fixed point of this new map becomes repelling in a period doubling bifurcation. Beyond this bifurcation, one can restrict $L_\lambda^4 = (L_\lambda^2)^2$ to a suitable interval and iterate this procedure, obtaining what is known as a *cascade of period doublings*, as the next result describes.

Theorem 1.12 ([41], Theorem 11.2.2.). *There exists an increasing sequence of parameter values $(\lambda_n)_{n \geq 1}$, with $\lambda_1 = 3$ and $\lambda_2 = 1 + \sqrt{6}$, such that for any $\lambda \in (\lambda_n, \lambda_{n+1}]$ the quadratic map L_λ has*

- *one attracting periodic orbit of period 2^n ;*
- *two repelling fixed points 0 and x_λ*
- *one repelling periodic orbit of period 2^k for each $k = 1, 2, \dots, n - 1$.*

Furthermore, at $\lambda = \lambda_n$, the attracting periodic orbit undergoes a period-doubling bifurcation.

The appearance of infinitely many period doubling can be derived in much greater generality; in this Notes we are not really interested in the analysis of this phenomenon. We refer to [27] for a deep analysis of the dynamical system (6) when $\lambda \in [3, 4]$, to [41, §11.3] for an investigation about the universality of this phenomenon.

We concentrate on the presence and the growing the number of n -periodic points, when the parameter is high enough. The following result holds.

Proposition 1.13. *For $\lambda \geq 4$ we have $P_n(L_\lambda) = 2^n$.*

Proof. Since the map L_λ is a second order polynomial, L_λ^n is a 2^n -polynomial in $\mathbb{R}$. Hence, equation $L_\lambda^n(x) = x$ has at most 2^n solutions in $\mathbb{R}$, so that $P_n(L_\lambda) \leq 2^n$.

In order to prove the inverse inequality, we first remark that, being $\lambda \geq 4$, then $L_\lambda(1/2) \geq 1$ and the entire interval $[0, 1]$ is contained both in $L_\lambda([0, 1/2])$ and in $L_\lambda([1/2, 1])$. Then we define two closed intervals

$$\Delta_0 = \left[0, \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\lambda}}\right] \subseteq [0, 1/2] \text{ and } \Delta_1 = \left[\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\lambda}}, 1\right] \subseteq [1/2, 1],$$

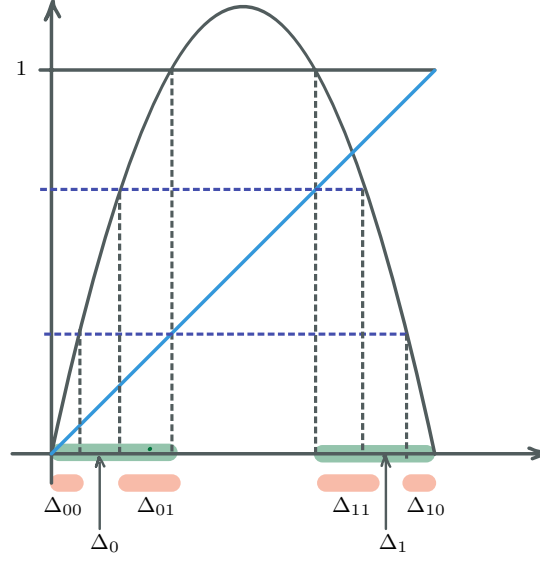


Figure 5: The green segments on the horizontal axis are Δ_0 and Δ_1 : the restrictions $L_\lambda|_{\Delta_0}$ and $L_\lambda|_{\Delta_1}$ are surjective on $[0, 1]$. The four red ones are selected in such a way that their image under L_λ^2 is the entire segment $[0, 1]$.

be such that $L_\lambda(\Delta_0) = L_\lambda(\Delta_1) = [0, 1]$. Since L_λ is a continuous function, there exists a fixed point for L_λ in each of these two intervals. Of course, the fixed point in Δ_0 is 0, but the one in Δ_1 is necessarily in its interior, hence away from the boundary of $[0, 1]$ (actually, it is x_λ).

Consider now the preimages of Δ_0 and Δ_1 under L_λ : they consist of two intervals each, so there are four intervals whose images under L_λ^2 are exactly $[0, 1]$. Each contains a fixed point for L_λ^2 , again every one except 0 being in the interior of the corresponding interval. So, no two of these points coincide.

Repeating this argument, at step n we obtain 2^n intervals whose images under L_λ^n are $[0, 1]$, and therefore each of which contains at least one fixed point, giving at least 2^n distinct orbits of period n for L_λ (see Figure 5). Merging this result with $P_n(L_\lambda) \leq 2^n$, we conclude the proof. $\square$

Remark 1.14. The argument in the proof of Proposition 1.13 to show that $P_n(L_\lambda) \geq 2^n$ applies to any continuous function

$$f: [0, 1] \rightarrow \mathbb{R}, \text{ with } f(0) = f(1) = 0 \text{ and such that } \exists c \in (0, 1), \text{ with } f(c) \geq 1.$$

When such a function is monotonic on $[0, c]$ as well as on $[c, 1]$ is said to be *unimodal*. Thus, if f is a unimodal function then $P_n(f) \geq 2^n$. The sharp equality holds when f is expanding, that is $|f(x) - f(y)| > |x - y|$, on each interval of $f^{-1}((0, 1))$ (see [41, Proposition 7.1.5].)

From Proposition 1.13, we immediately deduce that, in analogy with the linear expanding map f_2 , the logistic map L_λ presents an exponential growth rate

$$p(L_\lambda) = \log 2, \text{ for every } \lambda \geq 4.$$

Coding the action of L_λ . The value of the exponential growth rate is not the unique analogy between these two dynamical systems. Indeed, let us go back to the construction in

the proof of Proposition 1.13. We have introduced the closed and disjoint subsets of $[0, 1]$, Δ_0 and Δ_1 , such that

$$\Delta_0 \cup \Delta_1 = \{x \in [0, 1] : L_\lambda(x) \in [0, 1]\}$$

or, equivalently, $L_\lambda^{-1}([0, 1]) = \Delta_0 \cup \Delta_1$. Similarly,

$$L_\lambda^{-2}([0, 1]) = \Delta_{00} \cup \Delta_{01} \cup \Delta_{10} \cup \Delta_{11}$$

where

$$\Delta_{ij} = \{x \in \Delta_i : L_\lambda(x) \in \Delta_j\} = \Delta_i \cap L_\lambda^{-1}(\Delta_j), \quad i, j \in \{0, 1\}$$

are closed, with non-empty interior, and disjoint (see once more Figure 5). Given a finite sequence of 0s and 1s, $s_0 s_1 \dots s_N$, $s_k \in \{0, 1\}$, $k = 0, \dots, N$, we can define the closed, with non-empty interior set

$$\Delta_{s_0 s_1 \dots s_N} = \Delta_{s_0} \cap L_\lambda^{-1}(\Delta_{s_1}) \cap \dots \cap L_\lambda^{-N}(\Delta_{s_N}),$$

and the chain of strict inclusions holds

$$\Delta_{s_0 s_1 \dots s_N} \subset \Delta_{s_0 s_1 \dots s_{N-1}} \subset \Delta_{s_0 s_1} \subset \Delta_{s_0}.$$

Hence, given a sequence

$$\underline{s} = (s_n)_{n \in \mathbb{N}}, \quad s_n \in \{0, 1\}.$$

by the nested interval Theorem, the set

$$\Delta_{\underline{s}} = \bigcap_{k=0}^{+\infty} L_\lambda^{-k}(\Delta_{s_k})$$

is not empty.

It turns out that the positive invariant set for the evolution map L_λ , when $\lambda \geq 4$, that is

$$\Lambda = \left\{ x \in [0, 1] : L_\lambda^k(x) \in [0, 1] \text{ for any } k \in \mathbb{N} \right\}, \quad (7)$$

is the union of $\Delta_{\underline{s}}$, varying the infinite 2-symbols sequence $\underline{s}$. Furthermore Λ is a one-dimensional Cantor set. Next definition will be usefull in the next sections.

Definition 1.15. Let $x_0 \in \Lambda$. Then the *itinerary* of x_0 is the sequence $h(x_0) = (s_0 s_1 s_2 \dots)$ where

$$s_j = k \quad \Longleftrightarrow \quad L_\lambda^j(x_0) \in \Delta_k.$$

1.3 On the definition of chaos

In 1993 Robert Devaney wrote in his book [33, Chapter 10]

“There are many possible definitions of chaos. In fact, there is no general agreement within the scientific community as to what constitutes a chaotic dynamical system. However, this will not deter us from offering one possible definition.”

His “offer” is commonly used today and essentially states that a continuous map $F : X \rightarrow X$, or a dynamical system (F, X) , is termed chaotic if it has the following three properties:

- (c1) It moves from any region of the space to any other region; hence the evolution function F mixes the space X .

- (c2) Periodic points are dense in X . Periodic orbits of F appear everywhere in the phase space, they are actually dense in X . Although global behaviour may be complicated, the system is full of small “regular islands”.
- (c3) Arbitrarily close initial states eventually diverge by at least a fixed amount.

The first one is a *transitivity property* and it is quite delicate to treat. Furthermore, as we will see in this paragraph, the first two imply the third, at least under some reasonable assumptions on the space X . We also notice that the first two properties are topological conditions and do not really need a metric structure on X , while the third requires a proximity that is naturally connected to metric spaces. Let us start to be more precise, in order to give the definition of chaotic map proposed by R. Devaney when (X, d_X) is a metric space.

Definition 1.16. Let X be a topological space. The map $F : X \rightarrow X$ is *transitive* if for any pair of open subsets $U, V \subset X$ there exists $n \in \mathbb{N}$ such that $F^n(U) \cap V \neq \emptyset$. When (X, d_X) is a metric space then transitivity reads as

$$\forall \varepsilon > 0, \forall x, y \in X, \exists n \in \mathbb{N} \text{ such that } F^n(x) \in B_\varepsilon(y).$$

Definition 1.17. Let (X, d_X) be a metric space. The map $F : X \rightarrow X$ *depends sensitively on initial conditions* if there exists $\Delta > 0$ such that for every $x \in X$ and for every $\varepsilon > 0$

$$\exists y \in X, \exists n \in \mathbb{N} \text{ such that } d_X(x, y) < \varepsilon \text{ and } d_X(F^n(x), F^n(y)) > \Delta.$$

Remark 1.18. Let us first outline that the *sensitive constant*, Δ , does not depend on x nor on ε . Furthermore, when a map F depends sensitively on initial conditions a very small error, quantified by ε , in the initial condition can lead to a macroscopic discrepancy in the evolution of the dynamics, Δ . The quantity Δ tells us at what scale these errors show up.

Example 1.19. Let us show that linear expanding maps (see §1.2.1) depend sensitively on initial conditions. We focus on f_2 , but we can argue similarly for f_m . First, we need a metric on $[0, 1)$; we choose

$$d|_{[0,1)}(x, y) = \min\{|x - y|, 1 - |x - y|\}.$$

Then, fix $x \in [0, 1)$ and $\varepsilon > 0$. Now choose $y \in [0, 1)$ such that

$$d|_{[0,1)}(x, y) = r < \min\{\varepsilon, 1/8\}.$$

Since f_2 expands any initial error by a factor of 2 at every iteration, we have that

$$d|_{[0,1)}(f_2(x), f_2(y)) = 2d|_{[0,1)}(x, y),$$

and, increasing the number of iterations,

$$d|_{[0,1)}(f_2^n(x), f_2^n(y)) = 2d|_{[0,1)}(f_2^{n-1}(x), f_2^{n-1}(y))$$

until $d|_{[0,1)}(f_2^n(x), f_2^n(y)) \leq 1/2$. Take now $\hat{n} \in \mathbb{N}$ such that

$$2^{\hat{n}}r \geq \frac{1}{4} \quad \text{and} \quad 2^{\hat{n}-1}r < \frac{1}{4}.$$

We can conclude that $\Delta = 1/4$ is the sensitive constant of f_2 . Similarly, the expanding map f_m exhibits sensitive dependence on initial conditions with $\Delta = 1/2m$.

Definition 1.20. Let (X, d_X) be a metric space and $F : X \rightarrow X$ be a continuous function. The map F or, equivalently, the dynamical system (F, X) is *chaotic (in the sense of Devaney)* if:

- (c1) it is transitive;
- (c2) periodic points for F are dense in X ;
- (c3) it depends sensitively on initial conditions.

We now propose a slightly different definition of chaotic dynamical systems, following [41, §7.2]. First, the transitivity property (c1) is reformulated in the notion of topological transitivity.

Definition 1.21. The map $F : X \rightarrow X$ is *topologically transitive* if it admits a dense orbit. When F is not invertible, the dense orbit is understood as a positive orbit $(F^n(x))_{n \in \mathbb{N}}$, for some $x \in X$. When F is invertible, the dense orbit can be $(F^n(x))_{n \in \mathbb{Z}}$, $x \in X$.

The following notion of chaotic map does not require the assumption of sensitive dependence.

Definition 1.22. Let X be a topological space and $F : X \rightarrow X$ be continuous. The map F or, equivalently, the dynamical system (F, X) is chaotic if:

- (c1') it is topologically transitive;
- (c2) periodic points for F are dense in X .

Remark 1.23. Let us observe that Definition 1.22 of chaotic map does not need any requirements on the space X . This poorness in the assumptions makes any trivial map with a single periodic orbit chaotic. As we will see in Theorem 1.26, trivial maps are the only maps which are chaotic according to Definition 1.22 but not to Definition 1.20.

Under an additional assumption on X , topological transitivity and transitivity are equivalent notions, as the next result states.

Lemma 1.24 ([41], Proposition 7.2.2.). *Let X be a metric space, $F : X \rightarrow X$ be a continuous map, and consider the following conditions*

- (1) F is topologically transitive, that is, it has a (complete) dense orbit;
- (2) F has a dense positive orbit;
- (3) for any pair of open subsets $U, V \subset X$ there exists $k \in \mathbb{Z}$ such that $F^k(U) \cap V \neq \emptyset$;
- (4) F is transitive.

Then (1) implies (3). Furthermore, if X has no isolated points, then (1) implies (4). Finally, if X is separable, then (3) implies (1) and (4) implies (2).

Corollary 1.25. *Let X be a separable metric space without isolated points and let $F : X \rightarrow X$ be continuous. Then, conditions (1)-(4) are equivalent and in particular the notions of transitivity and topological transitivity coincide.*

In the setting of metric spaces, Definition 1.22 definitely implies sensitive dependence on initial conditions as far as the entire space X does not consist of a single periodic orbit.³

Theorem 1.26. *Let (X, d_X) be a metric space. A chaotic map, F in the sense of Definition 1.22 exhibit sensitive dependence on initial conditions, except when the entire space consists of a single periodic orbit.*

³This result has been proved by Banks et al. in [3] in the context of compact metric spaces; we proposed here the minimal assumptions on X , following the book [40, Theorem 7.2.12].

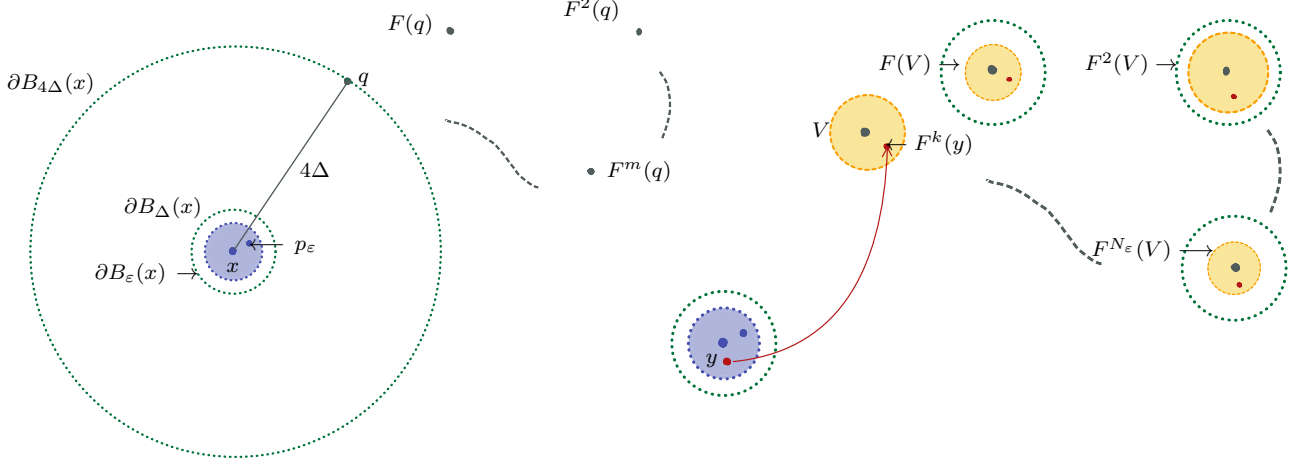


Figure 6: Left: the point x does not belong to the periodic orbits generated by p and q and is far at least 4Δ from the entire orbit of q . $\varepsilon \in (0, \Delta)$ and the grey ball is $B_{\varepsilon}(x)$. $p_{\varepsilon} \in B_{\varepsilon}(x)$ is a N_{ε} -periodic point. Right: V is the open neighbourhood of q that tracks q up to Δ for at least N_{ε} steps. By transitivity there exists $y \in B_{\varepsilon}(x)$ that evolves in V . The points y and p_{ε} , both in $B_{\varepsilon}(x)$ after N steps are further than 2Δ . The red dots represent the orbit of y .

Proof. When the space X does not consists of a single periodic orbit, then there are at least two distinct periodic orbits without common points (indeed, assumption (c2) holds). Hence, there exists a pair of periodic points $p, q \in X$ such that

$$\Delta := \frac{1}{8} \min\{d_X(F^n(p), F^m(q)) : n, m \in \mathbb{Z}\} > 0.$$

The quantity Δ is strictly positive since each distance $d_X(F^n(p), F^m(q))$ is strictly positive and there are a finite number of such distances.

If X consists exactly of two periodic orbits, then Δ is a sensitivity constant for F and the proof ends

Otherwise, let $x \in X$, not an iteration of p , nor of q . Then the orbit of one of these two points keeps distance at least 4Δ from x , otherwise, by triangular inequality, the definition of Δ is violated. Assume, w.l.o.g., this point is q , so that

$$d_X(x, F^m(q)) > 4\Delta, \quad \forall m \in \mathbb{Z}. \quad (8)$$

Take an arbitrary $\varepsilon \in (0, \Delta)$. We claim the existence of a point $z \in B_{\varepsilon}(x)$ such that

$$d_X(F^k(x), F^k(z)) > \Delta, \quad \text{for some } k \in \mathbb{Z}.$$

By density of periodic points, there exists a periodic point $p_{\varepsilon} \in B_{\varepsilon}(x)$. Let N_{ε} its period. The set

$$V = \bigcap_{i=0}^{N_{\varepsilon}} F^{-i}(B_{\Delta}(F^i(q)))$$

is an open set containing q and consists of such points whose first N_{ε} iterates track those of q up to Δ .

Since topological transitivity implies transitivity, we have that there exists $k \in \mathbb{Z}$ such that $F^k(B_{\varepsilon}(x)) \cap V \neq \emptyset$, hence

$$\exists y \in B_{\varepsilon}(x) : F^k(y) \in V. \quad (9)$$

Let now

$$j \in \left(\frac{k}{N_\varepsilon}, \frac{k}{N_\varepsilon} + 1 \right] \cap \mathbb{Z} \quad \text{and} \quad N = N_\varepsilon j,$$

then $0 < N - k \leq N_\varepsilon$ and $F^N(p_\varepsilon) = p_\varepsilon$. Furthermore, the following inequality holds

$$\begin{aligned} d_X(F^N(p_\varepsilon), F^N(y)) &= d_X(p_\varepsilon, F^N(y)) \\ &\geq -d_X(p_\varepsilon, x) - d_X(F^N(y), F^{N-k}(q)) + d_X(F^{N-k}(q), x). \end{aligned}$$

The first term is bigger than $-\Delta$, since $p_\varepsilon \in B_\varepsilon(x) \subset B_\Delta(x)$. The last one is bigger than 4Δ , since Eq. (8) holds. Concerning the middle term, let us observe that $F^N(y) = F^{N-k}(F^k(y)) \in F^{N-k}(V)$, by means of Eq.(9). Since $N - k \leq N_\varepsilon$ and $F^{N-k}(V) \subset B_\Delta F^{N-k}(q)$, this term is bigger than $-\Delta$. Hence we conclude that

$$d_X(F^N(p_\varepsilon), F^N(x)) + d_X(F^N(x), F^N(y)) \geq d_X(F^N(p_\varepsilon), F^N(y)) \geq 2\Delta$$

and at least one between $F^N(p_\varepsilon)$ and $F^N(y)$ is more than Δ far from $F^N(x)$. This concludes the proof. $\square$

Example 1.27. Linear expanding maps considered in §1.2.1 are definitely cahotic, with respect to both Definitions 1.20 and 1.22. Indeed, we have already shown that periodic points are dense (in Proposition 1.6) and that such maps are transitive (see the discussion at page 6, just below Proposition 1.6).

Since the phase space $X = \mathbb{S}^1$ is a separable metric space with no isolated points, Corollary 1.25 and Theorem 1.26 apply, and such maps are then chaotic.

Remark 1.28. There are examples of dynamical systems which are not cahotic, but that satisfy some of the conditions (c1)/(c1')-(c2)-(c3). For example, the rotation on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$

$$T_r : \mathbb{S}^1 \rightarrow \mathbb{S}^1, \quad T_r(x) = x + r, \quad \text{with } r \in \mathbb{R} \setminus \mathbb{Q},$$

satisfies (c1'), but not (c2), nor (c3).

Similarly, also the property of sensitive dependence on initial conditions can appear without chaoticity. For example, the *linear twist map on the cylinder*

$$T : \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{S}^1 \times [0, 1], \quad T(x, y) = (x + y, y),$$

exhibits a sensitive dependence on initial conditions. Indeed, let $\Delta = 1/4$, fix any point (x, y) and any neighbourhood U of it. Because irrationals are dense in $[0, 1]$, we can choose $(x', y') \in U$ with $y' - y$ irrational and x' sufficiently close to x . Here we put on the cylinder the metric

$$d((x, y), (x', y')) = \max(\text{dist}_{\mathbb{S}^1}(x, x'), |y - y'|).$$

For such a choice, the sequence $n(y - y')$ is dense in $\mathbb{S}^1$, therefore the sequence

$$\begin{aligned} d(T^n(x, y), T^n(x', y')) &= \max(\text{dist}_{\mathbb{S}^1}(x + ny, x' + ny'), |y - y'|) \\ &= \text{dist}_{\mathbb{S}^1}(x + ny, x' + ny') \end{aligned}$$

is dense in $[0, 1/2]$, and for some n is greater than Δ .

The map T also admits a dense set of periodic points, since every point with rational y is periodic. Indeed, let $y = p/q$ with $p, q \in \mathbb{Z}^*$, then

$$T^q \left(x, \frac{p}{q} \right) = (x + p, y) = (x, y).$$

Although assumptions (c2) and (c3) are satisfied, the map T is not cahotic because it is not transitive (conditions (c1) nor (c1') are satisfied), since the y -coordinate is invariant under T and every orbit of a point (x, y) stays in the horizontal circle $\mathbb{S}^1 \times \{y\}$.

1.4 Coding a chaotic dynamical system: the dynamics of symbols

In §1.2 we have introduced the idea of coding a dynamical system, understanding its orbits as sequences of a finite number of symbols from a certain alphabet. In both cases we have analyzed, the linear doubling map f_2 and the logistic map L_λ , the alphabet was made up only of two symbols, 0 and 1. Anyway, we can deal with an alphabet of $N \geq 2$ symbols similarly. In this paragraph we discuss the concept of coding in general.

We consider an alphabet of N symbols, say

$$\mathcal{A} = \{0, 1, \dots, N-1\},$$

and we define the spaces of infinite and bi-infinite sequences made up of symbols in $\mathcal{A}$ respectively as

$$\Sigma_N^R = \{(\omega_n)_{n \in \mathbb{N}} : \omega_n \in \mathcal{A}\}, \quad \Sigma_N = \{(\omega_n)_{n \in \mathbb{Z}} : \omega_n \in \mathcal{A}\}.$$

Let us focus on Σ_N^R , since the study of Σ_N is essentially the same (we will do it in Remark 1.31). For a fixed arbitrary $\lambda > 1$, we can endow Σ_N^R with the metric

$$d_\lambda(\omega, \omega') = \sum_{i \in \mathbb{N}} \frac{\beta(\omega_i, \omega'_i)}{\lambda^i}, \quad \text{where } \beta(\omega_i, \omega'_i) = \begin{cases} 0, & \text{if } \omega_i = \omega'_i, \\ 1, & \text{otherwise.} \end{cases} \quad (10)$$

In order to give a characterization of the balls of the metric space (Σ_N^R, d_λ) in terms of cylindric sets, we restrict the range of λ to $\lambda > 2$. Indeed, fix a finite n -sequence $\alpha_0 \dots \alpha_{n-1}$ and consider the *cylinder* of sequences having the first n component equal to $\alpha_0 \dots \alpha_{n-1}$, that is

$$C_{\alpha_0 \dots \alpha_{n-1}}^R = \{\omega \in \Sigma_N^R : \omega_i = \alpha_i, i = 0, \dots, n-1\}.$$

If $\omega \in C_{\alpha_0 \dots \alpha_{n-1}}^R$, then

$$\begin{aligned} d_\lambda(\alpha, \omega) &= \sum_{i \geq n} \frac{\beta(\alpha_i, \omega_i)}{\lambda^i} = \frac{1}{\lambda^n} \sum_{i \geq n} \frac{\beta(\alpha_i, \omega_i)}{\lambda^{i-n}} \\ &\leq \frac{1}{\lambda^n} \sum_{i \geq n} \frac{1}{\lambda^{i-n}} = \frac{1}{\lambda^n} \sum_{i \geq 0} \frac{1}{\lambda^i} = \frac{1}{\lambda^{n-1}} \frac{1}{\lambda - 1} < \frac{1}{\lambda^{n-1}}, \end{aligned}$$

where the last inequality holds since we have assumed that $\lambda > 2$. We have shown that

$$C_{\alpha_0 \dots \alpha_{n-1}}^R \subseteq B_{\frac{1}{\lambda^{n-1}}}(\alpha)$$

where α has the first n elements equal to $\alpha_0 \dots \alpha_{n-1}$. In order to prove the other inclusion, we consider $\omega \notin C_{\alpha_0 \dots \alpha_{n-1}}^R$, then

$$\omega_i \neq \alpha_i, \quad \text{for some } i \in \{0, \dots, n-1\}.$$

Hence,

$$d_\lambda(\alpha, \omega) \geq \frac{1}{\lambda^{n-1}} \quad \text{and} \quad \omega \notin B_{\frac{1}{\lambda^{n-1}}}(\alpha).$$

In summary, we have the following result.

Proposition 1.29. *Let $\lambda > 2$ and consider the space (Σ_N^R, d_λ) . Then for any $\alpha \in \Sigma_N^R$ there holds*

$$B_{\frac{1}{\lambda^{n-1}}}(\alpha) = C_{\alpha_0 \dots \alpha_{n-1}}^R.$$

We now consider the *right-shift map*

$$\sigma : \Sigma_N^R \rightarrow \Sigma_N^R, \quad \sigma(\omega_0\omega_1\ldots\omega_n\ldots) = (\omega_1\omega_2\ldots\omega_n\ldots), \quad (11)$$

also termed *Bernoulli shift*. The function σ is evidently continuous and the following result holds.

Theorem 1.30. *The dynamical system (Σ_N^R, σ) is chaotic.*

Proof. We verify the two conditions of Definition 1.22.

Let us start verifying (c1'), that is to show the presence of a dense orbit in (Σ_N^R, σ) , i.e.

$$\exists \omega \in \Sigma_N^R : \forall \varepsilon > 0, \forall \alpha \in \Sigma_N^R, \exists m = m(\alpha, \varepsilon) \text{ such that } \sigma^m(\omega) \in B_\varepsilon(\alpha).$$

The required ω can be defined as the sequence containing any possible string of any length, in some order. For instance we choose

$$\omega = \omega_1\omega_2\omega_3\ldots,$$

where, for any entire $n \geq 1$, ω_n is the finite concatenation of all possible n -sequences of 0s and 1s.

Take now any $\varepsilon > 0$, $\alpha \in \Sigma_N^R$, and $n \in \mathbb{N}$ such that

$$\frac{1}{\lambda^{n-1}} < \varepsilon. \quad (12)$$

The iteration m is then the one for which the first n elements of both α and ω coincide (such an m surely exists since ω_n contains any possible string of length n). This means that

$$\sigma^m(\omega) \in C_{\alpha_0\ldots\alpha_{n-1}}^R \subset B_\varepsilon(\alpha),$$

and (c1') holds. To prove (c2), that is, the density of periodic points for σ in Σ_N^R , let us consider any $\varepsilon > 0$ and any $\alpha \in \Sigma_N^R$. Taking once more $n \in \mathbb{N}$ satisfying Eq.(12) we have that the periodic point which repeats the string $[\alpha_0\ldots\alpha_{n-1}]$ belong to the cylinder $C_{\alpha_0\ldots\alpha_{n-1}}^R$ and hence to $B_\varepsilon(\alpha)$.

This prove the density of periodic points and concludes the proof. $\square$

We observe that Theorem 1.26 applies to (Σ_N^R, σ) , so that this dynamical system is chaotic also in the sense on Definition 1.20.

Remark 1.31. When we consider the set of bi-infinite sequences, Σ_N , we endow it with the same metric d_λ defined in Eq. (10). In this case, fix a finite $(2n-1)$ -sequence

$$\alpha_{-(n-1)}\ldots\alpha_0\ldots\alpha_{n-1}$$

and consider the *bi-cylinder* of sequences having the central $2n-1$ components equal to $\alpha_{-n+1}\ldots\alpha_0\ldots\alpha_{n-1}$, that is

$$C_{\alpha_{-n+1}\ldots\alpha_0\ldots\alpha_{n-1}} = \{\omega \in \Sigma_N : \omega_i = \alpha_i, i = -n+1, \ldots, n-1\}.$$

With the same procedure followed above, we can prove the result of Proposition 1.29, restricting λ to the range $\lambda > 3$.

The Bernoulli shift σ can also be defined on Σ_N and the dynamical system (Σ_N, σ) is chaotic; the proof of this result follows the one of Theorem 1.30.

Let us conclude this paragraph extending this construction to the case when the alphabet is countable, but not finite. Hence let us take $\mathcal{A} = \mathbb{N}$ and consider the set of infinite, or bi-infinite, sequences

$$\Sigma_{\mathbb{N}}^R \text{ and } \Sigma_{\mathbb{N}}.$$

The function d_λ introduced in (10) is still a metric on such sets, and the construction of cylindric neighborhoods remains the same, as far as $\lambda > 2$ (or $\lambda > 3$ when considering bi-infinite sequences).

The proof of Theorem 1.30 can be adapted as far as there exists a sequence $\omega \in \Sigma_{\mathbb{N}}^R$ that contains any finite string of symbols. Let us make some considerations to be convinced of this fact:

- the set of worlds of a fixed length n made with symbols in $\mathbb{N}$, $\mathcal{A}^n$, is countable, since it is a finite product of countable sets;
- the set of all finite words made with symbols in $\mathbb{N}$,

$$\bigcup_{n \in \mathbb{N}} \mathcal{A}^n$$

is still countable, since it is a countable union of countable sets.

We conclude that also in this case we can define a sequence, ω , which contains all possible finite sequences of symbols in $\mathcal{A}$. The cahoticity of both $(\Sigma_{\mathbb{N}}^R, \sigma)$ and $(\Sigma_{\mathbb{N}}, \sigma)$ follows.

The dynamics of the shift map σ on the set of infinite or bi-infinite sequences of symbols in an alphabet (finite or countable) is usually referred to as the *dynamics of symbols*.

1.5 Conjugacy and semi-conjugacy of maps

In Section 1.2, when dealing with the doubling linear map f_2 or with the discrete logistic map L_λ ($\lambda \geq 4$), we have observed that sequences of 0s and 1s can be used to represent points in the phase space of such dynamical systems. Such sequences are often termed *codes* of the represented points. The aim of this paragraph is to formalize this concept.

Definition 1.32. Let $g : X \rightarrow X$ and $f : Y \rightarrow Y$ be maps on metric spaces X and Y . Let $h : X \rightarrow Y$ be a continuous and onto map such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow h & & \downarrow h \\ Y & \xrightarrow{f} & Y \end{array}$$

that is $h \circ g = f \circ h$. Then f is said to be a *factor* of g (or *semi-conjugate to g*) via the *semi-conjugacy map* h . If h is a homeomorphism (continuous, with continuous inverse), then f and g are *conjugate* and h is a *conjugacy map*.

Example 1.33. The map $f : [0, +\infty) \rightarrow [0, +\infty)$, $f(x) = x$, is a *factor* of $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = -x$ via the continuous map $h : \mathbb{R} \rightarrow [0, +\infty)$, $h(x) = |x|$. Let us observe that h is not one-to-one, hence not invertible.

The maps $f, g : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 8x$, $g(x) = 2x$, are conjugate via the homeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}$, $h(x) = x^3$.

Example 1.34. The coding construction at the end of §1.2.1, at page 6, together with the definition of the metric space of infinite sequences proposed in Section 1.4, implies the existence of a natural semi-conjugacy between σ and f_2 . More precisely, let us consider the diagram

$$\begin{array}{ccc} \Sigma_2^R & \xrightarrow{\sigma} & \Sigma_2^R \\ \downarrow h & & \downarrow h \\ [0, 1) & \xrightarrow{f_2} & [0, 1) \end{array}$$

The function h associates to any infinite sequence of 0s and 1s

$$\underline{s} = (s_n)_{n \in \mathbb{N}}, \quad s_n \in \{0, 1\},$$

a unique number $x \in [0, 1)$, whose binary representation is $\underline{s}$. We recall that a sequence with a queue of 0s corresponds to a number with a finite representation and that two sequences of the form

$$(xxxx)01111\dots \quad \text{and} \quad (xxxx)10000\dots$$

have the same image under h (here $(xxxx)$ stays for any finite array of 0s and 1s). The function h is then well defined and surjective, provided that we identify $0.1111\dots$ with $1 \pmod{1} = 0$.

The continuity of h and the commutativity of the diagram follow easily.

Example 1.35. The coding construction of the logistic map, at page 10, implies the existence of a natural conjugacy between σ and L_λ , at least for high values of λ . Let us recall the itinerary function introduced in Definition 1.15 at page 11:

$$h : \Lambda \rightarrow \Sigma_2^R, \quad h(x_0) = (s_0 s_1 s_2 \dots), \quad \text{where } s_j = k \iff L_\lambda^j(s_0) \in \Delta_k,$$

and Λ is the invariant set for L_λ defined in (7). We show here that, if $\lambda > 2 + \sqrt{5}$, then h is a homeomorphism.

For these values of λ we have that

$$\exists K > 1 : |L'_\lambda(x)| > K, \quad \forall x \in \Lambda.$$

To prove that h is one-to-one, let $x, y \in \Lambda$ and suppose $h(x) = h(y)$. This means that, for every $n \in \mathbb{N}$, $L_\lambda^n(x)$ and $L_\lambda^n(y)$ lie on the same side of $\frac{1}{2}$. Assume that $x \neq y$. Since $|L'_\lambda(x)| > K > 1$, each iteration expands the distance between $L_\lambda^n(x)$ and $L_\lambda^n(y)$ by a factor of at least K so, definitely, $L_\lambda^n(x)$ and $L_\lambda^n(y)$ cannot lie on the same side of $\frac{1}{2}$. This contradicts the fact that $h(x) = h(y)$.

We have already proved that h is onto at page 10, by means of the Nested Intervals Theorem.

Since the inverse of an homeomorphism is still an homeomorphism, *conjugacy between maps is an equivalence relation*. Definitely, the notion of conjugacy is natural and central: two conjugate maps share the qualitative properties of their dynamics, and they are obtained from one another by a continuous change of coordinates. Hence all properties that are independent of such changes of coordinates persist in conjugate functions.

Also the weaker condition of semi-conjugacy guarantees that some properties of the starting function are inherited by the factor, as the next result states.

Lemma 1.36. *Let $f : Y \rightarrow Y$ be a factor of $g : X \rightarrow X$ via the semi-conjugacy $h : X \rightarrow Y$. Then*

(f1) *for any $k \geq 2$, f^k is a factor of g^k via h ;*

(f2) if x is a periodic point for g with minimal period n , then $y = h(x)$ is a periodic point for f with minimal period m , where $m|n$; in particular, if x^* is a fixed point for g , then $h(x^*)$ is a fixed point for f .

(f3) h maps the orbits under g to the orbits under f ;

(f4) if g is transitive on X , then f is transitive on Y ;

(f5) if g is topologically transitive on X , then f is topologically transitive on Y ;

(f6) if periodic points of g are dense in X , then periodic points of f are dense in Y .

In particular, if g is chaotic, then f is chaotic (according to Definition 1.22).

Proof. The proof of (f1) is immediate, indeed for any $k \geq 2$ we have

$$\begin{aligned} h \circ g^k &= (h \circ g) \circ g^{k-1} = (f \circ h) \circ g^{k-1} \\ &= f \circ (h \circ g^{k-1}) = \dots \\ &= f^k \circ h. \end{aligned}$$

To prove (f2), take a n -periodic point for g ; using (f1) we have $h \circ g^n(x) = f^n \circ h(x)$ which implies $h(g^n(x)) = f^n(h(x))$ and $h(x) = f^n(h(x))$. The point $y = h(x)$ is then a n -periodic point for f . The minimal period of y is then a divisor of n .

Concerning (f3), let $(x_n)_{n \in \mathbb{N}}$ be the orbit of $x_0 \in X$ under g , i.e. $x_{n+1} = g(x_n)$. For any n , let $y_n = h(x_n) \in Y$. Then

$$y_{n+1} = h(x_{n+1}) = h(g(x_n)) = f(h(x_n)) = f(y_n),$$

hence $(y_n)_n$ is an orbit for f , image of $(x_n)_n$ via h .

Finally, to prove (f4), take two open sets $U_1, U_2 \subset Y$ and consider the open sets $V_1, V_2 \subset X$, $V_i = h^{-1}(U_i)$, $i = 1, 2$. Since g is transitive, there exist $x \in V_1$ and $k \geq 1$ such that $g^k(x) \in V_2$. Then, for the chain of equality just above,

$$h(g^k(x)) = f^k(h(x))$$

so that $y = h(x) \in U_1$ and $f^k(y) = h(g^k(x)) \in U_2$. The map f is then transitive.

Properties (f5) and (f6) follow from the fact that h maps a dense set in X to a dense set in Y , in combination with (f2) and (f3). $\square$

Let us now consider two conjugate maps $g : X \rightarrow X$ and $f : Y \rightarrow Y$, via the homeomorphism $h : X \rightarrow Y$. Since in this case f is a factor of g and g is a factor of f the next result follows straightforwardly.

Corollary 1.37. *Let $f : Y \rightarrow Y$ and $g : X \rightarrow X$ be conjugate maps via the homeomorphism $h : X \rightarrow Y$. Then*

(c1) for any $k \geq 2$, f^k is conjugate to g^k via h ;

(c2) x is a n -periodic point for g if and only if $y = h(x)$ is a n -periodic point for f ;

(c3) $(x_n)_n$ is an orbit for g if and only if $(h(x_n))_n$ is an orbit for h ;

(c4) g is chaotic (according to Definition 1.22) if and only if f is chaotic.

Let us conclude this paragraph outlining that chaoticity passes from a map to its factors, but not from a factor to its extensions. Indeed, the semi-conjugacy can create transitivity by collapsing invariant components. To understand this fact, take any chaotic map $f : Y \rightarrow Y$ and consider $X = Y \times \{0, 1\}$, and

$$g : X \rightarrow X, \quad g(y, i) = (f(y), i).$$

Of course, the map g is not transitive and neither topologically transitive, but the projection $h : X \rightarrow Y$, $h(y, i) = y$ is a semi-conjugacy from g to f .

In the next paragraph, we introduce a new concept, the one of *topological entropy* that will allow to deduce *complexity* of the extension g starting from chaoticity of its factor f .

1.6 Topological entropy: a quantitative measure for complexity

In the previous paragraphs, we have encountered some *indicators of the complexity of a dynamical system*: density of the set of periodic points, (topological) transitivity, and chaos. These objects propose a *qualitative measures of the complexity* of a dynamical system and they are the same in conjugated dynamical systems. The only quantitative measure of complexity we have introduced up to now is the growth rate of periodic orbits (at the end of §1.1, at page 4; see also its computation in Propositions 1.6 and 1.13). A step beyond such notion is to measure the growth of all orbits, in some sense. This is what we are doing in this paragraph: *we introduce a numerical invariant related to the orbit growth*, termed *topological entropy*.

Being a numerical invariant, two dynamical systems with different topological entropy can not be conjugated; moreover, being a number, it gives a finer distinction between dynamical systems than a qualitative (yes/no) invariant. Furthermore, this quantitative complexity indicator, represents the exponential growth rate for the number of orbits segments distinguishable with arbitrary fine (but finite) precision. In a sense, the topological entropy describes, with a single number, the total exponential complexity of the orbit structure. Chaotic systems are distinguished by having positive entropy, and the topological entropy is no less than the growth rate of periodic orbits. Therefore, it is appropriate to view entropy as a quantitative measure of the amount of chaos in a dynamical system.

In this paragraph we mainly follow the exposition in [41][§8.2].

Let us consider a compact metric space (X, d) , a continuous map $F : X \rightarrow X$, and define the sequence of metrics (termed Bowen metrics or *dynamic distances*)

$$d_n^F(x, y) := \max_{0 \leq i \leq n-1} d(F^i(x), F^i(y)), \quad n = 1, 2, \dots$$

Of course, fixed x and y , d_n^F increases in n . Denoting the open ball

$$B_F(x, r, n) = \{y \in X : d_n^F(x, y) < r\},$$

it turns out that $B_F(x, r, n+1) \subseteq B_F(x, r, n)$, for any n .

Definition 1.38. A set $E \subset X$ is (n, r) -dense if

$$X \subseteq \bigcup_{x \in E} B_F(x, r, n).$$

The r -capacity of d_n^F , denoted with $S_d(F, r, n)$, is the minimal cardinality of an (n, r) -dense set, or, equivalently, the cardinality of a minimal (n, r) -dense set.

In other words, the r -capacity of d_n^F is the minimal number of initial conditions whose behaviour, up to time n , approximates the behaviour of *any* initial condition, up to r . Let us observe that the compactness assumption on X ensures that the r -capacity is finite.

Let us now consider the exponential growth rate of the r -capacity of d_n^F

$$h_d(F, r) := \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log S_d(F, r, n)$$

Since both $S_d(F, r, n)$ and h_d increase when $r \rightarrow 0$, we can give the following definition.

Definition 1.39. The *topological entropy* of F is

$$h_{top}(F) := \lim_{r \rightarrow 0} h_d(F, r).$$

Remark 1.40. The definition of h_{top} presents a double limit: the first one is in n and the second in r . Anyway, the import one is the limit in n : it is there that the dynamics enters. In many interesting cases (see the examples below), the second limit is trivial, since $h_d(F, r)$ does not depend on r .

Remark 1.41. Topological entropy does not depend on the metric d , as long as one change to an equivalent metric. As a consequence topological entropy is an invariant under topological conjugacy (for details, see [41][Proposition 8.2.2.]).

The next result allows us to infer exponential complexity in the orbit structure of a dynamical system from the presence of a sufficiently complex factor.

Proposition 1.42. *Let (X, d_X) , (Y, d_Y) be metric spaces, and assume X compact. If $f : Y \rightarrow Y$ is a factor of $g : X \rightarrow X$ via $h : X \rightarrow Y$, then*

$$h_{top}(g) \geq h_{top}(f).$$

Proof. h is uniformly continuous, then, for any $r > 0$ there exists $\delta > 0$ such that if $d_X(x_1, x_2) < \delta$ then $d_Y(h(x_1), h(x_2)) < r$. Hence, for any $x \in X$

$$h(B_g(x, \delta, n)) \subset B_f(h(x), r, n),$$

so that $S_{d_X}(g, \delta(r), n) \geq S_{d_Y}(f, r, n)$. The thesis follows taking the limits in n and r . $\square$

Topological entropy is often difficult to calculate, it is then convenient to have alternative definitions to be able to chose the more convenient one. There are indeed several quantities similar to $S_d(F, r, n)$ that can be used for this porpoise. Let us introduce the following one:

- $D_d(F, r, n)$, the minimal number of sets whose diameter in the metric d_n^F is less then r and whose union covers X ;
- $N_d(F, r, n)$, the maximal number of points in X with pairwise d_n^F -distances at least r . Such set of points is commonly termed (n, r) -separated, and generate the maximal number of orbit segments of length n that are distinguishable with precision r .

Topological entropy can be defined starting from D_d or N_d , and taking the limit on n and r . We omit the details, as they are purely technical and of little relevance to the purposes of these Notes.

We conclude this paragraph with a few examples illustrating the computation of topological entropy.

Example 1.43. Fixed any angle α , the translation on $\mathbb{S}^1$,

$$T : \mathbb{S}^1 \rightarrow \mathbb{S}^1, \quad T(\vartheta) = \vartheta + \alpha,$$

has zero topological entropy. Indeed, the distance between any iteration of two points does not vary with the number of iterations. Hence, the r -capacity of d_n^T does not depend on n and its exponential growth rate is zero.

Example 1.44. Let us consider the linear expanding map f_2 (see §1.2.1)

$$f_2 : [0, 1) \rightarrow [0, 1), \quad x \mapsto f_2(x) = 2x \pmod{1},$$

to have compactness of X , let us take $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ as a domain with distance

$$d(x, y) = \min\{|x - y|, 1 - |x - y|\}.$$

We claim that

$$h_{\text{top}}(f_2) = \log 2.$$

The first step is to consider, for any $n \geq 2$, the Bowen metric

$$d_n^{f_2}(x, y) = \max_{0 \leq i \leq n-1} d(f_2^i(x), f_2^i(y)),$$

and observe that if $d(x, y) = \delta$, then

$$d(f_2^k(x), f_2^k(y)) = \min\left(2^k \delta \pmod{1}, 1 - (2^k \delta \pmod{1})\right).$$

In particular,

$$\text{if } d(x, y) = \delta < \frac{1}{2^{k+1}} \quad \text{then} \quad d(f_2^k(x), f_2^k(y)) = 2^k \delta,$$

thus

$$d_n^{f_2}(x, y) < \varepsilon \quad \Longleftrightarrow \quad d(f_2^{n-1}(x), f_2^{n-1}(y)) = 2^{n-1} \delta < \varepsilon \quad \Longleftrightarrow \quad \delta < \frac{\varepsilon}{2^{n-1}}.$$

Bowen balls of radius ε are then segments of length $\frac{2\varepsilon}{2^{n-1}}$

$$B_F(x, \varepsilon, n) = \left(x - \frac{\varepsilon}{2^{n-1}}, x + \frac{\varepsilon}{2^{n-1}}\right) \pmod{1}.$$

Hence, the ε -capacity of $d_n^{f_2}$ is

$$S_d(f_2, \varepsilon, n) = \frac{2^{n-1}}{2\varepsilon} + 1.$$

Fixed any small ε , we have that

$$S_d(f_2, \varepsilon, n) \sim 2^n,$$

from which the thesis follows.

Example 1.45. Let us consider the right-shift σ on the space of infinite sequences with two symbols

$$\Sigma_N^R = \{(\omega_n)_{n \in \mathbb{N}} : \omega_n \in \{0, 1, \dots, N-1\}\},$$

endowed with the metric

$$d_\lambda(\omega, \omega') = \sum_{i \in \mathbb{N}} \frac{\beta(\omega_i - \omega'_i)}{\lambda^i},$$

for some $\lambda > 1$ (see §1.4). It is not difficult to understand that (Σ_N^R, d_λ) is compact. We claim that

$$h_{\text{top}}(\sigma) = \log N.$$

The Bowen metric is

$$d_{\lambda, n}^\sigma(\omega, \omega') = \max_{0 \leq i \leq n-1} d_\lambda(\sigma^i(\omega), \sigma^i(\omega')).$$

To compute the r -capacity of $d_{\lambda,n}^\sigma$ let us observe that whenever two sequences α and ω coincide on the first k elements then

$$d_\lambda(\alpha, \omega) = \sum_{i \geq k} \frac{\beta(\omega_i - \omega'_i)}{\lambda^i} \leq \frac{1}{\lambda^k} \frac{\lambda}{\lambda - 1}$$

and, assuming $k \geq n$,

$$d_{\lambda,n}^\sigma(\alpha, \omega) \leq \frac{1}{\lambda^{k-n}} \frac{\lambda}{\lambda - 1}$$

Take now $r > 0$ and observe that, fixed n there exists $k = k(r)$ such that

$$\frac{1}{\lambda^{k-n}} \frac{\lambda}{\lambda - 1} < r.$$

Hence a minimal covering with $d_{\lambda,n}^\sigma$ -ball consists of balls centered in sequences which differ each other in the first k elements. Hence the r -capacity is N^k with $k = n + c(r)$, where $c(r)$ is a positive constant that depends on r (when $r \rightarrow 0$, it diverges to $+\infty$). Hence

$$h_d(\sigma, r) = \varlimsup_n \frac{1}{n} \log N^k = \log N \varlimsup_n \frac{n + c(r)}{n} = \log N,$$

and the claim follows.

Example 1.46. Let us consider the linear twist map on the cylinder $X = \mathbb{S}^1 \times [0, 1]$, that is

$$T : X \rightarrow X, \quad T(x, y) = (x + y, x).$$

Since the y -coordinate is constant along orbits, on each horizontal circle $\mathbb{S}^1 \times \{y\}$ the map is a rotation by angle y . The space X is then decomposed into invariant sets $\mathbb{S}^1 \times \{y\}$, for each $y \in [0, 1]$. When $y \in \mathbb{Q}$, motions are periodic on the corresponding fiber.

Equipping X with the metric

$$d((x, y), (x', y')) = |y - y'| + d_{\mathbb{S}^1}(x, x'),$$

and being

$$T^n(x, y) = T^n(x', y') = ((x - x') + n(y - y'), y - y'),$$

we have

$$d(T^n(x, y), T^n(x', y')) \leq |y - y'| + d_{\mathbb{S}^1}(x, x') + n|y - y'|.$$

The growth rate of this last quantity is n , hence just linear. The topological entropy of this dynamical system is then 0, even if it is not trivial.

2 Some models from Celestial Mechanics

It is a matter of fact: singularities in the equations of motion of a classical mechanical system usually play a dominant role in the global phase portrait of the system. With the term *singularity*, we mean a point, or set of points, where the system is not defined: for instance, the position of a star in a planet-star system is a singular point; more generally, *Kepler-type potentials* (meaning homogeneous functions with a singularity at one point) are examples of potentials inducing a singular force. If we consider $N \geq 2$ huge and fixed bodies with a small one moving among them under their gravitational attractions, we obtain the so called *N-center problem*. Here, the dynamics is much more complex with respect to the 1-center problem and the situation becomes more and more involved when all bodies move: in this case we talk about the *N-body problem*, a challenging and well known dynamical system. The complex global orbit structure induced by the presence of singularities has been studied starting from Henry Poincaré. Since then, many researchers contributed, and continue to contribute today, to this study.

In this Notes we aim to analyze the presence of a hidden chaotic dynamic in two declinations of the Kepler problem. The first one is the *anisotropic Kepler problem*. This model has been studied in the pioneristic works of Martin Gutzwiller (see [37, 38, 39]) in the study of bound states of an electron near a donor impurity of a semiconductor. The second one is a classical Birkoff billiard with a Keplerian mass inside (the so called *Keperian billiard*, see [19, 52, 49, 56, 57, 59]).

The presence of a complex dynamics in models arising from Celestial Mechanics has been studied in many other situations that we recall here, without claiming to be exhaustive. Concerning the *N-center problem*, we refer to the work of Bolotin and Negrini, [14, 17, 18], but also to the one of Knauf and Taimanov [46, 47, 48], or to the more recent ones that take also into account some interesting generalizations of the classical setting [54, 58, 23, 5, 6]. Concerning the restricted 3-body problem, we mention the papers [35, 36].

2.1 The planar anisotropic Kepler problem

This section mainly follows the exposition proposed by Robert Devaney in the Lecture Notes [32, Section 1.2]; some of Devaney's results have been improved by Casasayas and Llibre in [20, 21] (see also the book [22]). Some references concerning the planer anisotropic Kepler problem are [28, 29, 30, 31, 9, 1].

The *planar anisotropic Kepler problem* is a one parameter family of classical mechanical systems with two degrees of freedom which depends analytically on a real parameter $\mu \geq 1$. When $\mu = 1$, it reduces to the ordinary Kepler problem, which is completely integrable and the orbit structure is well understood (see Remarks 2.4 and 2.6). When $\mu > 1$, the anisotropy introduced into the system destroys not only its symmetry, but also the orbit structure. Practically, we consider a point mass with position $q = (q_1, q_2)$ moving in the euclidean plane under the action of an attracting force induced by the potential

$$V_\mu(q) = -\frac{1}{\langle q, q \rangle_M^{1/2}}, \quad q \in \mathbb{R}^2 \setminus \{0\}.$$

At the denominator we find the weighted scalar product in $\mathbb{R}^2$

$$\langle q, q \rangle_M = q^T M q = \mu q_1^2 + q_2^2,$$

with positive definite matrix

$$M = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}.$$

Since V_μ has a singularity at the origin, the configuration space is

$$Q = \mathbb{R}^2 \setminus \{0\}.$$

Furthermore, the assumption $\mu \geq 1$ is not restrictive and we should make an analogue construction for $\mu \in (0, 1]$. We are interested in the study of the conservative dynamical system,

$$q'' = -\nabla V_\mu(q)$$

equivalent to

$$\begin{cases} q' = p, \\ p' = -\nabla V_\mu(q) \end{cases} \quad (13)$$

with *total energy* (or *Hamiltonian*)

$$H(p, q) = \frac{1}{2}|p|^2 + V_\mu(q).$$

Since H is an integral for the system, level sets of H , i.e.

$$H^{-1}(e) = \left\{ (p, q) \in \mathbb{R}^2 \times Q : \frac{1}{2}|p|^2 + V_\mu(q) = e \right\}, \quad e \in \mathbb{R},$$

are invariant sets, in the tangent bundle $TQ = \mathbb{R}^2 \times Q$, for the dynamical system. In particular, fixed e , being $|p|^2 \geq 0$ the particle is confined to move in the *Hill's region*

$$\{q \in Q : V_\mu(q) \leq e\},$$

which is an ellipse centered at the singularity when e is negative, otherwise is the whole Q .

2.1.1 Regularizing the flow via McGehee coordinates and the collision manifold

Let us now introduce a new set of space-time coordinates proposed by Richard McGehee in his landmark paper published on *Inventiones Mathematicae* in 1974 (see [50]). These variables allow to regularize the singular system (13), not removing the singularity (as the classical Levi-Civita regularisation does), but blowing it up and replacing it with an invariant manifold for the flow.

As a first step, we consider the *polar coordinates* (r, ϑ) such that

$$q = rs_\vartheta, \quad r > 0,$$

where

$$s_\vartheta = (\cos \vartheta, \sin \vartheta), \quad \vartheta \in \mathbb{R}/2\pi\mathbb{Z}.$$

Then since $r = |q|^{1/2}$ we obtain that r' is the radial component of p , indeed

$$r' = \frac{1}{r} \langle q, p \rangle = \langle s_\vartheta, p \rangle.$$

Furthermore, since $q = rs_\vartheta$ we have

$$p = q' = r's_\vartheta + rs'_\vartheta = \langle s_\vartheta, p \rangle s_\vartheta + r\vartheta' s_\vartheta^\perp,$$

where $s_\vartheta^\perp = (-\sin \vartheta, \cos \vartheta)$. Then the angular component of p is $r\vartheta' = \langle s_\vartheta^\perp, p \rangle$, and

$$\vartheta' = \frac{1}{r} \langle s_\vartheta^\perp, p \rangle.$$

The conservation of H and the singularity of V_μ at $q = 0$ imply that $|p| \rightarrow \infty$ when $r = |q| \rightarrow 0$, with rate $r^{-1/2}$. In order to slow down the velocity when $r \rightarrow 0$ we introduce the *rescaled radial and angular components of p*

$$v = r^{1/2} \langle s_\vartheta, p \rangle, \quad u = r^{1/2} \langle s_\vartheta^\perp, p \rangle.$$

We can then write the Hamiltonian system (13) in terms of the variables (r, ϑ, v, u) as

$$\begin{cases} r' &= \frac{1}{r^{1/2}} v, \\ \vartheta' &= \frac{1}{r^{3/2}} u, \\ v' &= \frac{r'}{2r^{1/2}} \langle s_\vartheta, p \rangle + r^{1/2} \vartheta' \langle s_\vartheta^\perp, p \rangle + r^{1/2} \langle s_\vartheta, p' \rangle \\ &= \frac{1}{r^{3/2}} \left(\frac{1}{2} v^2 + u^2 + V_\mu(s_\vartheta) \right), \\ u' &= \frac{r'}{2r^{1/2}} \langle s_\vartheta^\perp, p \rangle - r^{1/2} \vartheta' \langle s_\vartheta, p \rangle + r^{1/2} \langle s_\vartheta^\perp, p' \rangle \\ &= \frac{1}{r^{3/2}} \left(-\frac{1}{2} v u - \langle s_\vartheta^\perp, \nabla V_\mu(s_\vartheta) \rangle \right), \end{cases} \quad (14)$$

where in the third line we have used the (-1)-homogeneity of V_μ and Euler's theorem, which give the identities

$$r^{1/2} \langle s_\vartheta, p' \rangle = r^{-1/2} \langle r s_\vartheta, -\nabla V_\mu(r s_\vartheta) \rangle = r^{-1/2} V_\mu(r s_\vartheta) = r^{-3/2} V_\mu(s_\vartheta).$$

Similar computations yield the last line of (14). Let us observe that System (14) is still singular at the origin of the configuration space, that is $r = 0$. The last variable change consists in the introduction of a *new time variable*

$$\tau = \tau(t), \quad \text{such that} \quad \frac{d\tau}{dt} = r^{-3/2},$$

that permits to remove this singularity. In conclusion, the following result holds (the *dot* means the derivative with respect to τ).

Proposition 2.1. *The anisotropic Kepler problem (13) in McGehee coordinates is given by*

$$\begin{cases} \dot{r} = r v \\ \dot{\theta} = u \\ \dot{v} = u^2 + \frac{1}{2} v^2 + \tilde{V}_\mu(\theta) \\ \dot{u} = -\frac{1}{2} v u - \tilde{V}_\mu'(\theta) \end{cases} \quad (15)$$

where the potential $\tilde{V}_\mu(\theta)$ is

$$\tilde{V}_\mu(\theta) = -\frac{1}{\sqrt{\mu \cos^2(\theta) + \sin^2(\theta)}}, \quad (16)$$

(see Figure 7, left). Furthermore, the energy relation $H(p, q) = e$ reads

$$\frac{1}{2}(u^2 + v^2) + \tilde{V}_\mu(\theta) = r e. \quad (17)$$

The vector field of the dynamical system in the new set of variables, (15), is analytic and, in particular, not singular at $r = 0$. Furthermore, the first equation implies that the set

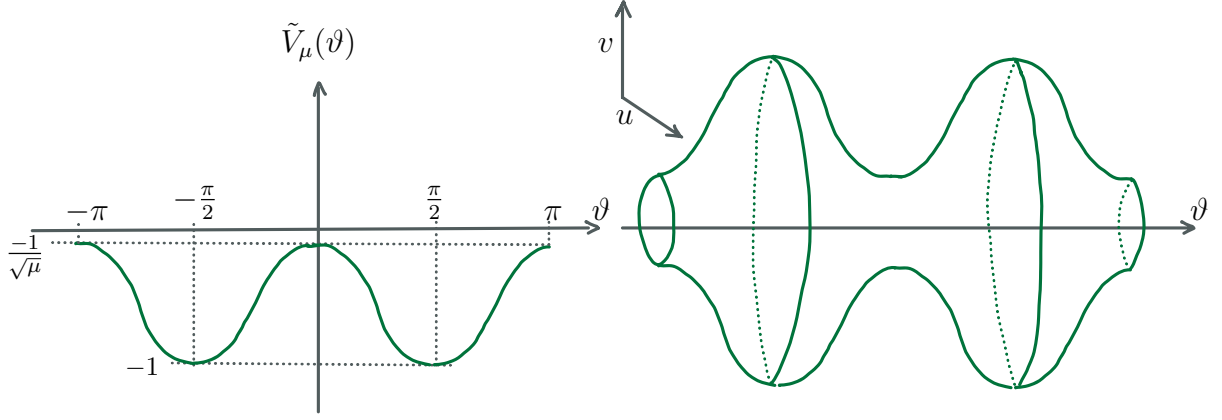


Figure 7: Left: the graph of $\tilde{V}_\mu(\vartheta)$, given by Eq. (16), on $[-\pi, \pi]$. This function has two non-degenerate minima, at level -1 , and two non-degenerate maxima at level $-1/\sqrt{\mu}$. Right: The collision manifold Λ is a surface of revolution contained in $\{r = 0\}$. Indeed, for any ϑ , the variables u and v describes a circle of radius $(-2\tilde{V}_\mu(\vartheta))^{1/2}$.

$\{r = 0\}$ is invariant for the flow, and, for any fixed energy value, e , the motion in $\{r = 0\}$ is confined to

$$\Lambda = \left\{ (r, \vartheta, v, u) : r = 0 \text{ and } \frac{1}{2}(u^2 + v^2) + \tilde{V}_\mu(\vartheta) = 0 \right\}. \quad (18)$$

The set Λ is a closed and bounded submanifold of $\{r = 0\}$, being the 0-level set of the smooth function $\frac{1}{2}(u^2 + v^2) + \tilde{V}_\mu(\vartheta)$ (see Figure 7, right). These arguments give sense to the next definition.

Definition 2.2. The set Λ defined in Eq. (18) is termed *collision manifold* for the System (15).

Remark 2.3. From the energy relation (17), we deduce that the Hill's region with energy level e in McGehee coordinates reads

$$\{\vartheta : \tilde{V}_\mu(\vartheta) \leq re\}.$$

Since $\tilde{V}_\mu$ is negative, the Hill's region is the whole configuration space when $e \geq 0$; otherwise, when $e < 0$, it is bounded, its boundary does not depend on e and coincides with the collision manifold Λ .

Remark 2.4 (Kepler problem - Act 1). When $\mu = 1$ the anisotropic problem reduced to the Kepler problem. In this case $\tilde{V}_\mu(\vartheta) = 1$, M is the identity matrix, bounded Hill's regions are circles, and the regularized system in McGehee coordinates is

$$\begin{cases} \dot{r} = rv \\ \dot{\theta} = u \\ \dot{v} = u^2 + \frac{1}{2}v^2 - 1 \\ \dot{u} = -\frac{1}{2}vu, \end{cases}$$

while the energy relation reads

$$\frac{1}{2}(u^2 + v^2) - 1 = re.$$

The collision manifold reduces to the cylinder

$$\Lambda_K = \{(r, \vartheta, v, u) : r = 0, u^2 + v^2 = 2\}.$$

2.1.2 The dynamic on the collision manifold

The regularized flow (15) restricted to Λ reads

$$\begin{cases} \dot{v} = \frac{1}{2}u^2, \\ \dot{\theta} = u, \\ \dot{u} = -\frac{1}{2}vu - \tilde{V}'_\mu(\theta). \end{cases} \quad (19)$$

Concerning its equilibrium points, the following result holds.

Proposition 2.5. *The vector field on Λ admits exactly eight equilibrium points in each ϑ -period: two sinks, two sources and four saddles. The property of these rest points are summarized in the next table (see Figure 8):*

Equilibrium point	Characteristic exponents		Type
	On Λ	Off Λ	
$(-\sqrt{2}, -\frac{\pi}{2}, 0)$	$\frac{\sqrt{2}}{4} \pm \frac{\sqrt{2}}{4}\sqrt{9-8\mu}$	$-\sqrt{2}$	Source
$(-\sqrt{2}\mu^{-\frac{1}{4}}, 0, 0)$	$\frac{\sqrt{2}}{4}\mu^{-\frac{1}{4}} \pm \frac{\sqrt{2}}{4}\mu^{-\frac{1}{4}}\sqrt{\frac{9\mu-8}{\mu}}$	$-\sqrt{2}\mu^{-\frac{1}{4}}$	Saddle
$(-\sqrt{2}, \frac{\pi}{2}, 0)$	$\frac{\sqrt{2}}{4} \pm \frac{\sqrt{2}}{4}\sqrt{9-8\mu}$	$-\sqrt{2}$	Source
$(-\sqrt{2}\mu^{-\frac{1}{4}}, \pi, 0)$	$\frac{\sqrt{2}}{4}\mu^{-\frac{1}{4}} \pm \frac{\sqrt{2}}{4}\mu^{-\frac{1}{4}}\sqrt{\frac{9\mu-8}{\mu}}$	$-\sqrt{2}\mu^{-\frac{1}{4}}$	Saddle
$(\sqrt{2}, -\frac{\pi}{2}, 0)$	$-\frac{\sqrt{2}}{4} \pm \frac{\sqrt{2}}{4}\sqrt{9-8\mu}$	$\sqrt{2}$	Sink
$(\sqrt{2}\mu^{-\frac{1}{4}}, 0, 0)$	$-\frac{\sqrt{2}}{4}\mu^{-\frac{1}{4}} \pm \frac{\sqrt{2}}{4}\mu^{-\frac{1}{4}}\sqrt{\frac{9\mu-8}{\mu}}$	$\sqrt{2}\mu^{-\frac{1}{4}}$	Saddle
$(\sqrt{2}, \frac{\pi}{2}, 0)$	$-\frac{\sqrt{2}}{4} \pm \frac{\sqrt{2}}{4}\sqrt{9-8\mu}$	$\sqrt{2}$	Sink
$(\sqrt{2}\mu^{-\frac{1}{4}}, \pi, 0)$	$-\frac{\sqrt{2}}{4}\mu^{-\frac{1}{4}} \pm \frac{\sqrt{2}}{4}\mu^{-\frac{1}{4}}\sqrt{\frac{9\mu-8}{\mu}}$	$\sqrt{2}\mu^{-\frac{1}{4}}$	Saddle

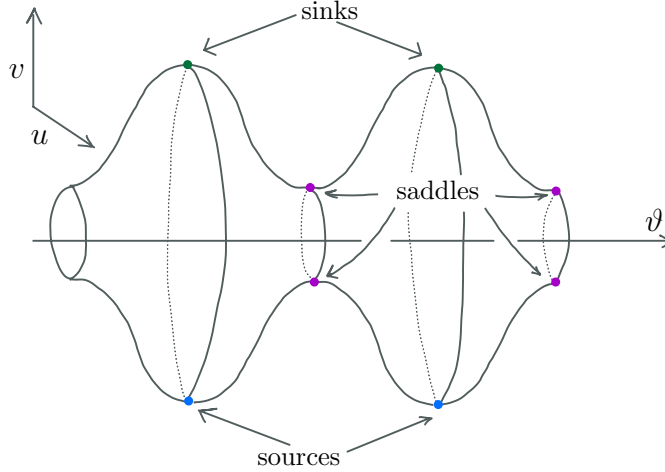


Figure 8: Up to periodicity, the regularized flow has 8 hyperbolic equilibria on the collision manifold Λ : two sinks (in green), two sources (blue) and four saddles (purple).

Proof. Equilibrium points on Λ are

$$\left\{ (v_0, \vartheta_0, u_0) : u_0 = 0, \tilde{V}'_\mu(\vartheta_0) = 0, v_0 = \pm\sqrt{-2\tilde{V}_\mu(\vartheta_0)} \right\}.$$

Hence, choosing $\vartheta \in [-\pi, \pi)$, we have

- if $\theta_0 = -\frac{\pi}{2}$ they are $(\pm\sqrt{2}, -\frac{\pi}{2}, 0)$,
- if $\theta_0 = 0$ they are $(\pm\sqrt{2}\mu^{-\frac{1}{4}}, 0, 0)$,
- if $\theta_0 = \frac{\pi}{2}$ they are $(\pm\sqrt{2}, \frac{\pi}{2}, 0)$,
- if $\theta_0 = \pi$ they are $(\pm\sqrt{2}\mu^{-\frac{1}{4}}, \pi, 0)$.

In order to classify such stationary points we linearize System (15) at $(0, v_0, \vartheta_0, 0)$ obtaining the matrix

$$A = \begin{pmatrix} v_0 & 0 & 0 & 0 \\ 0 & v_0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\tilde{V}_\mu''(\theta_0) & -\frac{1}{2}v_0 \end{pmatrix},$$

while the matrix giving the linearization of the vector field on Λ is the 2×2 submatrix

$$A' = \begin{pmatrix} 0 & 1 \\ -\tilde{V}_\mu''(\theta_0) & -\frac{1}{2}v_0 \end{pmatrix}.$$

The eigenvalues of A' at a rest point $(v_0, \theta_0, 0)$ are

$$-\frac{1}{4}v_0 \pm \frac{1}{2}\sqrt{\frac{1}{4}v_0^2 - 4\tilde{V}_\mu''(\theta_0)},$$

and the other two eigenvalues of A are both equal to $v_0 = \pm\sqrt{-2\tilde{V}_\mu(\theta_0)} \neq 0$.

Let us explicitly compute the eigenvalues of A' . From

$$\tilde{V}_\mu''(\theta) = (1 - \mu) \frac{(\cos^2 \theta - \sin^2 \theta)(\mu \cos^2 \theta + \sin^2 \theta) - 3(1 - \mu) \cos^2 \theta \sin^2 \theta}{(\mu \cos^2 \theta + \sin^2 \theta)^{\frac{5}{2}}}$$

we obtain

$$\tilde{V}_\mu''\left(\pm\frac{\pi}{2}\right) = \mu - 1, \quad \text{and} \quad \tilde{V}_\mu''(0) = \tilde{V}_\mu''(\pi) = -\mu^{-\frac{3}{2}}(\mu - 1).$$

Since $\tilde{V}_\mu(\pm\pi/2) = -1$, we have

$$\sqrt{-\frac{1}{2}\tilde{V}_\mu(\theta_0) - 4\tilde{V}_\mu''(\theta_0)} = \sqrt{\frac{1}{2} - 4(\mu - 1)} = \sqrt{\frac{1}{2}(9 - 8\mu)},$$

and the eigenvalues when $\vartheta_0 = \pm\pi/2$ are

$$-\frac{v_0}{4} \pm \frac{\sqrt{2}}{4}\sqrt{9 - 8\mu}.$$

We then conclude that the eigenvalues of $(-\sqrt{2}, \pm\pi/2, 0)$ have both strictly positive real parts for any $\mu > 1$; in particular, when $\mu > 9/8$ they have a non-vanishing complex part. The eigenvalues of $(\sqrt{2}, \pm\pi/2, 0)$ have both strictly negative real parts for any $\mu > 1$, and, when $\mu > 9/8$, they have a non-vanishing complex part. Hence, Hartmann-Grobmann theorem applies to the four equilibria $(\pm\sqrt{2}, \pm\pi/2, 0)$.

When $\theta_0 = 0$ or $\theta_0 = \pi$, $\tilde{V}_\mu(\theta_0) = -\mu^{-\frac{1}{2}}$, so that the eigenvalues of A' are

$$-\frac{v_0}{4} \pm \frac{\sqrt{2}}{4}\mu^{-\frac{1}{4}}\sqrt{\frac{9\mu - 8}{\mu}}.$$

Since $\mu > 1$, both eigenvalues are real and

$$\frac{9\mu - 8}{\mu} > 1.$$

Furthermore, since in these two cases $v_0 = \pm\sqrt{2\mu^{-\frac{1}{2}}}$, the eigenvalues are

$$\pm \frac{\sqrt{2}}{4} \mu^{-\frac{1}{4}} \left(1 \pm \sqrt{\frac{9\mu - 8}{\mu}} \right)$$

and hence each of the four stationary points $(\pm\sqrt{2}, 0, 0)$ and $(\pm\sqrt{2}, \pi, 0)$ have a pair of non-zero real eigenvalues with opposite sign. We conclude that Hartmann-Grobmann theorem can be applied also in these case. $\square$

Remark 2.6 (Kepler problem - Act 2). When $\mu = 1$, i.e. when the anisotropic problem reduces to the Kepler problem, the flow on the collision manifold Λ_K (see Remark 2.4) reduces to

$$\begin{cases} \dot{v} = \frac{1}{2}u^2 \\ \dot{\theta} = u \\ \dot{u} = -\frac{1}{2}vu, \end{cases}$$

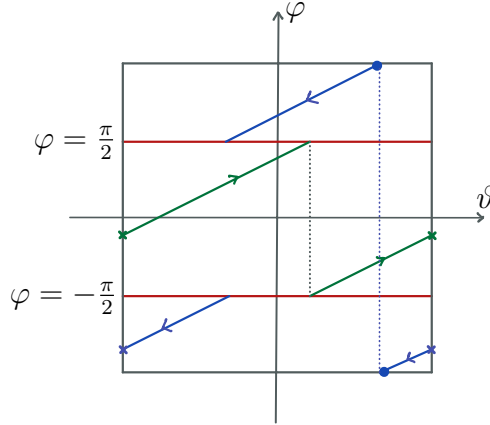


Figure 9: The collision manifold for the Kepler problem in the torus $\mathbb{S}^1 \times \mathbb{S}^1$. There are two circles of equilibrium points (in red): $(\vartheta, \pi/2)$ are stable, while $(\vartheta, -\pi/2)$ are unstable. In green and blue two examples of orbits on Λ_K .

Furthermore, Λ_K can be described by a pair of angles: ϑ and ψ , where

$$u = \sqrt{2} \cos \psi, \quad v = \sqrt{2} \cos \psi.$$

The equations of motion solved by such angles actually are

$$\begin{cases} \dot{\vartheta} = \sqrt{2} \cos \psi \\ \dot{\psi} = \frac{1}{\sqrt{2}} \sin \psi. \end{cases}$$

There are two circles of equilibrium points, $\{(\vartheta, \pm\pi/2) : \vartheta \in [-\pi, \pi)\}$ and any orbit belongs to a line of the family

$$\psi(\vartheta) = \frac{\vartheta}{2} + \text{const.}$$

Equilibria with $\psi = \pi/2$ are stable, the ones with $\psi = -\pi/2$ are unstable (see Figure 9).

Let us go further in the analysis of the regularized flow on Λ (System (18)). In order to do that, let us consider the function

$$g(v, \vartheta, u) = v,$$

that makes the vector field on the compact manifold Λ a *gradient-like vector field*. Indeed,

g strictly increases along any non-equilibrium orbit.

In order to understand this property, we observe that $\dot{v} = u^2/2$, hence $\dot{g} \geq 0$, and $\dot{g} > 0$ along any orbit segment that misses $u = 0$. Nevertheless, we claim that g still increases strictly also along any non-stationary orbit that meets $u = 0$. Indeed, when $u = 0$, then $\dot{u} = -\tilde{V}_\mu(\vartheta)$ and $v = \pm\sqrt{-2\tilde{V}_\mu(\vartheta)}$. Hence, $\dot{u}$ vanishes only at equilibrium points. Zeros of u are then isolated, making g strictly increasing. We can then deduce that

all non-equilibrium orbits on Λ both begin and end at stationary points and the system does not admit any non-trivial recurrence.

Let us now recall the classification of equilibrium points on Λ proved in Proposition 2.5: this result provides that every stationary point is hyperbolic and, at least locally, stable/unstable manifolds of equilibria are well defined. Furthermore,

- both sources are at level $v = -\sqrt{2}$;
- both sinks are at level $v = \sqrt{2}$;
- two of the four saddles are at level $v = -\sqrt{2}\mu^{-\frac{1}{4}} > -\sqrt{2}$;
- the remaining saddles are at level $v = \sqrt{2}\mu^{-\frac{1}{4}} < \sqrt{2}$.

We aim now to analyse the ultimate behaviour (in past and future) of the stable and unstable manifolds of the four saddles. We denote $W^s(p)$ the stable manifold of the point p , and $W^u(p)$ the unstable one.

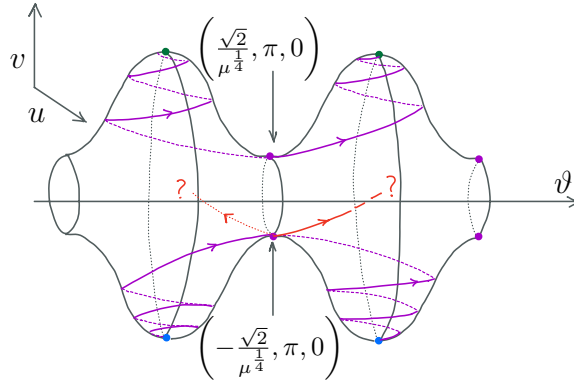


Figure 10: In purple: the two branches of $W^s(-\sqrt{2}\mu^{-\frac{1}{4}}, \pi, 0)$ emanate from the two sources $(-\sqrt{2}, \pm\pi/2, 0)$, and the two of $W^u(\sqrt{2}\mu^{-\frac{1}{4}}, \pi, 0)$ die in the two sinks $(\sqrt{2}, \pm\pi/2, 0)$. In red: a branch of $W^u(-\sqrt{2}\mu^{-\frac{1}{4}}, \pi, 0)$ matches with a stable branch of a saddle at most for countable set of values of μ (see Theorem 2.7).

Since v strictly increases on orbits (referring to Figure 11, this means that each orbit on Λ goes from above to below, being v conventionally fixed on the vertical axis), we deduce that

- every orbit in the unstable manifolds $W^u(\sqrt{2}\mu^{-\frac{1}{4}}, 0, 0)$ and $W^u(\sqrt{2}\mu^{-\frac{1}{4}}, \pi, 0)$ is forward asymptotic to a sink;
- every orbit in the stable manifolds $W^s(-\sqrt{2}\mu^{-\frac{1}{4}}, 0, 0)$ and $W^s(-\sqrt{2}\mu^{-\frac{1}{4}}, \pi, 0)$ is backward asymptotic to a source.

We can actually say more. Indeed, since the system on Λ , Eq. (19), is invariant under the reflection

$$(v, \vartheta, u) \rightarrow (v, -\vartheta, -u), \quad (20)$$

then each branch of $W^u(\sqrt{2}\mu^{-\frac{1}{4}}, 0 \text{ or } \pi, 0)$ (resp. of $W^s(-\sqrt{2}\mu^{-\frac{1}{4}}, 0 \text{ or } \pi, 0)$) dies in a *distinct sink* (resp. emanates from a *distinct source*).

We are then left to analyse the ultimate behaviour of the remaining invariant manifolds of the saddle points: the stable ones of $(\sqrt{2}\mu^{-\frac{1}{4}}, 0 \text{ or } \pi, 0)$ and the unstable of $(-\sqrt{2}\mu^{-\frac{1}{4}}, 0 \text{ or } \pi, 0)$. In fact, there are two possibilities: either the stable manifolds emanate from sources and the unstable ones fall in sinks, or one or more of these manifolds match up forming a *saddle-saddle connection*. Next theorem (due to Gutwiler, see [37]), will show that for an open and dense set of $\mu > 1$, such saddle connections do not exist.

Theorem 2.7 (Gutwiler, [37]). *There exists an open and dense subset $\mathcal{O}$ of $(1, \infty)$ such that, if $\mu \in \mathcal{O}$, then the vector field on Λ satisfies*

- *each branch of the stable manifolds of any saddle point is backward asymptotic to distinct source;*
- *each branch in the unstable manifold of any saddle point is forward asymptotic to a distinct sink.*

Remark 2.8. Gutwiler's Theorem has been improved by Josefina Casasayas, Jaume Llibre (see [22, Theorem 12]). These authors proved that Gutwiler result can be improved to *any* $\mu \in (1, +\infty)$. Furthermore, they show that the two branches of the stable manifold of a saddle with $\vartheta = \vartheta_0$ and $v = v_0 > 0$ born from two different sources with $\vartheta = \vartheta_0 \pm 3\pi/2$. Similarly, the two branches of the unstable manifold of a saddle with $\vartheta = \vartheta_0$ and $v = v_0 < 0$ die in two different sinks with $\vartheta = \vartheta_0 \pm 3\pi/2$.

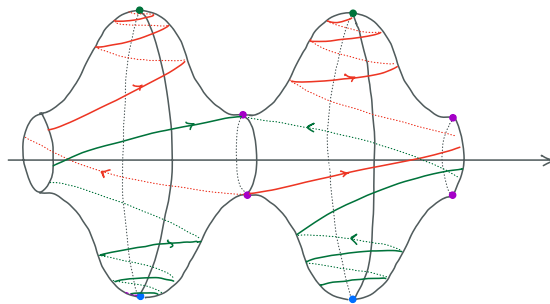


Figure 11: In green: the two branches of the stable manifold of a saddle with $v_0 > 0$. In red: the two branches of the unstable manifold of a saddle with $v_0 < 0$.

Proof. The fact that different branches of the same stable/unstable manifold emanate/die from/in different sources/sinks has been already proved using the reflection in Eq. (20).

We prove that, for an open and dense set of $\mu > 1$, $W^s(\sqrt{2}\mu^{-\frac{1}{4}}, \pi, 0)$ does not match up with $W^u(-\sqrt{2}\mu^{-\frac{1}{4}}, -\pi, 0)$, being the other situations exactly the same. As in Remark 2.6 in the context of the Kepler problem, we introduce the angular variable ψ , naturally induced by the energy relation

$$u = R_\mu(\vartheta) \cos \psi, \quad v = R_\mu(\vartheta) \sin \psi, \quad R_\mu(\vartheta) = \sqrt{-2\tilde{V}_\mu(\vartheta)}. \quad (21)$$

In terms of (ϑ, ψ) the system on Λ , (19), becomes

$$\begin{cases} \dot{\theta} = \sqrt{2}\Delta^{-\frac{1}{4}} \cos \psi \\ \dot{\psi} = \frac{1}{\sqrt{2}}\Delta^{-\frac{5}{4}}(1 - \mu) \sin \psi \cos \theta \sin \theta + \frac{1}{\sqrt{2}}\Delta^{-\frac{1}{4}} \cos \psi, \end{cases} \quad (22)$$

where

$$\Delta = \tilde{V}_\mu^{-2}(\vartheta) = \mu \cos^2 \theta + \sin^2 \theta.$$

Note that if $\mu = 1$ then $\Delta = 1$, and the variable change is the same proposed in Remark 2.6.

Let us now prove (22). The first line follows immediately replacing (21) in the second equation of (19). Concerning the second line, from the first equation of System (19) we have

$$\dot{v} = \Delta^{-\frac{1}{2}} \cos^2 \psi$$

and differentiating the expression of v in (21) one has

$$\begin{aligned} \dot{v} &= \Delta^{-\frac{1}{2}}(\sqrt{2}\Delta^{\frac{1}{4}} \cos \psi \dot{\psi} - \frac{\sqrt{2}}{2}(1 - \mu)\Delta^{-\frac{3}{4}} \sin \psi \cos \theta \sin \theta \dot{\theta}) \\ &= \sqrt{2}\Delta^{-\frac{1}{4}} \cos \psi \dot{\psi} - (1 - \mu)\Delta^{-\frac{3}{2}} \sin \psi \cos \psi \cos \theta \sin \theta. \end{aligned}$$

Comparing these two equations, one obtains the expression of $\dot{\psi}$ in (22).

In the new variables (ϑ, ψ) the rest point $(\sqrt{2}\mu^{-\frac{1}{4}}, \pi, 0)$ (resp. $(-\sqrt{2}\mu^{-\frac{1}{4}}, -\pi, 0)$) becomes $(\pi, \frac{\pi}{2})$ (resp. $(-\pi, -\frac{\pi}{2})$). Hence it will be proved that for an open and dense set of $\mu > 1$, $W^s(\pi, \frac{\pi}{2})$ doesn't match up with $W^u(-\pi, -\frac{\pi}{2})$.

When $\mu = 1$, we still refer to Remark 2.6, $W^u(-\pi, -\frac{\pi}{2})$ matches up exactly with $W^s(\pi, \frac{\pi}{2})$, indeed one branch of $W^u(-\pi, -\frac{\pi}{2})$ lies along the line

$$\psi = \frac{1}{2}\theta. \quad (23)$$

The goal now is to prove that there exists $\delta > 0$ such that for every $\mu \in (1, \delta)$ the stable manifold $W^s(\pi, \frac{\pi}{2})$ doesn't match up with the unstable manifold $W^u(-\pi, -\frac{\pi}{2})$.

Let $\zeta(\theta, \mu)$ denote the ψ -coordinate of this branch, at least for $-\pi < \theta \leq 0$. From Eq. (23) we have that $\zeta(0, 1) = 0$. Furthermore, for a sufficiently small $\delta > 0$, the following inequality holds

$$\frac{\partial}{\partial \mu} \zeta(0, \mu) > 0, \quad \text{for any } \mu \in (1, \delta). \quad (24)$$

As a consequence, $\zeta(0, \mu) > 0$ on $(1, \delta)$. With a similar argument, one may also show that the $W^s(\pi, \frac{\pi}{2})$ meets the line $\theta = 0$ slightly below the point $(0, 0)$.

From the expression of v in (21) we deduce that $v(0, \psi) < 0$ for $-\frac{\pi}{2} < \psi < 0$ and that $v(0, \psi) > 0$ for $0 < \psi < \frac{\pi}{2}$. This implies that, for δ small enough, the branches of $W^u(-\pi, -\frac{\pi}{2})$ and $W^s(\pi, \frac{\pi}{2})$ above don't match up.

By applying the same argument along the lines $\psi = \frac{1}{2}\theta - \pi$ for $0 < \theta < \pi$ and $\psi = \frac{1}{2}\theta + \pi$ for $-\pi < \theta < 0$, one sees that the remaining branches of the unstable manifold at $(-\pi, -\frac{\pi}{2}) = (\pi, -\frac{\pi}{2})$ and of the stable manifold at $(\pi, \frac{\pi}{2}) = (-\pi, \frac{\pi}{2})$ also don't match up for

δ small enough.

Hence one has that

$$W^u(-\pi, -\frac{\pi}{2}) \cap W^s(\pi, \frac{\pi}{2}) = \emptyset$$

at least for δ small enough.

This mismatch results can be extended $\forall \mu \in \mathcal{O} \subset [1, \infty)$, where $\mathcal{O}$ is an open and dense subset of $[1, \infty)$. In fact, since the vector field on Λ varies analytically with μ , it follows that $W^u(-\pi, -\frac{\pi}{2})$ and $W^s(\pi, \frac{\pi}{2})$ vary analytically with μ for $\mu > 1$. Hence if exists a value $\mu_0 \geq 1$ such that for $\mu = \mu_0$ these invariant manifolds match up then it exists a sufficiently small neighborhood I_{μ_0} of μ_0 such that $\forall \mu \in I_{\mu_0}$ the invariant manifolds does not match up. As a consequence the invariant manifolds above can match up for at most a discrete set of parameter values $\mu > 1$.

We conclude the proof showing inequality (24). Eliminating time from (22), one has that

$$\begin{aligned} \frac{d\psi}{d\theta} &= \frac{2^{-\frac{1}{2}} \Delta^{-\frac{5}{4}} (1 - \mu) \sin \psi \cos \theta \sin \theta + 2^{-\frac{1}{2}} \Delta^{-\frac{1}{4}} \cos \psi}{\sqrt{2} \Delta^{-\frac{1}{4}} \cos \psi} \\ &= \frac{2^{-2} \Delta^{-1} (1 - \mu) \sin \psi \sin(2\theta) + 2^{-1} \cos \psi}{\cos \psi} \\ &= \frac{(1 - \mu) \sin \psi}{4\Delta \cos \psi} \sin(2\theta) + \frac{1}{2}. \end{aligned}$$

Let us define

$$F(\theta, \psi, \mu) = \frac{(1 - \mu) \sin \psi}{4\Delta \cos \psi} \sin(2\theta) + \frac{1}{2}$$

and note that in terms of μ , the function $\zeta(\theta, \mu)$ satisfies the equation

$$\zeta(\theta, \mu) = \int_{-\pi}^{\theta} F(s, \zeta(s, \mu), \mu) ds.$$

Let

$$\zeta(\theta, \mu) = \zeta(\theta, 1) + (\mu - 1)\zeta'(\theta, 1) + O((\mu - 1)^2)$$

where $'$ indicates the differentiation with respect to μ .

From (23) we can derive that

$$\zeta(\theta, 1) = \frac{1}{2}\theta$$

and hence

$$\zeta(0, 1) = 0.$$

Let us now compute $\zeta'(\theta, 1)$:

$$\begin{aligned} \zeta'(\theta, 1) &= \int_{-\pi}^{\theta} \left(\frac{\partial F}{\partial \mu} + \frac{\partial F}{\partial \psi} \zeta'(s, 1) \right) ds \\ &= \int_{-\pi}^{\theta} -\frac{1}{4} \tan \frac{s}{2} \sin(2s) ds \\ &= \frac{1}{4} (\theta + \sin \theta (\cos \theta - 2) + \pi) \end{aligned}$$

and hence

$$\zeta'(0, 1) = \frac{\pi}{4} > 0.$$

Finally

$$\frac{\partial}{\partial \mu} \zeta(0, \mu) = \zeta'(0, 1) \implies \frac{\partial}{\partial \mu} \zeta(0, \mu) > 0,$$

and this completes the proof. $\square$

2.1.3 Collision and near-collision solutions

Let us return to the regularized system in McGehee coordinates, that is

$$\begin{cases} \dot{r} = rv \\ \dot{\theta} = u \\ \dot{v} = u^2 + \frac{1}{2}v^2 + \tilde{V}_\mu(\theta) \\ \dot{u} = -\frac{1}{2}vu - \tilde{V}'_\mu(\theta). \end{cases} \quad (25)$$

We aim now to deepen the analysis of the orbits that born from (or die into) the collision manifold or of the ones that transit near it.

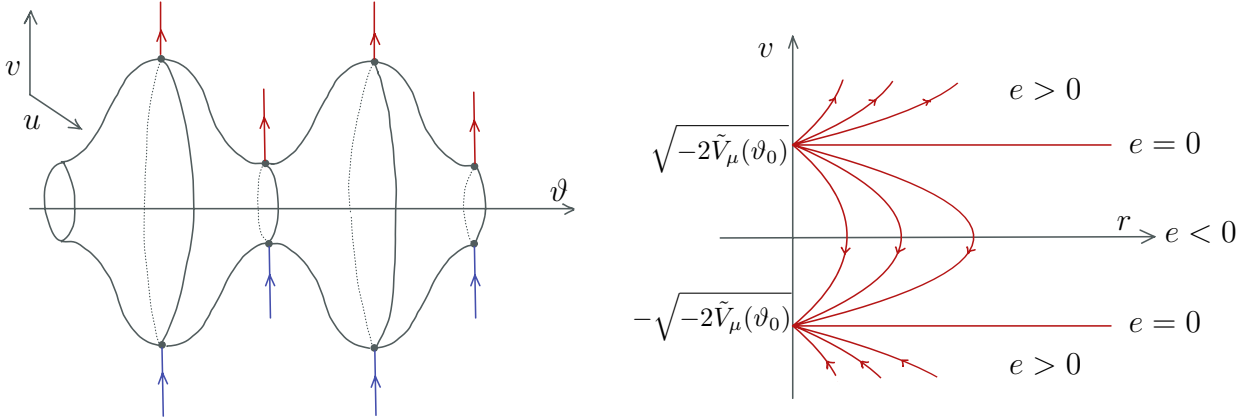


Figure 12: Left: In the (ϑ, v, u) -space homothetic motions belong to half-lines starting from equilibria on Λ . When $v_0 < 0$ homothetic motions enter into Λ and belong to the manifold $\mathcal{C}_{v_0, \vartheta_0}$; when $v_0 > 0$ they leave the collision manifold on half-lines belonging to $\mathcal{E}_{v_0, \vartheta_0}$. Depending on the energy level e , such motions can be bounded or not.

Right: In the (r, v) -plane is clear how homothetic motions depend on the energy level. In the picture, each orbit corresponds to a homothetic motion. When $e > 0$, orbits leave the singularity with $v_0 > 0$, along a ϑ_0 -ray, or reach it with $v_0 < 0$. For each fixed negative energy level, there is a unique homothetic orbit which begins and ends in collision. In the configuration space, such orbits born in collision with $v = \sqrt{-2\tilde{V}_\mu(\vartheta_0)}$ and, travelling along the ray $\vartheta = \vartheta_0$, meet the zero velocity curve, whereupon they fall back to collision.

The analysis of the flow on Λ , carried on in §2.1.2, implies that an orbit of (25) can reach the collision manifold if and only if it belongs to the stable manifold (outside Λ itself) of one of its rest points. Similarly, an orbit can leave from Λ just on the unstable manifold of one of them. Such rest points are listed in the table of Proposition 2.5. Hence, to each critical point, ϑ_0 , of $\tilde{V}_\mu(\vartheta)$ corresponds an invariant manifolds: when $v_0 < 0$ such manifold consists of collision orbits asymptotic to the rest point $(v_0, \vartheta_0, 0)$. When $v_0 > 0$, it consists of ejection orbits emanating from $(v_0, \vartheta_0, 0)$.

We will denote such manifolds respectively

$$\mathcal{C}_{v_0, \vartheta_0} \quad \text{and} \quad \mathcal{E}_{v_0, \vartheta_0}$$

In each of such eight manifolds there exists a family of special orbits termed *homothetic solution*. On such trajectories, just the variables r and v vary, while the other two remain fixed: $\vartheta = \vartheta_0$, with $\tilde{V}'_\mu(\vartheta_0) = 0$, and $u = 0$ (see Figure 12, left).

Homothetic orbits are then solutions belonging to the invariant planes $(r, \vartheta_0, v, 0)$. On such (r, v) -planes, orbits solve system

$$\begin{cases} \dot{r} = rv \\ \dot{v} = \frac{1}{2}v^2 + \tilde{V}_\mu(\vartheta_0) = re, \end{cases}$$

where in the second line we have used the energy relation. It is then straightforward to deduce the phase space in Figure 12, right, from which we deduce that homothetic motions are bounded if and only if $e < 0$.

Remark 2.9 (Kepler problem - Act 3). Let us go on with the analysis of the Kepler problem (i.e. when $\mu = 1$) in McGehee coordinate, which has been carried on in Remarks 2.4 and 2.6. In this case homothetic motions appears for any value of ϑ and consists in motions starting and/or ending in the collision, depending on the energy level e , tracing the ray ϑ^* (see Figure 13).

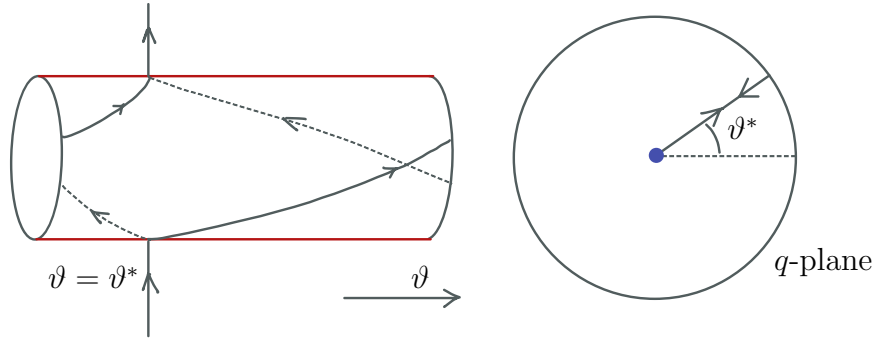


Figure 13: Left: the collision manifold Λ when $\mu = 1$ reduced to the cylinder Λ_K . Homothetic motions exists from the red edge above (or enters into the one below) along half-lines with $u = 0$ and $\vartheta = \vartheta^*$.

Right: for any ϑ^* , there exists a family of homothetic motions for the Kepler problem. In particular when $e < 0$ such motions are bounded.

The aim of the last part of this paragraph is to read off the local behaviour of solutions which pass near the origin, analysing the flow near Λ . First of all note that the flow in McGehee coordinates is everywhere analytic, hence an orbit which passes close to the origin follows a collision orbit on Λ until it reaches the neighbourhood of an unstable equilibria and it leaves it near its unstable manifold. Let us start understanding the simpler case of the Kepler problem.

Remark 2.10 (Kepler problem - Act 4). Consider a strongly eccentric ellipse with major axis on the direction $\vartheta = \vartheta^*$, that approaches the origin. In McGehee coordinates this motion approach Λ_K near the point $(\vartheta^*, -\pi/2)$. Afterwards, it has two possibilities depending upon which branch of the unstable manifold in Λ_K it follows. Along each branch, however, the variable θ is forced to change by 2π before yielding escape from a neighborhood of collision near an ejection orbit. The difference between the two branches is that in one case θ increases by 2π before leaving a neighborhood of the origin, while in the other one θ decreases by 2π . That is, in the first case such solutions are forced by θ to circle the origin in a counter-clockwise direction before escaping, while in the second case they circle the origin clockwise before escaping. This behavior is sketched in Figure 14.

Let us return to the anisotropic case and consider a non-collision trajectory that approaches the collision manifold Λ near a homothetic motion with $\vartheta_0 = 0$. Let us recall that

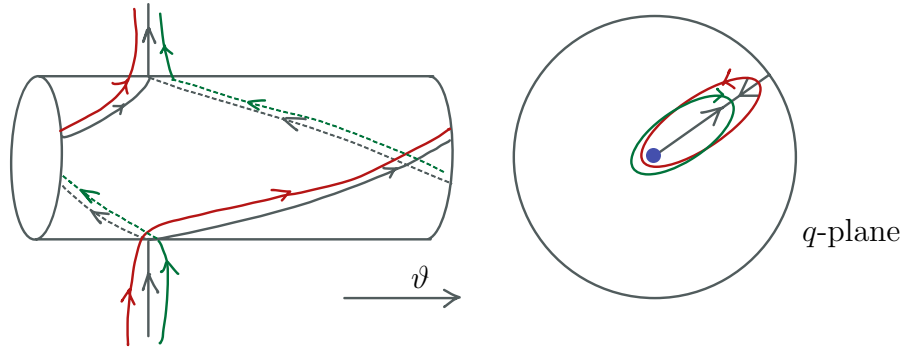


Figure 14: Left: Two orbits in the (ϑ, v, u) -space that approach Λ_K near the stable manifold of the equilibrium $(-\sqrt{2}, \vartheta^*, 0)$: the red one follows the branch of $W^u(-\sqrt{2}, \vartheta^*, 0)$ with increasing ϑ ; the green one follows the branch with decreasing ϑ . Right: Assuming $e < 0$, the two Keplerian ellipsis in the q -plane correspond to the red and green orbit in the left-picture.

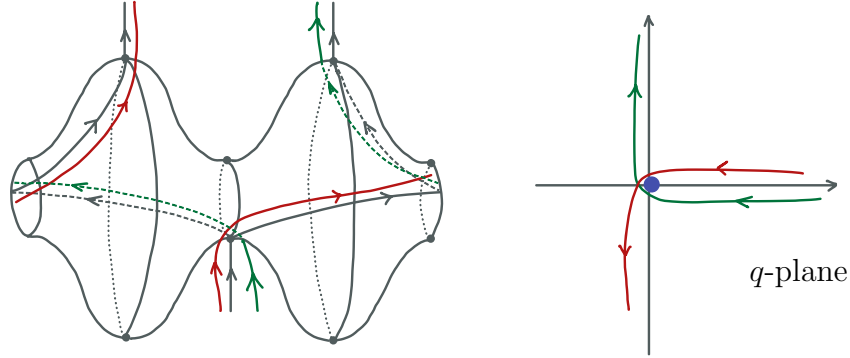


Figure 15: In these pictures the parameter μ is below $9/8$. The red and the green trajectories approach the collision manifold near the homothetic solution entering a saddle with $v_0 < 0$. Then they follow its unstable branches, the red trajectory follows the one with increasing ϑ , the green one the branch with decreasing ϑ . After that, the trajectories leave the neighborhood of Λ near a sink.

the point $(-\sqrt{2}\mu^{-\frac{1}{4}}, 0, 0)$ is a saddle on Λ and that it admits a stable manifold outside Λ . The improvement of Gutzwiller's Theorem (see Theorem 2.7 and Remark 2.8) asserts that the two unstable branches of this saddle are forward asymptotic to the two contiguous attracting equilibria at level $v = \sqrt{2}$. When $\mu \in (1, 9/8]$, the eigenvalues of such attracting equilibria are real, while when $\mu > 9/8$ the eigenvalues are complex. Hence, when a motion approaches the origin along the direction $\vartheta = 0$ it has two possible evolutions: it can leave it along the direction $\vartheta = -\pi/2$ or along $\vartheta = \pi/2$, depending on the branch of the unstable manifold of $(-\sqrt{2}\mu^{-\frac{1}{4}}, 0, 0)$ it follows. Depending on the value on μ , the orbits can oscillate along the q_2 axis. The situation is illustrated in Figures 15 and 16.

A similar analysis can be performed to describe close-to-collision trajectories that approach Λ near the equilibria at level $v = -\sqrt{2}$, or near the saddle at $\vartheta = \pi$.

Remark 2.11. Gutzwiller's Theorem, together with the improvement by Casasayas and Llibre imply that the anisotropic Kepler problem cannot be regularized in the sense of Easton (see [29], or the Easton's regularization by surgery in [34]).

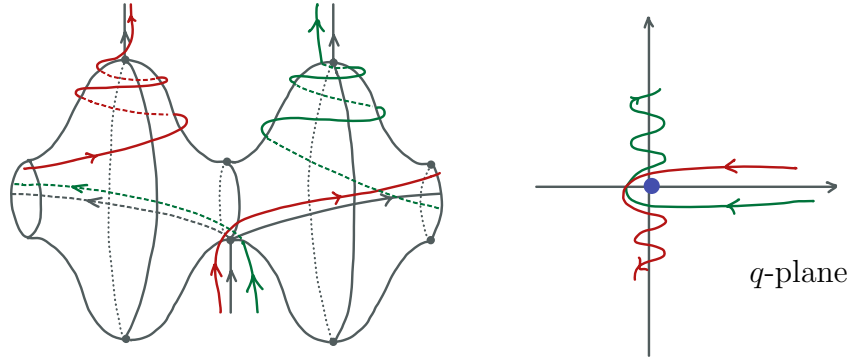


Figure 16: In these pictures the parameter μ is above $9/8$.

2.1.4 Chaotic behavior for $\mu > 9/8$

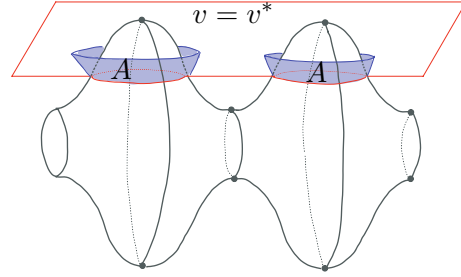
The analysis of the flow on Λ enables us to build infinitely many distinct collision/ejection and periodic solutions when $\mu > 9/8$. These constructions, which we make step-by-step supported by pictures, is based on the fact that the unstable manifold of each saddle with $v_0 > 0$, and the stable manifold of each saddle with $v_0 < 0$ have codimension 1.

Collision/ejection orbits. Let us fix $\mu > 9/8$, $e < 0$, and v^* in the open interval $(\sqrt{2}\mu^{-1/4}, \sqrt{2})$. Consider the set

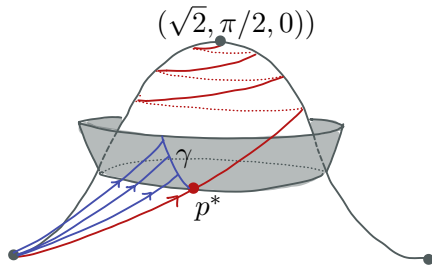
$$A = \{(r, v, \vartheta, u) \in H^{-1}(e) : r \in [0, \delta], v = v^*\}.$$

If δ is sufficiently small we have

$$A = [0, \delta] \times (\Lambda \cap \{v = v^*\}).$$



Let $p^* \in \Lambda$ be the intersection between the heteroclinic between $(\sqrt{2}\mu^{-1/4}, 0, 0)$ and $(\sqrt{2}, \pi/2, 0)$ (i.e. the branch of $W^u(\sqrt{2}\mu^{-1/4}, 0, 0)$ ending into $(\sqrt{2}, \pi/2, 0)$) and $\{v = v^*\}$.



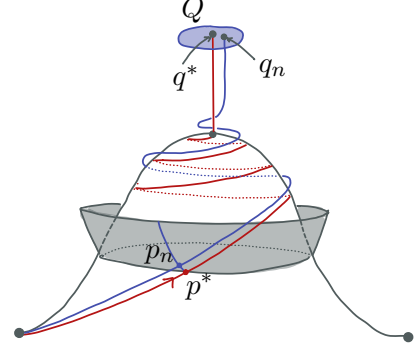
Since $W^u(\sqrt{2}\mu^{-1/4}, 0, 0)$, has codimension 1 and the mentioned heteroclinic orbit intersects transversally A (indeed, v increases strictly), we have that the intersection between A and $W^u(\sqrt{2}\mu^{-1/4}, 0, 0)$ is a curve $\gamma = \gamma(r)$, at least for δ small enough. Furthermore

$$\lim_{r \rightarrow 0} \gamma(r) = p^*.$$

Consider now the homothetic that leaves Λ from $(\sqrt{2}, \pi/2, 0)$. Since we have fixed $e < 0$, the Hill's region is bonded and the homothetic intersects $\partial H^{-1}(e)$ in a point, q^* .

Let Q be a neighbourhood of q^* in $\partial H^{-1}(e)$.

Under backwards time (observe that if $(q(t), p(t))$ solves the original Hamiltonian system, also $(q(-t), -p(-t))$ solves it), the orbits through each point in $q \in Q$ go back to Λ passing close to the sink $(\sqrt{2}, \pi/2, 0)$ and then move down spiralling near Λ . These orbits intersect A transversally, as far as Q is sufficiently small. The trace of these points of intersection with A form a spiral winding down to the circle $A \cap \Lambda$.

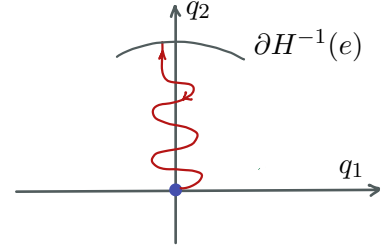


In particular there exists a sequence $(p_n)_n \subset A$ such that:

- $p_n \in \gamma$, for every n ,
- $\lim_n p_n = p^*$,
- p_n evolves to a point $q_n \in Q$.

Since $p_n \in \gamma = A \cap W^u(\sqrt{2}\mu^{-\frac{1}{4}}, 0, 0)$ for every n , to every p_n we can associate a collision motion such that:

- leaves the origin tangent to $\vartheta = 0$;
- oscillates around the $\vartheta = \pi/2$ direction;
- follows the homothetic up to the boundary of the Hill's region, near the homothetic;
- bounces on this boundary, and it goes back following the same path.



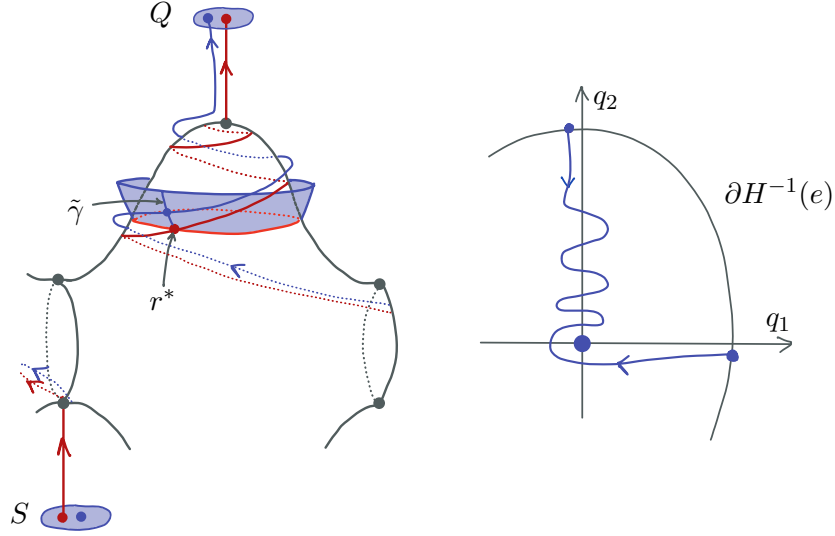
We then built infinite ejection-collision distinct orbits.

Periodic solutions. We now make a very similar construction to build periodic orbits.

We consider the point s^* , intersection between $\partial H^{-1}(e)$ and the homothetic motion that enters into Λ at the saddle $(-\sqrt{2}\mu^{-\frac{1}{4}}, 0, 0)$, and a neighborhood S , of s^* . When we follow the flow of a point in S , it passes close to $(-\sqrt{2}\mu^{-\frac{1}{4}}, 0, 0)$ and then it can follow one of its two unstable branches. Assume it follows the one that enters the sink $(\sqrt{2}, \pi/2, 0)$. This trajectory intersects A and ends in Q , near q^* . Reversibility in time allows for the construction of a periodic orbit.

To claim the existence of infinitely many periodic orbits of this kind, we consider the intersection point, r^* , between the branch of $W^u(-\sqrt{2}\mu^{-\frac{1}{4}}, 0, 0)$ on Λ ending in $(\sqrt{2}, \pi/2, 0)$ and $v = v^*$.

One can check easily that there exists an arc $\tilde{\gamma}$ of points in A that limits at r^* , comes from S and ends in Q . Each one of these points correspond to a different periodic orbit that touch the boundary of the Hill's region two times in a period.



Both families of orbits we have built (collisional and periodic) are characterized by a high number of intersections with the q_2 -axis. This makes a big gap with respect to periodic motions for the Kepler problem. Furthermore, these solutions suggest the possibility of codifying periodic orbits with respect to the number of crossings with the q_1 axis. This is the idea of Gutzwiller that allows one to find a dynamic of symbols hidden in the anisotropic Kepler problem.

Singular cross sections. In this paragraph we consider a transverse cross-section for the flow which takes into account the presence of the singularity. The idea is that a singular cross-section is crossed by all orbits except the ones that enter/exit to/from the singularity.

Consider a manifold M and a smooth vector field $X : M \rightarrow M$. Let $\Phi_t(\cdot)$ be the flow associated with the autonomous problem $\dot{x}(t) = X(x(t))$. This means that $\Phi_t(p)$ is the position in which the point p has moved after a time t .

Definition 2.12. A submanifold $S \subset M$ of codimension 1 is a *singular cross-section* for the field X (and for the associated flow Φ_t) if

1. X is transverse to S ;
2. every orbit of $\dot{x}(t) = X(x(t))$ meets S at least once;
3. there exist $C_1, \dots, C_n$ and $E_1, \dots, E_k$ submanifold of S of codimension 1 such that
 - $\forall p \in C_i, i = 1 \dots n, \Phi_t(p) \notin S$ for every $t > 0$;
 - $\forall p \in E_j, j = 1 \dots k, \Phi_t(p) \notin S$ for every $t < 0$;
4. for every $p \in S \setminus \bigcup_i C_i$ there exists $t_p > 0$ such that $\Phi_{t_p}(p) \in S$, and for every $q \in S \setminus \bigcup_j E_j$ there exists $t_q < 0$ such that $\Phi_{t_q}(p) \in S$.

Given a singular cross-section S for the flow Φ_t the following *first return* (or Poincaré) map is a well defined diffeomorphism

$$\mathcal{P} : S \setminus \bigcup_i C_i \rightarrow S \setminus \bigcup_j E_j \quad y \mapsto \Phi_{t_y}(y).$$

We can also think to $\mathcal{P} : S \rightarrow S$ as a map with singularities.

Let us now turn this notion in the context of the anisotropic Kepler problem

$$\begin{cases} q' = p, \\ p' = -\nabla V_\mu(q), \end{cases} \quad \text{with } H(p, q) = \frac{1}{2}|p|^2 + V_\mu(q). \quad (26)$$

Fix $e < 0$ and consider the following subsets of the elliptic energy level $H^{-1}(e)$

$$\begin{aligned} S^+ &= \{(p, q) \in H^{-1}(e) : p_2 = 0 \text{ and } q_2 > 0\}, \\ S^- &= \{(p, q) \in H^{-1}(e) : p_2 = 0 \text{ and } q_2 < 0\}. \end{aligned}$$

Proposition 2.13. $S_K = S^+ \cup S^-$ is a singular cross section for the flow associated to the anisotropic Kepler problem at a fixed energy level $e < 0$.

Proof. First, observe that S_K consists in initial conditions (p, q) whose momentum vector is parallel to the q_1 axis, indeed $p_2 = 0$. Hence, for every q in the bounded Hill's region $\{V_\mu(q) \leq e\}$ there are two possible momentum vectors $p = (p_1, 0)$, such that $p_1^2/2 + V_\mu(q) = e$. In other terms, both S^+ and S^- can be parametrized by an angle $\vartheta \in (0, \pi)$ (or $(\pi, 2\pi)$) and $p_1 \in \mathbb{R}$ with the energy-constraint

$$\frac{1}{2}p_1^2 = e - \frac{\tilde{V}_\mu(\vartheta)}{r},$$

where $\tilde{V}_\mu$ is the angular part of the potential defined in (16).

To see that the vector field is transverse to S_K , we consider $(q, p) \in S_K$, that is $q_2 \neq 0$ and $p_2 = 0$. In this point

$$p'_2 = -\frac{q_2}{|q|^3} \neq 0,$$

and transversality follows.

We now prove 3. and 4. of Definition 2.12. Let us start observing that at any $(p, q) \in S_K$ we have $q'_2 = 0$, and $q''_2 = p'_2 \neq 0$ has opposite sign of q_2 . The orbits then reach a relative maximal q_2 -value whenever they cross S^+ , a minimal one when they cross S^- .

Now we claim that a solution beginning at S^+ behaves just in few possible ways. Assume $q_2(0) > 0$ and $p(0) = (p_1(0), 0)$. Then in a right neighborhood of $t = 0$:

- $p'_2(0) = -q_2(0)/|q|^3 < 0$, and p_2 decreases;
- p_2 is negative and $q'_2(t) = p_2(t) < 0$;
- q_2 decreases (as far as $q_2(t)$ remains positive);
- $q''_2(t) = -q_2(t)/|q(t)|^3 < 0$ at such points, hence the solution must cross $q_2 = 0$
 - (a) at the origin, ending in a collision,
 - (b) in a different point.

Let us point out that at $q_2 = 0$ the solution can not be tangent to the axes $q_2 = 0$, just homothetic motions has this property.

In case (b), at some $\bar{t} > 0$ $q_2(\bar{t}) < 0$ and still $p_2(\bar{t}) < 0$. Hence, $p'_2(\bar{t}) > 0$ and p_2 increases. This situation persists in a right neighborhood of $\bar{t}$ and the solution pierces S^- . Hence, in this case, the orbit that started at S^+ go back to S_K .

When situation (a) occurs, the starting point necessarily belongs to a stable manifold of one of the equilibria on Λ . This orbit is termed *immediate collision orbit* from S_K . All immediate collision orbits with initial point in S_K form a finite union of submanifolds, given by the transverse intersection of the various stable manifolds with S_K . We then define

$$C = \{\text{immediate collision orbits from } S_K\}.$$

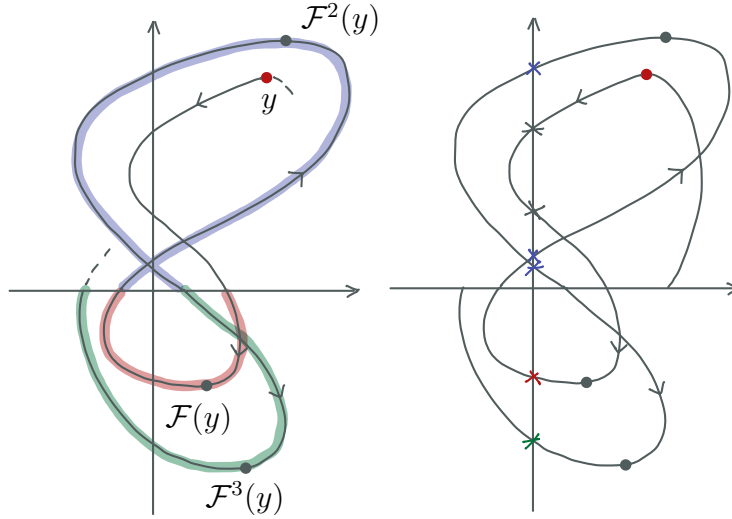


Figure 17: Left: The red point $y \in S^+$ and its evolution with the flow. The red orbit segment is the 1-st passage of y ; the blue one the 2-nd and green one the 3-rd. Right: with same colors the intersection of the orbit segments with the vertical axes. We can count: $s_0(y) = 2$, $s_1(y) = 1$, $s_2(y) = 3$, $s_3(y) = 1$.

Arguing similarly, we can claim that if an orbit starts in S^- , then after a certain amount of time it touches S_K , or it ends in C . Furthermore, we define the set of orbits that from a collision reaches the set S_K , that is

$$E = \{\text{immediate ejection orbits to } S_K\}.$$

We conclude showing point 2. of Definition 2.12, that is that every orbit (except the two homotetics in the q_1 -direction, that is the ones with $q_2 = 0$) meets S_K . Take (p, q) outside $q_2 = 0$; it is not restrictive to assume that $q_2 > 0$ (indeed if $q_2 < 0$, then since $p_2 \neq 0$, immediately $q_2 \neq 0$). Assume $p_2 \neq 0$ (otherwise we are already in S_K). When $p_2 > 0$, then q_2 increases and p_2 decreases up to S^+ . When $p_2 < 0$, then both q_2 and p_2 decrease and the particle reaches S^- . $\square$

Let us consider the following *first return map for the anisotropic Kepler problem*

$$\mathcal{F} : S_K \setminus C \rightarrow S_K \setminus E.$$

$\mathcal{F}$ takes a point $y \in S_K \setminus C$, hence in S^+ or in S^- , and associates $\mathcal{F}(y)$, the first intersection of the positive orbit of y with S_K , in S^- or in S^+ . Since S is a singular cross-section, $\mathcal{F}$ is a well defined diffeomorphism, and we can also extend $\mathcal{F} : S_K \rightarrow S_K$ understanding it as a singular map.

Definition 2.14. Let $y \in S_K$ and $j \in \mathbb{Z}$. The j -th passage of y is the orbit segment of y containing the point $\mathcal{F}^j(y)$ (when it exists). For any $j \in \mathbb{Z}$ such that $\mathcal{F}^j(y)$ exists, we define the natural number (see Figure 17)

$$s_j(y) = \# \text{ intersections of the } j\text{-th passage of } y \text{ with the } q_2\text{-axes.}$$

The sequence $s(y) = (s_j(y))_j$ terminates at some s_n with $n > 0$ if $F^n(y) \in C$ (here *terminates* means that the terms s_m with $m > n$ are not defined); it terminates at some

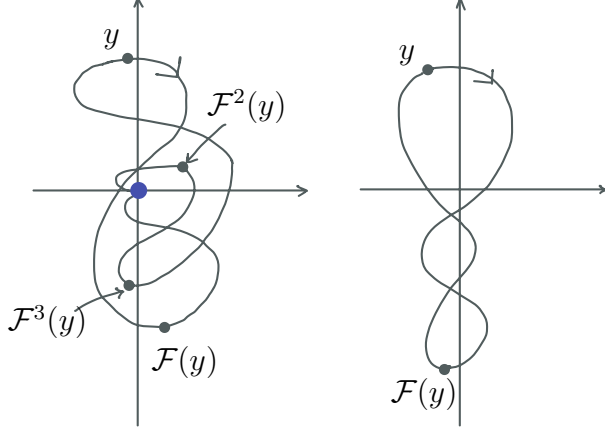


Figure 18: Left: The sequence $s(y)$ is $[\infty; 1, 2, 3, 2; \infty]$.

Right: the sequence $s(y)$ is here 2-periodic, indeed $s(y) = (\dots, 1, 5, 1, 5, 1, 5, \dots)$.

$-k < 0$ if $F^{-k}(y) \in E$ (in this case s_{-m} , $-m < -k$ are not defined). $s(y)$ takes then one of these forms

$$\begin{aligned} & (\dots s_{-2}, s_{-1}, s_0, s_1, s_2, \dots), \quad [-\infty; s_{-m}, \dots s_{-2}, s_{-1}, s_0, s_1, s_2, \dots s_n; \infty] \\ & (\dots s_{-2}, s_{-1}, s_0, s_1, s_2, \dots s_n; \infty] \text{ or } [-\infty; s_{-m} \dots, s_{-2}, s_{-1}, s_0, s_1, s_2, \dots). \end{aligned} \quad (27)$$

Hence, the sequence $s(y)$ gives a good qualitative picture of the behavior of the solutions through y . In particular, when $y \in C$ then it corresponds to a sequence in

$$\hat{C} = \{(\dots s_{-2}, s_{-1}, s_0; \infty] : s_j \in \mathbb{N}, j \leq 0\},$$

when $y \in E$ then it corresponds to a sequence in

$$\hat{E} = \{[-\infty; s_1, s_2, \dots) : s_j \in \mathbb{N}, j \geq 0\}.$$

If we term Λ the set of bi-infinite (possibly terminating) sequences of entire positive numbers as in (27), then the following diagram commutes

$$\begin{array}{ccc} S_K \setminus C & \xrightarrow{\mathcal{F}} & S_K \setminus E \\ \downarrow s & & \downarrow s \\ \Lambda \setminus \hat{C} & \xrightarrow{\sigma} & \Lambda \setminus \hat{E} \end{array}$$

We conclude the section with the following result.

Theorem 2.15 (Gutzwiller [38]). *The map s is a continuous surjection. Moreover, to any periodic sequence in $\Lambda \setminus \hat{C}$ correspond at least one periodic solution of the anisotropic Kepler problem.*

2.2 Keplerian billiards at high energies

Keplerian billiard are a special type of *billiards with potentials* that can be considered a generalization of classical *Birkhoff billiards*. In the last decades they have attracted the attention of a wider and wider community of mathematicians (for a quite exhaustive list of references, see [24] or [7]). Many results that hold for the classical case still persist when inside the billiard a non-constant potential is considered, but sometimes a new phenomenology can appear.

Let us start briefly recalling what a Birkhoff billiard is: in a billiard table, a bounded and planar domain with smooth boundary, a particle moves traveling along straight lines and with constant velocity. When it reaches the boundary, it reflects about the tangent line to the boundary, with an *elastic reflection* (see Figure 19, left). A natural first return map can be defined, and the dynamics is particularly well understood when the table is circular or elliptic, owing to their remarkable optical and geometric properties. In the circular case, the billiard map reduces to a rigid rotation in the phase space: the angle of reflection remains constant, resulting in a globally integrable system with all trajectories lying on invariant circles. Elliptic billiards also exhibit complete integrability: the phase space is globally foliated by smooth invariant curves, except for a 2-periodic orbit and its homoclinic connections (see Figure 20, right). In both cases, the existence of a family of invariant curves (and associated caustics) underpins the integrable structure. A central open question in the study of the integrability of Birkhoff billiards is the following:

*Among strictly convex, planar, smooth, bounded domains,
are ellipses the only integrable Birkhoff billiards?*

The conjecture, usually termed *Birkhoff-Poritsky Conjecture* (see [13, 53]), is that the answer is positive, but it is still open in general.⁴

In this Notes, we consider a generalization of Birkhoff billiards, and we propose some results in the flavor of Birkhoff-Poritsky conjecture. We take a billiard table with a mass located inside that generates a Keplerian potential. We will treat a classical version of the

⁴Several partial results and local rigidity theorems have been proved by Bialy [11], Avila, De Simoi and Kaloshin [2], Kaloshin and Sorrentino [44, 45], Huang, Kaloshin and Sorrentino [42], Bialy and Mironov [12]. Near ellipses, local versions of the conjecture have been affirmatively proven: small smooth perturbations of ellipses that remain integrable must be ellipses. For centrally symmetric tables, even stronger results are known, see [12].

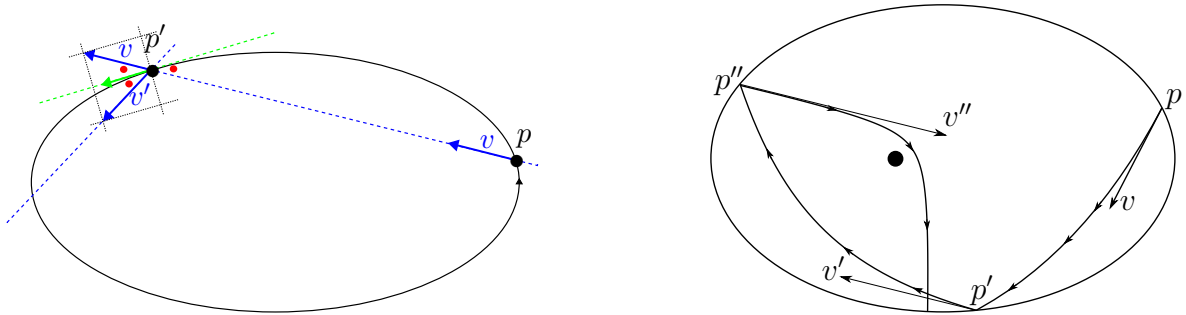


Figure 19: Left: In a classical billiard a particle move with constant velocity undergoing elastic reflections when it meets the boundary of the table.

Right: In a Keplerian billiard, a particle connects pair of points on the boundary following branches of Keplerian conics. The first return map takes a pair (p, v) and maps it into (p', v') .

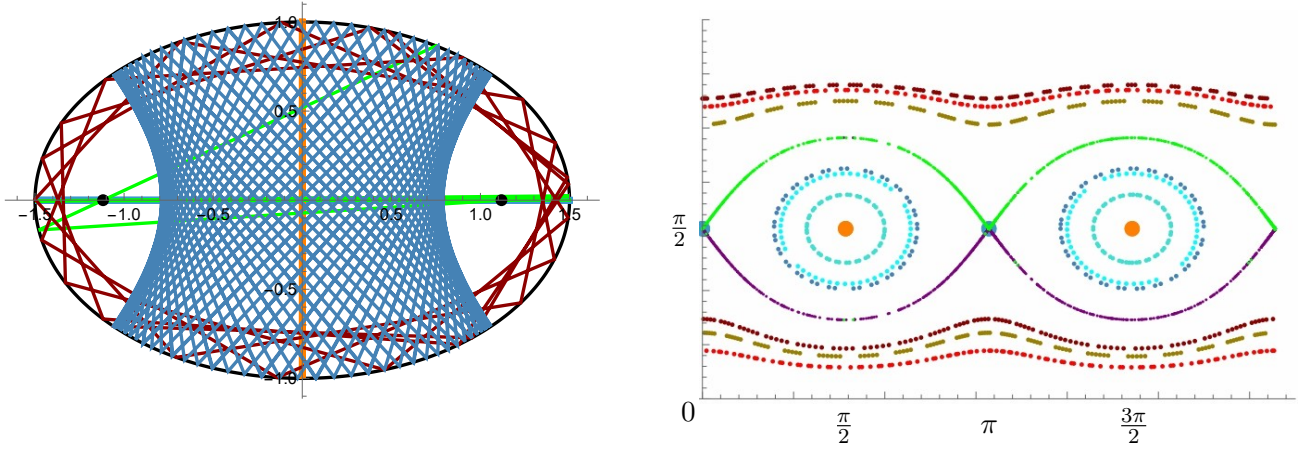


Figure 20: Left: Any trajectory in an elliptic billiard initially tangent to a given confocal conic (same foci of the elliptic boundary) remains tangent to it after each reflection. Such confocal conics are termed *caustics*. This structure arises naturally from the classical optical property of confocal conics.

Right: In an elliptic billiards the phase space is globally foliated by full-measure set of smooth invariant curves, except for a 2-periodic orbit and its homoclinic connections. Such curve are associated with the foliation by caustics to which billiard trajectories remain tangent under reflection. On the horizontal axis, a parameter describing the position on the boundary, on the vertical one, the angle between the tangent at such point and the velocity vector.

model, leaving any generalization to some recent papers on this subject (we refer to [8, 7, 15, 16, 26, 25, 56]).

We take a planar bounded and strictly convex domain, $D \subset \mathbb{R}^2$, and we consider an inner potential given by

$$V(z) = -\frac{\mu}{|z - c|}, \quad z \in D,$$

where $c \in \mathring{D}$ is a fixed point where a Keplerian center with mass $\mu > 0$ is located. A particle moves in D , under the influence of the gravitational force induced by V ; hence, the motion inside D consists in branches of Keplerian conics, which interact with the boundary ∂D with an *elastic reflection* (see Figure 19, right). At each bounce, the tangent component of the velocity is kept constant, while the normal one changes in sign. This model is usually referred to as the *(reflective) Keplerian billiard*.

Since the bounces are elastic and the induced force field conservative, the total energy of the moving particle

$$h = \frac{1}{2}|\dot{z}(t)|^2 + \frac{\mu}{|z(t) - c|}$$

is conserved along solutions and can take in principle any real value; nevertheless, since the results we expose hold for high values of the inner energy, we restrict ourselves to the case $h > 0$. Hence, the particle follows branches of Keplerian hyperbolae between two consecutive bounces. We observe that, with respect to a classical Birkhoff billiard, this model has two additional parameters: the energy h , which influences the shape of the trajectories, and the mass μ , both in its position and value.

In the context of Kepler billiards, the following result, due to Panov (see [52]), establishes a fundamental instance of integrability:

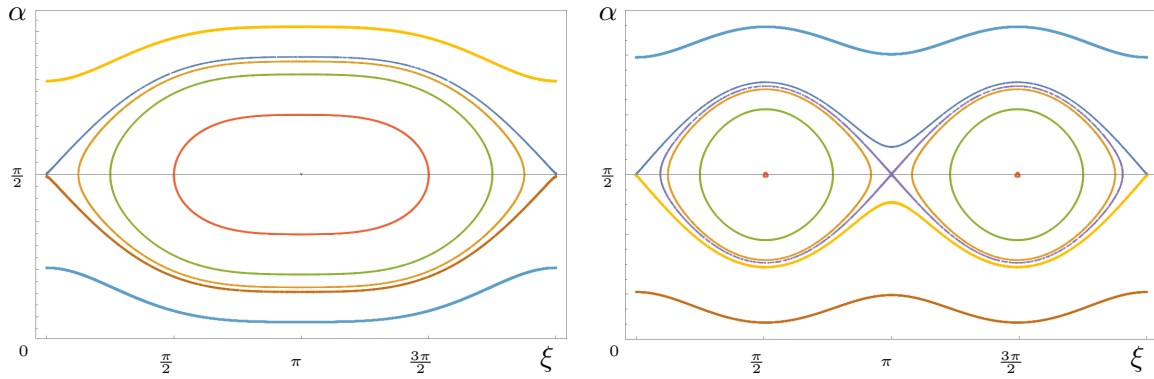


Figure 21: Phase portrait of a Keplerian billiard with the keplerian mass at its focus. Depending on the energy, the phase portrait changes. There exists a transition value, $\tilde{h}$, such that when $h < \tilde{h}$, then $(\pi, 0)$ is a centre, while when $h > \tilde{h}$, then $(\pi, 0)$ is a saddle and a pair of 2-periodic points appear.

*Elliptic Keplerian billiards are integrable at all energy levels
when the center of attraction is located at one of the foci of the ellipse*

(see also Figure 21 and its caption). Hence some natural questions arise:

- What happens if the center of mass is located in an elliptic domain, elsewhere than a foci?
- What happens if the domain D is not an ellipse?

The complete answers to these questions can be found in [8] and [4].

The aim of this section is to prove how the presence of chaos can be proved in a Keplerian billiard, when the domain satisfies a particular geometric condition.

2.2.1 First return map and generating functions

Assume, without loss of generality, that the bounded domain D contains the origin and that the mass is located there, i.e. $c = 0$. Furthermore, assume that ∂D admits a C^2 -parameterization by arc length $\gamma = \gamma(\xi)$, $\xi \in [0, L]$, with $\gamma(0) = \gamma(L)$. The dynamics of the Keplerian billiard is then described by a map F

$$F : (p, v) \rightarrow (p', v'),$$

that associate to a pair (p, v) , where $p = \gamma(\xi) \in \partial D$ and v is a velocity vector pointing inward D , a pair (p', v') where p' is the point reached by a Keplerian arc starting at p with velocity v and v' is the velocity of the particle, after a reflection (see Figure 19, right). The map F is usually termed *first return map*. Due to the asymptotic behavior of Keplerian arcs at high energy, it can be proved that the map F is always well defined, provided the energy is high enough (see [4, Lemma 2.1] if interested).

Anyway, as will be clear in a few, for our construction we just need that F is locally well defined. More precisely we need the good definition of the first return maps in intervals of the boundary that are neighborhoods of special points on ∂D . For this reason, we introduce the following definition.

Definition 2.16. A *central configuration* (in short, c.c.) is a value of the parameter $\bar{\xi} \in [0, L]$ such that $\bar{P} = \gamma(\bar{\xi})$ is a constrained critical point for $|\cdot|_{\partial D}$, that is, the position vector $\bar{P}$ is orthogonal to the boundary ∂D at $\bar{P}$.

A c.c. $\bar{\xi}$ is termed *strict* if $\bar{P}$ is a strict local maximum/minimum for $|\cdot|_{\partial D}$. A c.c. $\bar{\xi}$ is termed *non degenerate* if the second differential of the function $|\cdot|_{\partial D}$ is not degenerate at $\bar{P}$. Two c.c.s are *non-antipodal* if they correspond to a pair of points on ∂D which are not aligned with the origin.

Central configurations are of fundamental importance for the dynamics of a Keplerian billiard. Indeed, along their directions, one has equilibrium trajectories for the systems, called *homothetic motions*: they reflect on the mass on one side (through a Levi-Civita regularization process - see the proof of Lemma 2.24), and go back and forth along the half-line identified by $\overline{0\bar{P}}$ (of course, the segment $\overline{0\bar{P}}$ is entirely contained in D since we have assumed D strictly convex). We then understand that

If $\bar{\xi}$ is a c.c., then $(\gamma(\bar{\xi}), n_i)$ is a stationary point for the first return map,

where n_i is the inward pointing normal vector at $\gamma(\bar{\xi})$. For instance, in Figure 21 we see that a Keplerian billiard with a mass at one of its foci has two central c.c.s that correspond to the two vertex of the ellipse on the major axes. It can be proved that, on the one hand, the aphelion, $\bar{\xi} = 0$, remains a saddle, for any value of the energy; on the other hand, the perihelion, $\bar{\xi} = \pi$ is a center for low energy regime and bifurcates into a saddle and a pair of two periodic points when h is sufficiently large. The main result we prove in this chapter of the Notes has a geometric assumption on the domain D that it is linked to c.c.s.

Definition 2.17. A strictly convex planar domain with smooth boundary is termed *admissible* if it admits at least two non-antipodal strict central configurations. D is termed *strongly admissible* if these two c.c. are non-degenerate.

Theorem 2.18. *Assume that D is admissible. Then, for any sufficiently large h , there exists a subset of initial conditions for the first return map, X , such that $F : X \rightarrow X$ is semi-conjugate to the right shift on the set of bi-infinite sequences with two symbols, $\sigma : \Sigma_2 \rightarrow \Sigma_2$. Furthermore, if D is strongly admissible, then $F|_X$ is conjugate to $\sigma|_{\Sigma_2}$.*

The idea of the construction, that we follow here step-by-step, is to build complex orbits that shadow homothetic trajectories, passing from one to another using the Keplerian center as a scatterer. For this reason, we need the map F to be well defined just in suitable neighborhood of central configurations. Let us enter into the details, assuming, from now on, that the domain D is admissible. We use the following notation:

- $\bar{\xi}_0, \bar{\xi}_1$ are two non-antipodal strict c.c.s for D ;
- $I_0, I_1 \subset [0, L]$ are *non-antipodal neighborhoods* of $\bar{\xi}_0, \bar{\xi}_1$, i.e., each point in $\gamma(I_0)$ is non-antipodal to any point in $\gamma(I_1)$.

Take now $p_1, p_2 \in \gamma(I_0) \cup \gamma(I_1)$, possibly in the same interval, and consider the fixed-end problem

$$\begin{cases} z''(s) = -\nabla V(z(s)), & s \in [0, T], \\ \frac{1}{2}|z'(s)|^2 + V(z(s)) = h, & s \in [0, T], \\ z(0) = p_1, z(T) = p_2, \end{cases} \quad (28)$$

for some T . It is a classical result by Carl Jacobi (see [43], or the more recent book by Richard Battin [10, p. 274]), that when $p_1 \neq p_2$, since the two points are non-antipodal by assumption, System (28) admits exactly two smooth and distinct solutions, usually termed *direct* and *undirect arc connecting p_1 and p_2* and denoted

$$z_d(\cdot; p_1, p_2; h) \quad \text{and} \quad z_i(\cdot; p_1, p_2; h).$$

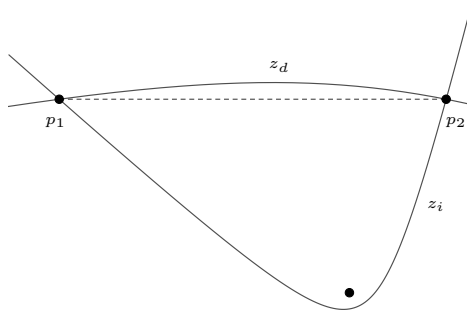


Figure 22: Direct and indirect Keplerian arcs with positive energy connecting two points in $\mathbb{R}^2 \setminus \{0\}$.

The first is homotopic to the segment $\overline{p_1 p_2}$ in $\mathbb{R}^2 \setminus \{(0,0)\}$; the second, connected to $\overline{p_1 p_2}$, forms a non-contractible loop in $\mathbb{R}^2 \setminus \{(0,0)\}$ (see Figure 22). When $p_1 = p_2$, the direct arc collapses to a point, while the indirect one is a collision-ejection solution that can be interpreted in the regularized Levi-Civita plane (see the proof of Lemma 2.24).

In this treatment, we will deal with indirect arcs connecting non-antipodal points on ∂D . First of all, let us notice that the indirect arc z_i , solution of (28), is entirely contained in the domain D , as far as h is sufficiently large. Indeed, z_i converges uniformly on $[0, T]$, when $h \rightarrow \infty$ to the broken line $p_1 0 \cup 0 p_2$ and D is strictly convex. The smoothness of the boundary ensures the next result.

Corollary 2.19. *There exists $\hat{h}$ such that for any $h > \hat{h}$ any indirect arc solution of (28) is entirely contained in D .*

From this property follows that the first return map is actually well defined, at least for high energies and for and near non-antipodal c.c.s.

Furthermore, we can define *Jacobi distance between p_1 and p_2* as

$$d(p_1, p_2; h) = \int_0^T |z'_i(s)| \sqrt{-V(z_i(s)) + h} ds, \quad (29)$$

where $z_i(\cdot) = z_i(\cdot; p_1, p_2; h)$. The integral quantity in (29) is the *Jacobi length of the arc z_i* and it is generally denoted with

$$L(z_i) = \int_0^T |z'_i(s)| \sqrt{-V(z_i(s)) + h} ds.$$

The Jacobi length is defined on the set of path connecting p_1 to p_2 and belonging to the Hill's region associated to the potential V , that is the set

$$H_{p_1 p_2} = \{z \in H^1([0, T], \mathbb{R}^2) : z(0) = p_1, z(T) = p_2, V(z(t)) \leq h \forall s \in [0, T]\}.$$

Remark 2.20. The uniqueness of an indirect arc as a solution of (28), $z_i(\cdot; p_1, p_2; h)$ implies that d is well defined and differentiable. In particular, when $p_1 = p_2$ the arc is a collision-ejection solution, and the integral representing its Jacobi length can be interpreted in its regularised formulation in the Levi-Civita plane.

Furthermore, the functions d is not a proper distance: as a matter of fact, when $p_1 = p_2$ it does not vanish but represent the non-zero Jacobi length of the homothetic arc, $2|p_1|$. Finally, the Jacobi length is invariant under reparametrizations of the path z .

It is useful to express the Jacobi distance as a function of the curve's parameter rather than of the endpoints in the plane. Given then $\xi_1, \xi_2 \in I_0 \cup I_1$ and $h > 0$, we can define the function

$$S(\xi_1, \xi_2; h) = d(p_1, p_2; h) = L(z_i). \quad (30)$$

We observe that this definition is coherent with the one of *generating function* in classical Birkhoff billiards (see the book of Serge Tabachnikov [55]).

Let us recall that the distance d is a C^∞ -functions of the endpoints in every set in which it is well defined; by the chain rule, this implies that S inherit the regularity of the curve γ , hence it is of class C^2 .

Denoting with ∂_a and ∂_b the partial derivatives respectively with respect to the first and second variable, one has

$$\begin{aligned}\partial_a S(\xi_1, \xi_2) &= \nabla_{p_1} d(\gamma(x_1), \gamma(x_2)) \cdot \dot{\gamma}(\xi_1) \\ \partial_b S(\xi_1, \xi_2) &= \nabla_{p_2} d(\gamma(x_1), \gamma(x_2)) \cdot \dot{\gamma}(\xi_2)\end{aligned}\tag{31}$$

where ∇_{p_1} and ∇_{p_2} denote respectively the gradient with respect to the first and the second point.

In order to compute these quantities, let us start by some classical results coming from Critical Points theory.

Lemma 2.21. *A path $z \in H_{p_1 p_2}$ is a critical point of $L(\cdot)$ if and only if it is a solution of the Euler-Lagrange equations*

$$\frac{d}{ds} \left(\sqrt{-V(z(s)) + h} \frac{z'(s)}{|z'(s)|} \right) + \frac{|z'(s)|}{2\sqrt{-V(z(s)) + h}} \nabla V(z(s)) = 0 \quad \text{a.e. in } [0, T]. \tag{32}$$

Proof. One has that z is a critical point for the Jacobi length if and only if for every $v \in H_0^1([0, T])$ one has that $dL(z)[v] = 0$. This is equivalent to require that the following chain of equalities holds:

$$\begin{aligned}0 &= \frac{d}{d\epsilon} L(z + \epsilon v)|_{\epsilon=0} = \left(\frac{d}{d\epsilon} \int_0^T |z'(s) + \epsilon v'(s)| \sqrt{-V(z(s) + \epsilon v(s)) + h} ds \right) \Big|_{\epsilon=0} \\ &= \int_0^T \frac{\sqrt{-V(z(s)) + h}}{|z'(s)|} z'(s) \cdot v'(s) - \frac{|z'(s)|}{2\sqrt{-V(z(s)) + h}} \nabla V(z(s)) \cdot v(s) ds \\ &= \int_0^T \left[-\frac{d}{ds} \left(\sqrt{-V(z(s)) + h} \frac{z'(s)}{|z'(s)|} \right) - \frac{|z'(s)|}{2\sqrt{-V(z(s)) + h}} \nabla V(z(s)) \right] \cdot v(s) ds,\end{aligned}$$

where in the last equation an integration by parts has been employed. As the identity must be true for every $v \in H_0^1([0, T])$, one has that the Euler-Lagrange equations (32) must hold for almost every $s \in [0, T]$. $\square$

We stress that if $u \in C^1([0, T]) \cap H_{p_1 p_2}$ is such that $-V(z(s)) < -h$ and $|z'(s)| > 0$ for every $s \in [0, T]$, by continuity one can infer that Eq. (32) holds everywhere in $[0, T]$. Making use of the previous result, it is possible to find a connection between solutions of problem (28) and critical points of $L(\cdot)$.

Lemma 2.22. *Let $\bar{z} \in H_{p_1 p_2}$ be a solution of problem (28) such that $-V(z(s)) < h$ for every $s \in [0, T]$. Then $\bar{z}$ is also a critical point of the Jacobi length L .*

Proof. Let us start by observing that, if $\bar{z}$ is a solution of problem (28) such that $V(\bar{z}(s)) > -h$ in $[0, T]$, then $\bar{z} \in C^2([0, T]) \cap H_{p_1 p_2}([0, T])$ and $|\bar{z}'(s)| > 0$ for every $s \in [0, T]$. We will then prove that $\bar{z}$ is a critical point for $L(\cdot)$ by verifying that it solves the Euler-Lagrange equations (32) for every $s \in [0, T]$. As a matter of fact, one has, for every $s \in [0, T]$,

$$\begin{aligned}&\frac{d}{ds} \left(\sqrt{-V(\bar{z}(s)) + h} \frac{\bar{z}'(s)}{|\bar{z}'(s)|} \right) \\ &= -\frac{\nabla V(\bar{z}(s)) \cdot \bar{z}'(s)}{2\sqrt{-V(\bar{z}(s)) + h}} \frac{\bar{z}'(s)}{|\bar{z}'(s)|} + \frac{\sqrt{-V(\bar{z}(s)) + h}}{|\bar{z}'(s)|} \bar{z}''(s) - \sqrt{-V(\bar{z}(s)) + h} \frac{\bar{z}'(s) \cdot \bar{z}''(s)}{|\bar{z}'(s)|^3} \bar{z}'(s),\end{aligned}$$

and the Euler-Lagrange equations follow from the first two lines in Eq. (28). The conclusion follows from Lemma 2.21. $\square$

We can now obtain the explicit expressions of the quantities involved in Eq. (31).

Lemma 2.23. *Let $\xi_1, \xi_2 \in I_0 \cup I_1$ and $h > 0$. Then*

$$\begin{aligned}\partial_a S(\xi_1, \xi_2; h) &= -\sqrt{-V(\gamma(\xi_1)) + h} \frac{z'_i(0; \gamma(\xi_1), \gamma(\xi_2); h)}{|z'_i(0; \gamma(\xi_1), \gamma(\xi_2); h)|} \cdot \dot{\gamma}(\xi_1) \\ \partial_b S(\xi_1, \xi_2; h) &= \sqrt{-V(\gamma(\xi_2)) + h} \frac{z'_i(T; \gamma(\xi_1), \gamma(\xi_2); h)}{|z'_i(T; \gamma(\xi_1), \gamma(\xi_2); h)|} \cdot \dot{\gamma}(\xi_2).\end{aligned}\tag{33}$$

Proof. Observe that the partial derivatives in Eq.(31) can be expressed as directional derivatives of L , for instance

$$\partial_a S(\xi_1, \xi_2) = \partial_{1, \dot{\gamma}(\xi_1)} d(\gamma(\xi_1), \gamma(\xi_2)) = \partial_{1, \dot{\gamma}(\xi_1)} L(z_i(\cdot; \gamma(\xi_1), \gamma(\xi_2))),$$

where, $\partial_{1, \sigma}$ denotes the directional derivative with respect to variations of the first endpoint in the direction of the unit vector σ .

Let us start by assuming $\xi_1 \neq \xi_2$. in this case $z_i(\cdot)$ is a classical solution of the associated Bolza problem such that $V(z_i(s)) > -h$ for every $s \in [0, T]$: by Lemma 2.22, the arc solves the associated Euler-Lagrange equations on $[0, T]$.

Let us now consider the new parametrization $s \mapsto t(s)$ given by

$$\begin{cases} \frac{dt}{ds} = \sqrt{2} \frac{-V(z_i(s)) + h}{L(z_i)} \\ t(0) = 0 \end{cases}$$

Observe that $t'(s) > 0$, for every s , and

$$\begin{aligned}t(L) &= \int_0^L t'(s) ds = \frac{\sqrt{2}}{L(z_i)} \int_0^L (-V(z_i(s)) + h) ds \\ &= \frac{\sqrt{2}}{L(z_i)} \int_0^L \sqrt{-V(z_i(s)) + h} \frac{1}{\sqrt{2}} |z'_i(s)| ds = \frac{1}{L(z_i)} \int_0^L \sqrt{-V(z_i(s)) + h} |z'_i(s)| ds \\ &= 1.\end{aligned}$$

Let, for every $t \in [0, 1]$,

$$\tilde{z}(t) = z_i(s(t))$$

and, denoting with a dot the derivative with respect to the new time-variable t , we have

$$\dot{\tilde{z}}(t) = z'_i(s(t)) \frac{1}{\sqrt{2}} \frac{L(z_i)}{-V(z_i(s)) + h}, \quad \text{hence } |\dot{\tilde{z}}(t)| = \frac{1}{\sqrt{2}} |z'_i(s(t))| \frac{L(z_i)}{-V(z_i(s)) + h},$$

so that

$$\forall t \in [0, 1] \quad |\dot{\tilde{z}}(t)| \sqrt{-V(\tilde{z}(t)) + h} = L(z_i).$$

Moreover, by the invariance of the Jacobi length under reparametrizations, one has the following three quantities which actually do not depend on t :

$$S(\xi_1, \xi_2; h) = L(z_i) = L(\tilde{z}).$$

We conclude computing

$$\partial_a S(\xi_1, \xi_2; h) = \partial_{1, \dot{\gamma}(\xi_1)} L(\tilde{z}).$$

Starting by (32), one can prove that the reparametrized curve $\tilde{z}$ satisfies the Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial \Sigma_2}{\partial \dot{z}} \right) = \frac{\partial \Sigma_2}{\partial z},$$

where $\Sigma_2(z(t)) = |\dot{z}(t)|^2 (V(z(t)) + h)$, namely, when $z = \tilde{z}$

$$\frac{d}{dt} [2(-V(\tilde{z}(t)) + h) \dot{\tilde{z}}(t)] = -|\dot{\tilde{z}}(t)|^2 \nabla V(\tilde{z}(t)). \quad (34)$$

Let us now compute $\partial_{1, \dot{\gamma}(\xi_1)} L(\tilde{z})$. Consider $\mathcal{L}(\tilde{z}) = L^2(\tilde{z})$, which is actually constant in t and hence

$$\mathcal{L}(\tilde{z}) = \int_0^1 \mathcal{L}(\tilde{z}(t)) dt.$$

Differentiating $\mathcal{L}(\tilde{z})$ with respect to the first endpoint, one has

$$\begin{aligned} 2L(\tilde{z}) \partial_{1, \dot{\gamma}(\xi_1)} L(\tilde{z}) &= \partial_{1, \dot{\gamma}(\xi_1)} \mathcal{L}(\tilde{z}) = \partial_{1, \dot{\gamma}(\xi_1)} \int_0^1 \mathcal{L}(\tilde{z}(t)) dt \\ &= \partial_{1, \dot{\gamma}(\xi_1)} \int_0^1 |\dot{\tilde{z}}(t)|^2 (-V(\tilde{z}(t)) + h) dt = \int_0^1 \partial_{1, \dot{\gamma}(\xi_1)} [|\dot{\tilde{z}}(t)|^2 (-V(\tilde{z}(t)) + h)] dt \\ &= \int_0^1 [2(-V(\tilde{z}(t)) + h) \dot{\tilde{z}}(t) \cdot \partial_{1, \dot{\gamma}(\xi_1)} \dot{\tilde{z}}(t) - |\dot{\tilde{z}}(t)|^2 \nabla V(\tilde{z}(t)) \cdot \partial_{1, \dot{\gamma}(\xi_1)} \tilde{z}(t)] dt. \end{aligned} \quad (35)$$

Moreover, multiplying (34) by $\partial_{1, \dot{\gamma}(\xi_1^E)} \tilde{z}(t)$ and integrating the result in $[0, 1]$, one obtains

$$\int_0^1 \frac{d}{dt} (2(-V(\tilde{z}(t)) + h) \dot{\tilde{z}}(t) \cdot \partial_{1, \dot{\gamma}(\xi_1)} \tilde{z}(t)) dt = - \int_0^1 |\dot{\tilde{z}}(t)|^2 \nabla V(\tilde{z}(t)) \cdot \partial_{1, \dot{\gamma}(\xi_1)} \tilde{z}(t) dt.$$

After an integration by part, observing that $\partial_{1, \dot{\gamma}(\xi_1)} \tilde{z}(0) = \dot{\gamma}(\xi_1)$ and $\partial_{1, \dot{\gamma}(\xi_1)} \tilde{z}(1) = 0$, we have

$$\begin{aligned} \int_0^1 [2(-V(\tilde{z}(t)) + h) \dot{\tilde{z}}(t) \cdot \partial_{1, \dot{\gamma}(\xi_1^E)} \tilde{z}(t) - |\dot{\tilde{z}}(t)|^2 \nabla V(\tilde{z}(t)) \cdot \partial_{1, \dot{\gamma}(\xi_1)} \tilde{z}(t)] dt = \\ -2(-V(\tilde{z}(0)) + h) \dot{\tilde{z}}(0) \cdot \dot{\gamma}(\xi_1). \end{aligned} \quad (36)$$

Comparing now (36) and (35), and recalling the expression of L , one gets the final expression

$$\partial_{1, \dot{\gamma}(\xi_1)} L(\tilde{z}) = - \frac{(-V(\tilde{z}(0)) + h) \dot{\tilde{z}}(0)}{L} \cdot \dot{\gamma}(\xi_1) = - \sqrt{(-V(\tilde{z}(0)) + h)} \frac{\dot{\tilde{z}}(0)}{|\dot{\tilde{z}}(0)|} \cdot \dot{\gamma}(\xi_1).$$

Returning now to the time parameter s , one obtains

$$\partial_a S(\xi_1, \xi_2) = - \sqrt{-V(\gamma(\xi_1)) + h} \frac{z'_i(0; \gamma(\xi_1), \gamma(\xi_2))}{|z'_i(0; \gamma(\xi_1), \gamma(\xi_2))|} \cdot \dot{\gamma}(\xi_1).$$

The same identity can be extended to the case $\xi_1 = \xi_2$ by observing that $V(z_i(s; \gamma(\xi_1), \gamma(\xi_1)))$ is strictly negative almost everywhere in $[0, T]$ and by taking into account the differentiable dependence of $z_i(\cdot; \gamma(\xi_1), \gamma(\xi_2))$ with respect to variations of the endpoints. $\square$

Take now two inner arcs that share a point on the boundary

$$z_{i,1}(t; \gamma(\xi_1), \gamma(\xi); h), \quad t \in [0, T_1], \quad \text{and} \quad z_{i,2}(t; \gamma(\xi), \gamma(\xi_2); h), \quad t \in [0, T_2].$$

We now fix the parameters ξ_1, ξ_2 , while ξ can be moved in order to verify the reflection law. This means that the tangent vector to the first arc at its final point $v = z_{i,1}(T_1)$ must me

reflected with respect to the tangent vector to the boundary, $\gamma'(\xi)$, into the tangent vector to the second arc at its initial point $w = z_{i,2}(0)$. This is equivalent to require that (the bounce is supposed to be elastic)

$$\left(\frac{v}{|v|} - \frac{w}{|w|} \right) \cdot \gamma'(\xi) = 0.$$

In this case we say that the first arc *reflects* in the second one at $\gamma(\xi)$. By Lemma 2.23, we have a variational carachterization of this reflection: $z_{i,1}$ reflects into $z_{i,2}$ at $\gamma(\xi)$ if and only if

$$\frac{d}{d\xi} (S(\xi_1, \xi; h) + S(\xi, \xi_2; h)) = 0. \quad (37)$$

Indeed, this equation reads

$$\begin{aligned} 0 &= \partial_b S(\xi_1, \xi; h) + \partial_a S(\xi, \xi_2; h) \\ &= \sqrt{-V(\gamma(\xi)) + h} \left(\frac{z'_{i,1}(T_1; \gamma(\xi_1), \gamma(\xi); h)}{|z'_{i,1}(T_1; \gamma(\xi_1), \gamma(\xi); h)|} - \frac{z'_{i,2}(0; \gamma(\xi), \gamma(\xi_2); h)}{|z'_{i,2}(0; \gamma(\xi), \gamma(\xi_2); h)|} \right) \cdot \dot{\gamma}(\xi). \end{aligned}$$

2.2.2 Asymptotic astimetes as $h \rightarrow \infty$

As already pointed out at page 48, when dealing with System (28), fixed two non-antipodal points in $p_1, p_2 \in \mathbb{R}^2 \setminus \{(0, 0)\}$ there exist a unique non-contractible Keplerian arc connecting them. We have termed it $z_1(\cdot; p_1, p_2; h)$. This arc need to be understood in a regularized sense if $p_1 = p_2$. We have also recalled that z_i converges uniformly, as $h \rightarrow \infty$ to the justapposition of segments $p_1 0 \cup 0 p_2$. In the next result we analyse the asymptotic behaviour of the Jacobi length of the arc z_i , $L(z_i)$.

Lemma 2.24. *Let $p_1, p_2 \in \mathbb{R}^2 \setminus \{(0, 0)\}$ be non-antipodal points, and $z_i(\cdot; p_1, p_2; h)$ be the (unique) indirect arc connecting them at energy h (in particular, z_i solves (28)). Then $L(z_i)$ has the following asymptotic behaviour*

$$L(z_i) = h(|p_1| + |p_2|) + \frac{\mu}{\sqrt{h}} \left(F(p_1, p_2; h) - \log \left(\frac{\mu}{2h} \right) \right),$$

where F is C^2 -bounded uniformly with respect to p_1 and p_2 , as $h \rightarrow \infty$.

Proof. We introduce the *Levi-Civita regularization*, and in particular we consider the transformation (in complex notation)

$$\frac{d}{ds} = \frac{1}{\sqrt{2}|z(s(\tau))|} \frac{d}{d\tau}, \quad w^2(\tau) = z(s(\tau))$$

Replacing in (28), that we recall here

$$\begin{cases} z''(s) = -\nabla V(z(s)), & s \in [0, T], \\ \frac{1}{2}|z'(s)|^2 + V(z(s)) = h, & s \in [0, T], \\ z(0) = p_1, z(T) = p_2, \end{cases}$$

we arrive to the harmonic repulsor-system for a suitable $\tilde{T}$

$$\begin{cases} w''(\tau) = w(\tau) & \tau \in [0, \tilde{T}] \\ \frac{1}{2}|w'(\tau)|^2 - \frac{1}{2}|w(\tau)|^2 = h' & \tau \in [0, \tilde{T}] \\ w(0) = w_1, w(\tilde{T}) = w_2 \end{cases} \quad (38)$$

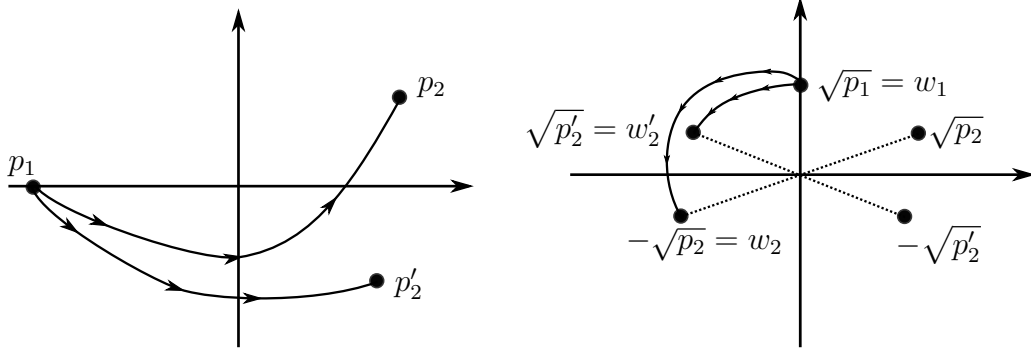


Figure 23: Left: Two keplerian arcs connecting non-antipodal point. Depending on the difference in the arguments between p_1 and p_2 (resp. p'_2), they correspond to indirect (resp. direct) arcs. Right: the same arcs in the Levi-Civita plane. Since $\det(p_1, p_2) < 0$, then $w_2 = -\sqrt{p_2}$; on the other hand, $\det(p_1, p'_2) > 0$ implies $w'_2 = \sqrt{p_2}$.

where $h' = \mu/2h$ and $w_1, w_2 \in \mathbb{C}$ are such that $w_1^2 = p_1$ and $w_2^2 = p_2$. Since the complex square determines a double covering over $\mathbb{C}$, we have two possible choices for every w_i , $i = 1, 2$. To obtain a solution corresponding to the indirect arc in the physical Kepler problem, we need to take $w_1 = \sqrt{p_1}$ and $w_2 = -\sqrt{p_2}$ (or, equivalently, $w_1 = -\sqrt{p_1}$ and $w_2 = \sqrt{p_2}$); with some geometric remarks we verify that, with this choice, $w_1 \cdot w_2 < 0$ (see also Figure 23).

Back to $\mathbb{R}^2$, the solution of (38) is

$$w(\tau) = Ae^\tau + Be^{-\tau},$$

where the vectors $A, B \in \mathbb{R}^2$ can be expressed in terms of w_1, w_2 and $\tilde{T}$ as

$$A = \frac{w_2 - w_1 e^{-\tilde{T}}}{e^{\tilde{T}} - e^{-\tilde{T}}}, \quad B = \frac{w_1 e^{\tilde{T}} - w_2}{e^{\tilde{T}} - e^{-\tilde{T}}},$$

and the energy conservation law gives $h' = -2A \cdot B$.

Computing the Jacobi length of z_i , one has then that (see [26] for more explicit computations)

$$\begin{aligned} L(z_i) &= \sqrt{h} \int_0^T |z'_i(s)| \sqrt{1 + \frac{\mu}{h} \frac{1}{|z_i(s)|}} ds = 2\sqrt{h} \int_0^{\tilde{T}} |w'(\tau)| \sqrt{2h' + |w(\tau)|^2} d\tau \\ &= 2\sqrt{h} \left[\frac{1}{2} \frac{e^{\tilde{T}} + e^{-\tilde{T}}}{e^{\tilde{T}} - e^{-\tilde{T}}} (|w_1|^2 + |w_2|^2) - \frac{2w_1 \cdot w_2}{e^{\tilde{T}} - e^{-\tilde{T}}} + \tilde{T} h' \right]. \end{aligned}$$

It is then necessary to express all the quantities depending on $\tilde{T}$ in terms of w_1 and w_2 . Let us start by computing

$$y(w_1, w_2; h') = \frac{e^{\tilde{T}} + e^{-\tilde{T}}}{2} = -\frac{w_1 \cdot w_2}{2h'} + \sqrt{\frac{1}{2h'} (|w_1|^2 + |w_2|^2) + \left(\frac{w_1 \cdot w_2}{2h'} \right)^2 + 1};$$

since $w_1 \cdot w_2 < 0$, the function $1/y$ is analytic on a right neighborhood of $h' = 0$ (which indeed

corresponds to $h \rightarrow +\infty$). Using the expression of y , one can then find

$$\begin{aligned}\tilde{T} &= \log \left(y + \sqrt{y^2 - 1} \right) \\ &= \log \left(\frac{|w_1 \cdot w_2|}{2} \right) - \log h' + \log \left(1 + \sqrt{1 + 2h' \frac{|w_1|^2 + |w_2|^2}{(w_1 \cdot w_2)^2} + \left(\frac{2h'}{w_1 \cdot w_2} \right)^2} \right) \\ &\quad + \log \left(1 + \sqrt{1 - \frac{1}{y^2}} \right) = -\log h' + g_1(w_1, w_2; h'),\end{aligned}\tag{39}$$

where g_1 is an analytic function. Furthermore, we have that

$$\begin{aligned}\frac{2}{e^{\tilde{T}} - e^{-\tilde{T}}} &= \frac{1}{\sqrt{y^2 - 1}} \\ &= \frac{\sqrt{2}}{|w_1 \cdot w_2|} h' \left(1 + \frac{h'(|w_1|^2 + |w_2|^2)}{(w_1 \cdot w_2)^2} + \sqrt{1 + \frac{2h'(|w_1|^2 + |w_2|^2)}{(w_1 \cdot w_2)^2} + \frac{4h'^2}{(w_1 \cdot w_2)^2}} \right)^{-1/2} \\ &= h' g_2(w_1, w_2; h'),\end{aligned}\tag{40}$$

where again g_2 is analytic. As for the coefficient of $(|w_1|^2 + |w_2|^2)$ one has that

$$\frac{e^{\tilde{T}} + e^{-\tilde{T}}}{e^{\tilde{T}} - e^{-\tilde{T}}} = \frac{1}{\sqrt{1 - \frac{1}{y^2}}};$$

this term is then analytic and its Taylor expansion around $h' = 0$ is of the form

$$\frac{e^{\tilde{T}} + e^{-\tilde{T}}}{e^{\tilde{T}} - e^{-\tilde{T}}} = 1 + h' g_3(w_1, w_2; h').\tag{41}$$

Taking together Eqs. (39), (40) and (41), and recalling that $h' = M/2E$, one obtains

$$\begin{aligned}L(z_i) &= 2\sqrt{h} \left(\frac{1}{2} (|w_1|^2 + |w_2|^2) + h' G(w_1, w_2; h') - h' \log h' \right) \\ &= \sqrt{h} (|p_1| + |p_2|) + \frac{M}{\sqrt{h}} \left(F(p_1, p_2; E) - \log \left(\frac{M}{2E} \right) \right),\end{aligned}$$

where F is C^2 -bounded with respect to p_1, p_2 as $E \rightarrow \infty$. $\square$

2.2.3 The importance of changing sign: an application of Miranda's Theorem

Let us return to the notion of admissibility for the domain D (see Definition 2.17 at page 48), and left $\bar{\xi}_0$ and $\bar{\xi}_1$ its non-antipodal strict c.c.s, with non-antipodal neighborhoods I_0 and I_1 . Since we are assuming that the two c.c.s are strict extremal for the function $|\gamma(\cdot)|$ and

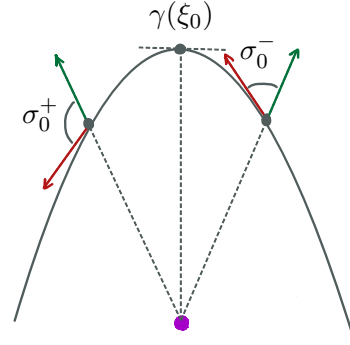
$$\frac{d}{d\xi} |\gamma(\xi)| = \frac{\gamma(\xi)}{|\gamma(\xi)|} \cdot \dot{\gamma}(\xi) = \cos \sigma_\xi,$$

(σ_ξ is the angle between $\gamma(\xi)$ and $\dot{\gamma}(\xi)$) we can select two compact intervals $[\alpha_i, \beta_i] \subset I_i$, $i = 0, 1$, such that:

- $\alpha_i < \bar{\xi}_i < \beta_i$, $i = 0, 1$;
- we denote σ_i^- the angle between $\gamma(\alpha_i)$ and $\dot{\gamma}(\alpha_i)$,
- we denote σ_i^+ the angle between $\gamma(\beta_i)$ and $\dot{\gamma}(\beta_i)$,
- it turns out that

$$\cos(\sigma_i^-) \cos(\sigma_i^+) < 0$$

(in the picture, $\bar{\xi}_0$ is a maximum, hence $\sigma_0^- < \pi/2$ and $\sigma_0^+ > \pi/2$)



Lemma 2.25. *Assume D is an admissible domain and let $[\alpha_i, \beta_i] \subset I_i$ as above. Then there exists $\bar{h}$ such that, for any $h > \bar{h}$ and for any $\xi_1, \xi_2 \in [\alpha_0, \beta_0] \cup [\alpha_1, \beta_1]$ (in the same interval or not), the quantity*

$$\frac{d}{d\xi} (S(\xi_1, \xi; h) + S(\xi, \xi_2; h)) = \partial_b S(\xi_1, \xi; h) + \partial_a S(\xi, \xi_2; h)$$

does not vanish, and takes opposite sign at the boundary of $[\alpha_i, \beta_i]$, $i = 0, 1$.

Proof. We use Lemma 2.24 to write the following asymptotic behaviours for $h \rightarrow \infty$

$$\begin{aligned} & \frac{d}{d\xi} (S(\xi_1, \xi; h) + S(\xi, \xi_2; h))|_{\xi=\alpha_i} \\ &= \frac{d}{d\xi} \left[h(|\gamma(\xi_1)| + 2|\gamma(\xi)| + |\gamma(\xi_2)|) + \right. \\ & \quad \left. + \frac{\mu}{\sqrt{h}} \left(F(\gamma(\xi_i), \gamma(\xi); h) + F(\gamma(\xi), \gamma(\xi_2); h) - 2 \log \left(\frac{\mu}{2h} \right) \right) \right]|_{\xi=\alpha_i} \\ &= 2h \cos(\sigma_i^-) + \frac{\mu}{\sqrt{h}} (\dots) \\ & \frac{d}{d\xi} (S(\xi_1, \xi; h) + S(\xi, \xi_2; h))|_{\xi=\beta_i} = 2h \cos(\sigma_i^+) + \frac{\mu}{\sqrt{h}} (\dots). \end{aligned}$$

Since $\cos(\sigma_i^\pm)$ take opposite signs, if h is sufficiently large, then also the two above quantities take opposite signs. We stress that, due to the uniform boundedness of F_1 with respect to the endpoints' parameters, the regularity of γ and the compactness of $[\alpha_i, \beta_i]$, $i = 0, 1$, one can find a uniform threshold value $\bar{h}$ such that the result hold. $\square$

Up to now, we have obtained two results that hold for h sufficiently large, and precisely Corollary 2.19 and Lemma 2.25. Let

$$h_0 = \max(\hat{h}, \bar{h}). \quad (42)$$

The results in the rest of this section will hold for $h > h_0$. The alphabet we use to construct a symbolic dynamic is $\{0, 1\}$, corresponding each symbol to one of the two strict central configurations, $\bar{\xi}_0$ and $\bar{\xi}_1$. Let us start considering the set of finite words

$$\mathbb{L} = \bigcup_{n \in \mathbb{N} \setminus \{0\}} \{0, 1\}^n. \quad (43)$$

For any fixed $\underline{\ell} = (\ell_0, \dots, \ell_{k-1}) \in \mathbb{L}$ we define the k -dimensional open rectangle

$$\mathbb{U}_{\underline{\ell}} = I_{\ell_0} \times I_{\ell_1} \times \dots \times I_{\ell_{k-1}}, \quad (44)$$

domain of (the parameters of) the bouncing points. Given $\underline{\xi} \in \mathbb{U}_{\underline{\ell}}$ and fixed $h > h_0$, there exists a unique concatenation of k indirect arcs

$$z_i(\cdot; \gamma(\xi_0), \gamma(\xi_1); h), z_i(\cdot; \gamma(\xi_1), \gamma(\xi_2); h), \dots, z_i(\cdot; \gamma(\xi_{k-1}), \gamma(\xi_0); h).$$

Such concatenation is trivially continuous, and been a loop, we can extend it by periodicity, obtaining the periodic function $z(\cdot; \underline{\xi}; h) : \mathbb{R} \rightarrow D$. For z to be an admissible trajectory for the dynamics, *the reflection law must be satisfied at every bouncing point*.

Let us introduce the *total Jacobi length* as

$$W_{\underline{\ell}}(\cdot; h) : \mathbb{U}_{\underline{\ell}} \rightarrow \mathbb{R}, \quad W_{\underline{\ell}}(\underline{\xi}; h) = S(\xi_0, \xi_1) + S(\xi_1, \xi_2) + \dots + S(\xi_{k-1}, \xi_0) \quad (45)$$

By means of the discussion at page 53, that yields to Eq. (37), we have that

$\hat{\underline{\xi}}$ solves $\nabla W_{\underline{\ell}}(\hat{\underline{\xi}}; h) = \underline{0}$ if and only if $z(\cdot; \hat{\underline{\xi}}; h)$ satisfies the reflection law at every bounce.

The searches for critical points of $W_{\underline{\ell}}(\cdot; h)$ is then equivalent to the search for periodic trajectories. Provided that h is large enough, the existence of a critical point of $W_{\underline{\ell}}(\cdot; h)$ is a straightforward consequence of the following classical result.

Theorem 2.26 (Poincaré-Miranda Theorem, [51]). *Let $F_1, \dots, F_d$ d -functions in the variables $(x_1, \dots, x_d)$ continuous on the d -dimensional hypercube*

$$R = \{(x_1, \dots, x_d) \mid |x_k| \leq L \text{ for every } k = 1, \dots, d\}$$

and such that for every $k = 1, \dots, d$

$$F_k(x_1, \dots, x_d)|_{x_k=-L} \cdot F_k(x_1, \dots, x_d)|_{x_k=L} < 0. \quad (46)$$

Then there exists at least a solution in $\overset{\circ}{R}$ of

$$F_k(x_1, \dots, x_d) = 0 \quad \text{for every } k = 1, \dots, d. \quad (47)$$

Proposition 2.27. *Given $\underline{\ell} \in \mathbb{L}$, the total Jacobi length $W_{\underline{\ell}}(\cdot; h)$ admits a critical point $\hat{\underline{\xi}}$ in $\overset{\circ}{\mathbb{U}}_{\underline{\ell}}$ provided $h > h_0$, where h_0 is introduced in Eq. (42).*

Proof. We claim to proof the existence of $\hat{\underline{\xi}} \in \overset{\circ}{\mathbb{U}}_{\underline{\ell}}$ such that

$$\nabla W_{\underline{\ell}}(\hat{\underline{\xi}}; h) = 0, \quad \text{or} \quad \partial_{\xi_j} W_{\underline{\ell}}(\hat{\underline{\xi}}; h) = 0, \text{ for every } j = 0, \dots, k-1.$$

Let $R = [\alpha_{\ell_0}, \beta_{\ell_0}] \times \dots \times [\alpha_{\ell_{k-1}}, \beta_{\ell_{k-1}}]$, and consider the function

$$F : \mathbb{U}_{\underline{\ell}} \rightarrow \mathbb{R}^k, \quad F_j(\underline{\xi}) = \partial_{\xi_j} W_{\underline{\ell}}(\underline{\xi}; h), \quad j = 0, \dots, k-1.$$

Since

$$\partial_{\xi_j} W_{\underline{\ell}}(\underline{\xi}; h) = \partial_b S(\xi_{j-1}, \xi_j) + \partial_a S(\xi_j, \xi_{j+1})$$

we have that

$$F_j(\underline{\xi})|_{\xi_j=\alpha_j} F_j(\underline{\xi})|_{\xi_j=\beta_j} < 0, \text{ for every } j = 0, \dots, k-1.$$

and Lemma 2.25 ensures the application of Poincaré-Miranda Theorem to obtain the required $\hat{\underline{\xi}} \in R \subset \overset{\circ}{\mathbb{S}}_{\underline{\ell}}$. $\square$

The main result of this section now follows straightforwardly.

Theorem 2.28. *Given $\underline{\ell} \in \mathbb{L}$ and $h > h_0$, there exists $\hat{\underline{\xi}} \in \overset{\circ}{\mathbb{U}}_{\underline{\ell}}$ and a periodic trajectory $z(\cdot; \hat{\underline{\xi}}; h)$ for the dynamic of the Keplerian billiard which realizes the word $\underline{\ell}$, in the sense that it connects $\gamma(\hat{\xi}_0), \dots, \gamma(\hat{\xi}_k)$.*

2.2.4 Construction of a symbolic dynamics

We are finally ready to prove the existence of a symbolic dynamics for our Keplerian reflective billiard in an admissible domain: we will construct a surjective and continuous application between a suitable set of initial conditions of trajectories and Σ_2 .

We now introduce the set A , which contains pairs (ξ, v) for which the first return map is well defined, such set is not empty by means of Theorem 2.28.

Let $A \subset [0, L] \times \mathbb{R}^2$ be the set of pairs (ξ, v) such that

- $\xi \in I_0 \cup I_1$;
- v enters D at $\gamma(\xi)$;
- the solution of the Kepler problem with initial condition $z(0) = \gamma(\xi)$ and $z'(0) = v$ is an indirect arc $z_i = z_i(\cdot; \gamma(\xi), \gamma(\xi'); h)$ on $[0, T]$, for some $T > 0$, such that $\xi' \in I_0 \cup I_1$ and $z'_i(T)$ is not parallel to $\dot{\gamma}(\xi')$.

Let v' be reflected of $z'_i(T)$ with respect to $\dot{\gamma}(\xi')$. Then the first return map

$$F: A \rightarrow F(A), \quad F(\xi, v) = (\xi', v'). \quad (48)$$

is well defined and is clearly a bijection. By recurrence, we construct the set X as the set on which all the positive and negative iterates of F are defined. Let

$$X_1^+ = A \quad \text{and} \quad X_1^- = A \cap F(A) = X_1^+ \cap F(X_1^+)$$

and observe that $X_1^- \neq \emptyset$, still by means of Theorem 2.28. Moreover both F and F^{-1} are well defined on X_1^- . Then, for any positive integer $k \geq 2$, we introduce the non-empty sets

$$X_k^+ = X_{k-1}^- \cap F^{-1}(X_{k-1}^-) \quad \text{and} \quad X_k^- = X_k^+ \cap F(X_k^+)$$

so that on X_k^- the iterates F^j , with $j \in \mathbb{Z}$ and $|j| \leq k$, are well defined (in our notation $F^0 = \text{Id}$). We are then ready to define the set of initial conditions in A that generate trajectories with an infinite number of transversal intersections with ∂D , namely

$$X = \bigcap_{k \in \mathbb{N}} X_k^-.$$

X is not empty since it contains at least initial conditions leading to periodic trajectories. Furthermore, on X are defined the iterates F^j , for any integer $j \in \mathbb{Z}$, and X is invariant for the first return map F . It is then convenient to consider the restriction

$$\mathcal{F} = F|_X.$$

We now introduce the map

$$\chi: X \rightarrow \mathcal{I}, \quad \chi(\xi, v) = r \iff \xi \in I_r, \quad r \in \{0, 1\}, \quad (49)$$

and the projection

$$\pi: X \rightarrow \pm\epsilon, \quad (\xi, v) \mapsto \pi(\xi, v) = (\ell_j)_{j \in \mathbb{Z}} \quad \text{with} \quad \ell_j = \chi(\mathcal{F}^j(\xi, v))$$

through which we are able to associate to every (ξ, v) the bi-infinite word realized by the trajectory of initial condition (ξ, v) .

We can then consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathcal{F}} & X \\ \downarrow \pi & & \downarrow \pi \\ \Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \end{array}$$

In order to prove Theorem 2.18, we are left to show that the map π is a continuous surjection.

Proposition 2.29. *If $h > h_0$, the projection map π is surjective.*

Proof. Let us take a sequence $\underline{\ell} = (\ell_j)_{j \in \mathbb{Z}} \in \Sigma_2$, and, for every $n \in \mathbb{N}$ consider the truncated sequences

$$\underline{\ell}^{(n)} = (\ell_{-n}, \dots, \ell_n),$$

which are elements of $\mathbb{L}$ of length $2n + 1$. Since $h > h_0$, by Theorem 2.28, for every $n \in \mathbb{N}$ there exists $\hat{\xi}^{(n)} \in \mathbb{S}_{\underline{\ell}^{(n)}}$ such that the corresponding trajectory $z^{(n)}(\cdot)$ realizes the word $\underline{\ell}^{(n)}$. Since the trajectories $z^{(n)}$ are periodic, without loosing in generality and possibly with a time translation, we can assume that

$$z^{(n)}(0) \in \gamma(I_{\ell_0}).$$

For every $n \in \mathbb{N}$, we now extend by periodicity $\hat{\xi}^{(n)}$ to obtain $\underline{\xi}^{(n)}$, with $\underline{\xi}^{(n)} = (\xi_k^{(n)})_{k \in \mathbb{Z}}$, a $(2n + 1)$ -periodic sequence.

Fixed k , the points $\xi_k^{(n)}$ belong definitely to the same compact interval prescribed by the sequence $\underline{\ell}$, that is

$$\forall k \in \mathbb{Z}, \exists N_k > 0, \exists i_k \in \{0, 1\} : \xi_k^{(n)} \in [\alpha_{i_k}, \beta_{i_k}], \forall n \geq N_k. \quad (50)$$

Through a diagonal process, we can construct an index sequence $(a_n)_n \subset \mathbb{N}$ with $a_n \rightarrow \infty$ such that

$$\xi_i^{(a_n)} \xrightarrow{n \rightarrow \infty} \bar{\xi}_i, \quad \text{for every } i \in \mathbb{Z},$$

where the sequence $\bar{\xi} = (\bar{\xi}_i)_i$ realizes the word $\underline{\ell}$, in the sense that $\bar{\xi}_j \in [\alpha_j, \beta_j]$.

Since $h > h_0$, we can then define the concatenation $\bar{z}: \mathbb{R} \rightarrow \mathbb{R}^2$ connecting the sequence of points $(\gamma(\bar{\xi}_i))_i$: our aim is now to verify that $\bar{z}$ is an admissible trajectory for the complete dynamics. This fact follows from the differentiable dependence of each arc and recalling that the concatenations $(z^{(a_n)})_n \subset (z^{(n)})_n$ satisfy the reflection law at every transition point.

Defining now $(\xi, v) = (\bar{\xi}_0, \bar{z}'(0))$ we have that $\pi(\xi, v) = \underline{\ell}$ and the surjectivity is proved. $\square$

In order to prove the continuity of π let us start with a preliminary lemma whose proof follows from the continuous dependence of Keplerian arcs with respect to variations of the endpoints and by the compactness of the intervals $[\alpha_j, \beta_j]$, $j \in \{0, 1\}$.

Lemma 2.30. *Let $h > h_0$; then there exists $C > 0$ such that for every $\xi_0, \xi_1 \in I_0 \cup I_1$*

$$T(\xi_0, \xi_1) \leq C,$$

where T is the time necessary to connect $\gamma(\xi_0)$ to $\gamma(\xi_1)$ with an indirect Keplerian arc (see Eq. (28)).

We are now ready to verify the continuity of the map π .

Proposition 2.31. *If $h > h_0$, the projection map π is continuous.*

Proof. Fix $(\xi_0, v_0) \in X$. In order to estimate the quantity $d(\pi(\xi, v), \pi(\xi_0, v_0))$ we define, for every $k \in \mathbb{Z}$, the k -th projection map

$$\pi_k: X \rightarrow \mathcal{I}, \quad \pi_k(\xi, v) = \chi\left(\mathcal{F}^k(\xi, v)\right),$$

where χ is defined as in Eq. (49), to have (see the definition of the distance in the set of bi-infinite sequences at page 16)

$$d(\pi(\xi, v), \pi(\xi_0, v_0)) = \sum_{k \in \mathbb{Z}} \frac{\beta(\pi_k(\xi, v), \pi_k(\xi_0, v_0))}{4^{|k|}}. \quad (51)$$

As the above series is always convergent, for any $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for every $(\xi, v) \in X$

$$\sum_{|k| \geq k_0} \frac{\rho(\pi_k(\xi, v), \pi_k(\xi_0, v_0))}{4^{|k|}} < \epsilon.$$

We will now prove that, if $\delta > 0$ is small enough and $|(\xi, v) - (\xi_0, v_0)| < \delta$, then

$$\sum_{|k| < k_0} \frac{\rho(\pi_k(\xi, v), \pi_k(\xi_0, v_0))}{4^{|k|}} = 0. \quad (52)$$

This is equivalent to require that, if $\underline{\ell} = \pi(\xi, v)$ and $\underline{m} = \pi(\xi_0, v_0)$, then $\ell_k = m_k$ for every $|k| < k_0$, namely, that the trajectories generated by (ξ, v) and (ξ_0, v_0) intersect the boundary ∂D in the same neighbourhoods in the first k_0 steps forward and backward. In view of Lemma 2.30 it is possible to find $a > 0$ such that both the trajectories cross ∂D at least $2k_0 + 1$ times within the time interval $[-a, a]$. The thesis then follows from the continuous dependence on the initial data on $[-a, a]$. $\square$

2.2.5 Some concluding remarks

- (i) In a simple way it is possible to characterise sequences corresponding to collisional motions (see [8, §2.5]). The exclusion of such sequences from the set of admissible bi-infinite words ensures that the trajectory you obtain is collisionless.
- (ii) Following the same construction proposed in these Notes, we can prove the presence of subsystems of orbits (semi)-conjugated to the Bernoulli shift on sequences with more than two symbols, possibly with a grammar (see [8]).
- (iii) Restricting our attention to ellipses, a simple computation shows that elliptic domains are not always admissible in the sense of Definition 2.17. It can be indeed proved that an ellipse is admissible for a.e. position on the mass inside it (see [7, §2]). Moreover, the points giving non-admissibility form one or two pairs of segments, depending on the eccentricity. This simple computation, together with some numerical simulations that provide chaos evidence also when the mass is located on such segments, suggests the search of more complex trajectories to shadow to construct a symbolic dynamics in convex Keplerian billiards. This construction is proposed in [4, §3] and yields to a different definition of admissible domain.
- (iv) A natural question then arises: which convex domains are admissible with respect to this new definition? The answer to this question can be found in the construction proposed in [4, §4]. There the authors prove that ellipses always exhibit a symbolic dynamics, except when the center is in one of the two foci. Furthermore, given any strictly convex domain and except at most for one position of the center, the Keplerian billiard always exhibit a dynamics of symbols. In the same papers, the authors propose the construction of an infinite-dimensional family of domains admitting such special points, via a string-type argument.
Let us observe that numerical evidences suggest that chaos persists also at such points.
- (v) We recall that the construction proposed here and in the mentioned papers holds for high energies. Numerical simulations suggest that this is just a technical obstruction.

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